Matrices

4.1 Introduction

A rectangular array of $m * n$ numbers consisting of m rows and n columns is termed as a matrix of order $m \times n$ and given as:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ or } A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

It may also be denoted as $A = [a_{ij}],\ i = 1 \dots m, j = 1 \dots n$

Null Matrix: A matrix with all zero elements is known as a null matrix or zero matrix.

Square matrix: A matrix having equal number of rows and columns is called a square matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \dots & \dots & \dots & \dots \\ a_{n1} & a_{m2} & \dots & a_{nn} \end{pmatrix} \text{ is a square matrix of order } n \times n$$

Sum of all elements in the principal diagonal of a square matrix A is known as '**Trace** $\boldsymbol{A}$' or '**Spur** $\boldsymbol{A}$'. $\therefore$ **Trace** $\boldsymbol{A = a_{11} + a_{22} + \cdots + a_{nn}}$

Identity or Unit Matrix: A square matrix having all principal diagonal elements unity and non-diagonal elements zero is called an identity matrix.

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ is an identity matrix of order 3}$$

Triangular Matrix: A square matrix in which all elements above or below principal diagonal are zero is called a triangular matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 4 & 2 & 0 \\ 2 & 6 & 1 \end{pmatrix} \qquad B = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 2 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Lower Triangular Matrix   Upper Triangular Matrix

Diagonal Matrix: A square matrix having all non-diagonal elements zero is called a diagonal matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ is a diagonal matrix of order 3}$$

Scalar Matrix: A diagonal matrix with all equal elements is called a scalar matrix.

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{pmatrix} \text{ is a scalar matrix of order 3}$$

Singular Matrix: If the determinant of a square matrix is zero i.e., $|A| = 0$, then it is known as a singular matrix.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -3 \\ -2 & 1 & 3 \end{pmatrix} \text{ is a singular matrix of order 3}$$

Transpose: The matrix A' or A^T obtained by interchanging rows and columns of a matrix A is known as its transpose.

$$A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ 0 & 2 & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 & 0 \\ 3 & -1 & 2 \\ 5 & 4 & 3 \end{pmatrix}$$

Symmetric and Skew-Symmetric Matrices:

A square matrix $A = [a_{ij}]$ is said to be symmetric if $A^T = A$ or $a_{ij} = a_{ji}\ \forall\ i, j$ and skew-symmetric if $A^T = -A$ or $a_{ij} = -a_{ji}\ \forall\ i, j$

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 2 & 4 \\ 3 & 4 & 3 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & -1 & 2 \\ 1 & 0 & -3 \\ -2 & 3 & 0 \end{pmatrix}$$

Symmetric Matrix — Skew- Symmetric Matrix

Results: 1. Diagonal elements of a skew-symmetric matrix are all zero as

$$a_{ii} = -a_{ii} \ \Rightarrow\ a_{ii} = 0$$

2. Any real matrix can be uniquely expressed as the sum of a symmetric and a skew-symmetric matrix as $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, where $(A + A^T)$ is symmetric, while $(A - A^T)$ is skew-symmetric

Orthogonal Matrix

A square matrix $A = [a_{ij}]$ is said to be orthogonal if $AA^T = I = A^TA$

Result: If A and B are two orthogonal matrices, then AB is also a orthogonal matrix.

Proof: $(AB)(AB)^T = (AB)B^TA^T \qquad \because\ (AB)^T = B^TA^T$

$= A(BB^T)A^T$

$= AIA^T \qquad \because B$ is an orthogonal matrix

$= AA^T = I \qquad \because A$ is an orthogonal matrix

4.2 Algebra of Matrices

Addition and Subtraction of Matrix: Addition or subtraction can be performed on two matrices if and only if they are of same order.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

Then $A \pm B = \begin{pmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} \\ a_{31} \pm b_{31} & a_{32} \pm b_{32} & a_{33} \pm b_{33} \end{pmatrix}$

Multiplication of Matrix by a Scalar: If we multiply a matrix A by a scalar k, then each element of the matrix is multiplied by k

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \qquad kA = \begin{pmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{pmatrix}$$

Multiplication of Two Matrices: Matrix product AB is possible only if number of columns in matrix A are same as number of rows in matrix B.

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \qquad B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{pmatrix}$$

$$C = AB = \begin{pmatrix} c_{11} = a_{11}b_{11} + a_{12}\,b_{21} + a_{13}b_{31} & c_{12} = a_{11}b_{12} + a_{12}\,b_{22} + a_{13}b_{32} \\ c_{21} = a_{21}b_{11} + a_{22}\,b_{21} + a_{23}b_{31} & c_{22} = a_{21}b_{12} + a_{22}\,b_{22} + a_{23}b_{32} \end{pmatrix}$$

Note that: (i) $A_{m\times n}B_{n\times k} = C_{m\times k}$

(ii) $AB \neq BA$ in general

(iii) $AB = 0$ does not necessarily imply that $A = 0$ or $B = 0$

(iv) $AB = 0$ does not necessarily imply that $BA = 0$

For example, $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ $AB = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ $BA = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

Example 1 If $A = \begin{pmatrix} \sin x & \cos x \\ \sin x & \cos x \end{pmatrix}$ $B = \begin{pmatrix} \sin x & \sin x \\ \cos x & \cos x \end{pmatrix}$ find AB and BA

Solution: $AB = \begin{pmatrix} \sin^2 x + \cos^2 x & \sin^2 x + \cos^2 x \\ \sin^2 x + \cos^2 x & \sin^2 x + \cos^2 x \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$BA = \begin{pmatrix} 2\sin^2 x & \sin 2x \\ \sin 2x & 2\cos^2 x \end{pmatrix}$

Example 2 Express the matrix $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ 0 & 2 & 3 \end{pmatrix}$ as the sum of symmetric and skew-symmetric matrices.

Solution: $A = \begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ 1 & 2 & 3 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 & 1 \\ 3 & -1 & 2 \\ 5 & 4 & 3 \end{pmatrix}$

$$\frac{1}{2}(A+A^T)=\begin{pmatrix}1 & \frac{5}{2} & 3\\ \frac{5}{2} & -1 & 3\\ 3 & 3 & 3\end{pmatrix}, \qquad \frac{1}{2}(A-A^T)=\begin{pmatrix}0 & \frac{1}{2} & 2\\ \frac{-1}{2} & 0 & 1\\ -2 & -1 & 0\end{pmatrix}$$

$$\therefore A=\begin{pmatrix}1 & \frac{5}{2} & 3\\ \frac{5}{2} & -1 & 3\\ 3 & 3 & 3\end{pmatrix}+\begin{pmatrix}0 & \frac{1}{2} & 2\\ \frac{-1}{2} & 0 & 1\\ -2 & -1 & 0\end{pmatrix}$$

: Symmetric Skew-Symmetric

4.3 Minors, Cofactors, Determinants and Adjoint of a matrix

Minors associated with elements of a square matrix

A minor of each element of a square matrix is the unique value of the determinant associated with it, which is obtained after eliminating the row and column in which the element exists.

For a 2×2 matrix $A=\begin{pmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}$

$M_{11}= a_{22}$, $M_{12}= a_{21}$, $M_{21}= a_{12}$, $M_{22}= a_{11}$

For a 3×3 matrix $A=\begin{pmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix}$

$M_{11}=\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix}$, $M_{12}=\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix}$,…, $M_{33}=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}$

Cofactors associated with elements of a square matrix

The cofactor of each element is obtained on multiplying its minor by $(-1)^{i+j}$.

$C_{ij} = (-1)^{i+j}M_{ij}$

Determinant of a square matrix

Every square matrix is associated with a determinant and is denoted by det (A) or $|A|$.

$$\det(A)=|A|=\begin{vmatrix}a_{11} & a_{12} & \dots & a_{1n}\\ a_{21} & a_{22} & \dots & a_{2n}\\ \dots & \dots & \dots & \dots\\ a_{n1} & a_{n2} & \dots & a_{nn}\end{vmatrix}$$

Determinant of order n can be expanded by any one row or column using the formula

$|A| = \sum_{j=1}^{n} a_{ij}\ C_{ij}$, where C_{ij} is the cofactor corresponding to the element a_{ij} .

A determinant of order 2 is evaluated as:

$|A|=\begin{vmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{vmatrix}=a_{11}a_{22}-a_{12}a_{21}$

A determinant of order 3 is evaluated as:

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} M_{ij}$$

$$= a_{11} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} - a_{12} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + a_{13} \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$$

$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

Note: A determinant may be evaluated using any row or column, value remains the same.

Properties of Determinants

- Value of a determinant remains unchanged if rows and columns are interchanged i.e. $|A| = |A^T|$
- If any two rows or columns are interchanged, the value of determinant is multiplied by (-1)
- The value of determinant remains unchanged if k times elements of a row (column) is added to another row (column).
- If elements in any row (column) in a determinant are multiplied by a scalar k, then value of determinant is multiplied by k. Thus, if each element in the determinant is multiplied by k, value of determinant of order n multiplies by k^n i.e., $|kA| = k^n|A|$
- If A and B are square matrices of same order, then $|AB| = |A||B|$

Adjoint of a square matrix

The adjoint of a square matrix A of order n is the transpose of the matrix of cofactors of each element. If C_{11}, C_{12}, C_{13} ,..., C_{nn} be the cofactors of elements a_{11}, a_{12}, a_{13} ,..., a_{nn} of the matrix A. Then adjoint of A is given by

$$adj(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

4.4 Inverse of a Matrix

The inverse of a square matrix A of order n, denoted by A^{-1} is such that

$AA^{-1} = A^{-1}A = I_n$ where I_n is an identity matrix of order n.

A matrix is invertible if and only if matrix is non-singular i.e., $|A| \neq 0$. There are many methods to find inverse of a square matrix.

4.4.1 Inverse of a matrix using adjoint

Working rule to find inverse of a matrix using adjoint:

1. Calculate $|A|$
   1. If $|A| = 0$, inverse does not exist

ii. if $|A| \neq 0$, go to step 2

2. Find $adj(A)$ and compute the inverse using the formula $A^{-1} = \frac{adj(A)}{|A|}$

Example 3 Find inverse of the matrix $\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

Solution: Let $A = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$

$$|A| = 1(16 - 9) - 3(4 - 3) + 3(3 - 4) = 1(7) - 3(1) + 3(-1) = 1$$

$$adj(A) = \begin{pmatrix} C_{11} & C_{12} & \dots & C_{1n} \\ C_{21} & C_{22} & \dots & C_{2n} \\ \dots & \dots & \dots & \dots \\ C_{n1} & C_{n2} & \dots & C_{nn} \end{pmatrix}^T = \begin{pmatrix} C_{11} & C_{21} & \dots & C_{n1} \\ C_{12} & C_{22} & \dots & C_{n2} \\ \dots & \dots & \dots & \dots \\ C_{1n} & C_{2n} & \dots & C_{nn} \end{pmatrix}$$

$C_{11} = (-1)^2(16 - 9) = 7$ $C_{12} = (-1)^3(4 - 3) = -1$ $C_{13} = (-1)^4(3 - 4) = 1$

$C_{21} = (-1)^3(12 - 9) = -3$ $C_{22} = (-1)^4(4 - 3) = 1$ $C_{23} = (-1)^5(3 - 3) = 0$

$C_{31} = (-1)^4(9 - 12) = -3$ $C_{32} = (-1)^5(3 - 3) = 0$ $C_{33} = (-1)^6(4 - 3) = 1$

$$\therefore adj(A) = \begin{pmatrix} 7 & -1 & 1 \\ -3 & 1 & 0 \\ -3 & 0 & 1 \end{pmatrix}^T = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{1}\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

4.4.2 Inverse of a matrix by using Gauss-Jordan method

To find the inverse of a matrix using Gauss-Jordan method, we take an augmented matrix $(A : I)$ and transform it into another augmented matrix $(I : A)$ using elementary row (column) transformations.

Elementary row (column) transformations: As the name suggests, row (columns) operations are executed on matrices according to certain set of rules such that the transformed matrix is equivalent to the original matrix. These rules are:

- Any two rows (columns) are interchangeable i.e., $R_i \leftrightarrow R_j$ or $C_i \leftrightarrow C_j$
- All the elements of any row (column) can be multiplied by any non-zero number k i.e., $R_i \rightarrow kR_i$
- All the elements of a row (column) can be added one to one to corresponding scalar multiples of another row (column) i.e., $R_i \rightarrow R_i + kR_j$

Working rule to find inverse of a matrix using Gauss-Jordan method:

1. Prepare an augmented matrix $(A : I)$

2. Using elementary row transformations make element (1,1) of the augmented matrix as 1, and using this make all other elements in the 1st column zero.
3. Now make the element (2,2) as 1, using row transformations and remaining elements in the 2nd column zero.
4. Continue the process until the augmented matrix is transformed to $(I : A^{-1})$

Note: (i) Do not apply row and column transformations to the same matrix while using Gauss-Jordan method.

(ii) While using column transformations make element (1,1) of the augmented matrix as 1, and using this make all other elements in the 1st row zero and similarly proceed for other rows.

(iii) In the process of forming identity matrix, ensure that previously formed zeros and ones are not altered while applying row (column) transformations. For this while making element (2,2) as 1, do not use R_1 (C_1) and while making element (3,3) as one neither use R_1 (C_1) nor R_2 (C_2).

Example 4 Find the inverse of the following matrices using Gauss-Jordan method

(i) $\begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{pmatrix}$ (ii) $\begin{pmatrix} 3 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$

Solution: (i) Let $A = \begin{pmatrix} 3 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 3 \end{pmatrix}$

Augmented matrix is: $\begin{pmatrix} 3 & 1 & 3 & : & 1 & 0 & 0 \\ 3 & 1 & 4 & : & 0 & 1 & 0 \\ 4 & 1 & 3 & : & 0 & 0 & 1 \end{pmatrix}$

Transforming element at (1,1) position to one

$R_1 \to -R_1 + R_2$ $\begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 3 & 1 & 4 & : & 0 & 1 & 0 \\ 4 & 1 & 3 & : & 0 & 0 & 1 \end{pmatrix}$

Making element at (2,1) and (3,1) positions as 0

$R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$ $\begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 4 & : & 3 & 1 & -3 \\ 0 & 1 & 3 & : & 4 & 0 & -3 \end{pmatrix}$

Elements at (2,2) and (1,2) are already 1 and 0, so making element at (3,2) position to zero

$R_3 \to R_3 - R_2$ $\begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 4 & : & 3 & 1 & -3 \\ 0 & 0 & -1 & : & 1 & -1 & 0 \end{pmatrix}$

Now transforming element at (3,3) position to one

$$R_3 \to -R_3 \qquad \begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 4 & : & 3 & 1 & -3 \\ 0 & 0 & 1 & : & -1 & 1 & 0 \end{pmatrix}$$

Element at (1,3) is 0, so Transforming element at (2,3) position to zero

$$R_2 \to R_2 - 4R_3 \quad \begin{pmatrix} 1 & 0 & 0 & : & -1 & 0 & 1 \\ 0 & 1 & 0 & : & 7 & -3 & -3 \\ 0 & 0 & 1 & : & -1 & 1 & 0 \end{pmatrix} = (I : A^{-1})$$

$$\therefore \ \mathrm{A}^{-1} = \begin{pmatrix} -1 & 0 & 1 \\ 7 & -3 & -3 \\ -1 & 1 & 0 \end{pmatrix}$$

(ii) Let $A = \begin{pmatrix} 3 & -1 & 2 \\ 2 & -1 & 3 \\ -1 & 1 & 1 \end{pmatrix}$

Augmented matrix is: $\begin{pmatrix} 3 & -1 & 2 & : & 1 & 0 & 0 \\ 2 & -1 & 3 & : & 0 & 1 & 0 \\ -1 & 1 & 1 & : & 0 & 0 & 1 \end{pmatrix}$

Transforming element at (1,1) position as 1

$$R_1 \to R_1 - R_2 \qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 2 & -1 & 3 & : & 0 & 1 & 0 \\ -1 & 1 & 1 & : & 0 & 0 & 1 \end{pmatrix}$$

Transforming element at (2,1) and (3,1) positions as 0

$$R_2 \to R_2 - 2R_1 \text{ , } R_3 \to R_3 + R_1 \quad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & -1 & 5 & : & -2 & 3 & 0 \\ 0 & 1 & 0 & : & 1 & -1 & 1 \end{pmatrix}$$

Transforming element at (2,2) position as 1

$$R_2 \to -R_2 \qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & 1 & -5 & : & 2 & -3 & 0 \\ 0 & 1 & 0 & : & 1 & -1 & 1 \end{pmatrix}$$

Element at (1,2) is 0, so transforming element at (3,2) position to zero

$$R_3 \to R_3 - R_2 \qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & 1 & -5 & : & 2 & -3 & 0 \\ 0 & 0 & 5 & : & -1 & 2 & 1 \end{pmatrix}$$

Now making element at (3,3) position 1

$$R_3 \to \frac{1}{5} R_3 \qquad \begin{pmatrix} 1 & 0 & -1 & : & 1 & -1 & 0 \\ 0 & 1 & -5 & : & 2 & -3 & 0 \\ 0 & 0 & 1 & : & -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

Now transforming elements at (1,3) and (2,3) positions 0

$$R_1 \to R_1 + R_3\ ,\ R_2 \to R_2 + 5R_3 \begin{pmatrix} 1 & 0 & 0 & : & \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \\ 0 & 1 & 0 & : & 1 & -1 & 1 \\ 0 & 0 & 1 & : & -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix} = (I : A^{-1})$$

$$\therefore\ \mathrm{A}^{-1} = \begin{pmatrix} \frac{4}{5} & -\frac{3}{5} & \frac{1}{5} \\ 1 & -1 & 1 \\ -\frac{1}{5} & \frac{2}{5} & \frac{1}{5} \end{pmatrix}$$

4.5 Solution of System of Linear Simultaneous Equations

Here we will be discussing some direct methods of solving a system of linear equations.

4.5.1 Matrix Method

Working rule to solve a system of equations using matrix method

1. Write the system of equations as $AX = B$
2. Calculate $|A|$
   i. If $|A| = 0$, system of equations can not be solved using matrix method
   ii. if $|A| \neq 0$, go to step 3
3. Find $adj(A)$ and compute the inverse using the formula $A^{-1} = \frac{adj(A)}{|A|}$
4. Solution of the system of equations is given by $X = A^{-1}B$

Example 4 Solve the system of equations using matrix method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ ,\ B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

$|A| = 1(12 + 30) - 2(9 - 10) + 1(-18 - 8) = 18 \neq 0$

$C_{11} = (-1)^2(12 + 30) = 42$ $C_{12} = (-1)^3(6 + 6) = -12$ $C_{13} = (-1)^4(10 - 4) = 6$

$C_{21} = (-1)^3(9 - 10) = 1$ $C_{22} = (-1)^4(3 - 2) = 1$ $C_{23} = (-1)^5(5 - 3) = -2$

$C_{31} = (-1)^4(-18 - 8) = -26$ $C_{32} = (-1)^5(-6 - 4) = 10$ $C_{33} = (-1)^6(4 - 6) = -2$

$$\therefore adj(A) = \begin{pmatrix} 42 & -12 & 6 \\ 1 & 1 & -2 \\ -26 & 10 & -2 \end{pmatrix}^T = \begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}$$

$$A^{-1} = \frac{adj(A)}{|A|} = \frac{1}{18}\begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}$$

$$X = A^{-1}B = \frac{1}{18}\begin{pmatrix} 42 & 1 & -26 \\ -12 & 1 & 10 \\ 6 & -2 & -2 \end{pmatrix}\begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 42(5) + 1(-4) - 26(10) \\ -12(5) + 1(-4) + 10(10) \\ 6(5) - 2(-4) - 2(10) \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -54 \\ 36 \\ 18 \end{pmatrix}$$

$\therefore x = -3, y = 2, z = 1$

4.5.2 Gauss Elimination Method

Working rule to solve system of equations using Gauss Elimination method

1. Write the system of equations as $AX = B$
2. Write the matrix in augmented form as $C = (A: B)$
3. Reduce matrix A in $C = (A: B)$ to echelon form using row transformations
4. Solve the system of equations $AX = B$ by backward substitution method.

Example5 Solve the system of equations using Gauss Elimination method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix}, \; X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \;, B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

Augmented matrix $C = \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 2 & 4 & -6 & : & -4 \\ 1 & 5 & 3 & : & 10 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as 0

$$R_2 \to R_2 - 2R_1 \,, R_3 \to R_3 - R_1 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & -2 & -10 & : & -14 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (2,2) to one

$$R_2 \to R_2/-2 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (3,2) to zero

$$R_3 \to R_3 - 2R_2 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & -9 & : & -9 \end{pmatrix}$$

$\therefore$ Corresponding system of equations is given as

$$x + 3y + 2z = 5 \quad \ldots(1)$$
$$4y + 5z = 7 \quad \ldots(2)$$
$$-9z = -9 \quad \ldots(3)$$

Solving by back substitution

③⇒$z = 1$, using $z = 1$ in ② ⇒$y = 2$, using $y = 2,\ z = 1$ in ①⇒$x = -3$

∴ $x = -3,\ y = 2,\ z = 1$ is the required solution of given system of equations

4.5.3 Gauss Jordan Elimination Method

Working rule to solve system of equations using Gauss Jordan Elimination method

1. Write the system of equations as $AX = B$
2. Write the matrix in augmented form as $C = (A\!:\!B)$
3. Apply elementary row transformations to reduce the matrix A in $C = (A\!:\!B)$ to unit matrix
4. Last column of the transformed matrix augmented matrix gives vector X.

Example6 Solve the system of equations using Gauss Jorden Elimination method

$$x + 3y + 2z = 5$$
$$2x + 4y - 6z = -4$$
$$x + 5y + 3z = 10$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 4 & -6 \\ 1 & 5 & 3 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}\ ,B = \begin{pmatrix} 5 \\ -4 \\ 10 \end{pmatrix}$$

Augmented matrix $C = \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 2 & 4 & -6 & : & -4 \\ 1 & 5 & 3 & : & 10 \end{pmatrix}$

Transforming element at (2,1) and (3,1) positions as 0

$$R_2 \to R_2 - 2R_1\ ,\ R_3 \to R_3 - R_1 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & -2 & -10 & : & -14 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming element at (2,2) to one

$$R_2 \to R_2/-2 \quad \begin{pmatrix} 1 & 3 & 2 & : & 5 \\ 0 & 1 & 5 & : & 7 \\ 0 & 2 & 1 & : & 5 \end{pmatrix}$$

Transforming elements at (1,2) and (3,2) to zero

$$R_1 \to R_1 - 3R_2, R_3 \to R_3 - 2R_2 \quad \begin{pmatrix} 1 & 0 & -13 & : & -16 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & -9 & : & -9 \end{pmatrix}$$

Transforming element at (3,3) to one

$$R_3 \to R_3/-9 \quad \begin{pmatrix} 1 & 0 & -13 & : & -16 \\ 0 & 1 & 5 & : & 7 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

Transforming elements at (1,3) and (2,3) to zero

$$R_1 \to R_1 + 13R_3, R_2 \to R_2 - 5R_3 \quad \begin{pmatrix} 1 & 0 & 0 & : & -3 \\ 0 & 1 & 0 & : & 2 \\ 0 & 0 & 1 & : & 1 \end{pmatrix}$$

$\therefore x = -3, y = 2, z = 1$ is the required solution of given system of equations

4.6 Rank of a Matrix

The rank of a matrix A is the order of the highest ordered non-zero minor in A. It is denoted by $\rho(A)$.

Example6 Find the rank of the following matrices:

(i) $A = \begin{pmatrix} 1 & 4 \\ 3 & 7 \end{pmatrix}$ (ii) $A = \begin{pmatrix} 1 & 3 \\ 3 & 9 \end{pmatrix}$ (iii) $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{pmatrix}$

Solution: (i) Here $\begin{vmatrix} 1 & 4 \\ 3 & 7 \end{vmatrix} = -5 \neq 0 \;\; \therefore \; \rho(A) = 2$

(ii) Here $\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0 \qquad \therefore \; \rho(A) = 1$

(iii) Here $\begin{vmatrix} 1 & 2 & 3 \\ 1 & 4 & 2 \\ 2 & 6 & 5 \end{vmatrix} = 0 \quad \therefore \; \rho(A) \neq 3$

Next consider $\begin{vmatrix} 1 & 2 \\ 1 & 4 \end{vmatrix} = 2 \neq 0 \;\; \therefore \; \rho(A) = 2$

4.6.1 Rank of a matrix using Normal form

The normal form of a matrix is one of the following:

$\mathrm{I_r}$, $(\mathrm{I_n}, 0)$, $\begin{pmatrix} \mathrm{I_n} \\ 0 \end{pmatrix}$, $(\mathrm{I_n} \quad 0)$ or $\begin{pmatrix} \mathrm{I_n} & 0 \\ 0 & 0 \end{pmatrix}$ where $\mathrm{I_n}$ is the identity matrix of order n.

Changing to normal form, n is the rank of the given matrix.

Example7 Find the rank of the following matrices by reducing them to normal form:

(i) $A = \begin{pmatrix} 3 & -3 & 4 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$ (ii) $\mathrm{A} = \begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Solution: (i) Transforming element at (1,1) position to unity

Applying $R_1 \to \frac{1}{3} R_1$, we get $\mathrm{A} \sim \begin{pmatrix} 1 & -1 & 4/3 \\ 2 & -3 & 4 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (2,1) position to zero

$R_2 \to R_2 - 2R_1$ A ~ $\begin{pmatrix} 1 & -1 & 4/3 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (1,2) and (1,3) position to zero

$C_2 \to C_2 + C_1$, $C_3 \to C_3 - \frac{4}{3}C_1$ A ~ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (2,2) position to one

$R_2 \to -R_2$ A ~ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & -1 & 1 \end{pmatrix}$

Transforming element at (3,2) position to zero

$R_3 \to R_3 + R_2$ A ~ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -4/3 \\ 0 & 0 & -1/3 \end{pmatrix}$

Transforming element at (2,3) position 0

$C_3 \to C_3 + \frac{4}{3}C_2$ A ~ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{pmatrix}$

Transforming element at (3,3) position 1

$C_3 \to -3C_3$ A ~ $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ Hence rank of the given matrix is 3.

(ii) Here A = $\begin{pmatrix} 1 & 3 & 4 & 5 \\ 1 & 2 & 6 & 7 \\ 1 & 5 & 0 & 1 \end{pmatrix}$

Transforming element at (2,1) and (2,2) position to zero

Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$ we get A ~ $\begin{pmatrix} 1 & 3 & 4 & 5 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Transforming element at (1,2) , (1,3) and (1,4) position 0

$C_2 \to C_2 - 3C_1$, $C_3 \to C_3 - 4C_1$, $C_4 \to C_4 - 5C_1$ A ~ $\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 2 & 2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Making element at (2,2) position 1

$R_2 \to -R_2$ $\quad$ $A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 2 & -4 & -4 \end{pmatrix}$

Making element at (3,2) position 0

$R_3 \to R_3 - 2R_2$ $\quad$ $A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Making element at (2,3) and (2,4) position 0

$C_3 \to C_3 + 2C_2$, $C_4 \to C_4 + 2C_2$ $\quad$ $A \sim \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Hence rank of the given matrix is 2.

4.6.2 Rank of a matrix using Echelon form

Echelon Form: A matrix is said to be in Echelon form if:

(i) The number of zeros in succeeding row are greater than previous row

(ii) The first non-zero entry in each non-zero row is equal to unity.

Working rule: Transform the matrix to echelon form. The number of non-zero rows in echelon form becomes the rank of the matrix.

Example8 Find the rank of the following matrices by reducing them to echelon form:

(i) $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 7 \\ 3 & 6 & 10 \end{pmatrix}$ $\qquad$ (ii) $A = \begin{pmatrix} 1 & 2 & 3 & 2 \\ 2 & 3 & 5 & 1 \\ 2 & 3 & 4 & 5 \end{pmatrix}$

Solution: (i) Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 3R_1$ we get

$A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

Making element at (3,3) position 0

$R_3 \to R_3 - R_2$ $\quad$ $A \sim \begin{pmatrix} 1 & 2 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

Now the matrix is reduced to echelon form. Since the number of non-zero rows is 2, hence the rank of the given matrix is 2.

(ii) Transforming elements at (2,1) and (3,1) positions to zeros

Applying $R_2 \to R_2 - 2R_1$, $R_3 \to R_3 - 2R_1$, we get

$A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & -1 & -1 & -3 \\ 0 & -1 & -2 & 1 \end{pmatrix}$

Making element at (2,2) position 1

$$R_2 \to -R_2 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & -1 & -2 & 1 \end{pmatrix}$$

Making element at (3,2) position 0

$$R_3 \to R_3 + R_2 \quad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

Making element at (3,3) position 1

$$R_3 \to -R_3 \qquad A \sim \begin{pmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & 1 & 3 \\ 0 & 0 & 1 & -4 \end{pmatrix}$$

Now the matrix is reduced to echelon form. Since the number of non-zero rows is 3, hence the rank of the given matrix is 3.

4.7 Linear Dependence and Independence of Vectors

The set of vectors $X_{1,}X_{2,}X_{3,} \dots, X_n$ is said to be linearly dependent if there exist scalars $C_{1,}C_{2,}C_{3,} \dots, C_n$ not all zero , such that $C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = 0$ And they are linearly independent if $C_1X_1 + C_2X_2 + C_3X_3 + \cdots + C_nX_n = 0$

$$\Rightarrow C_i = 0 \ \forall i = 1,2,3,\dots,n$$

Example 8 Examine the following system of vectors for linear dependence. If dependent find the relation between them:

(i) $X_1 = [1 \quad -1 \quad 1]$, $X_2 = [2 \quad 1 \quad 1]$ and $X_3 = [3 \quad 0 \quad 2]$

(ii) $X_1 = [1 \quad 2 \quad 3]$ and $X_2 = [2 \quad -2 \quad 6]$

Solution: (i) Consider $C_1X_1 + C_2X_2 + C_3X_3 = 0$……….(1)

$$\Rightarrow C_1[1 \quad -1 \quad 1] + C_2[2 \quad 1 \quad 1] + C_3[3 \quad 0 \quad 2] = 0$$

$$\Rightarrow \; C_1 + 2C_2 + 3C_3 = 0$$

$$-C_1 + C_2 + 0C_3 = 0$$

$$C_1 + C_2 + 2C_3 = 0$$

$$\Rightarrow \begin{pmatrix} 1 & 2 & 3 \\ -1 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_2 \to R_2 + R_1$, $R_3 \to R_3 - R_1$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & -1 & -1 \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Applying $R_3 \to R_3 + \frac{1}{3}R_2$, we get

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\Rightarrow\ C_1 + 2C_2 + 3C_3 = 0$

$0C_1 + 3C_2 + 3C_3 = 0$

Let $C_3 = k$ $\Rightarrow\ C_2 = -k$ and $C_1 = -k$

Hence the given vectors are linearly dependent.

Putting these values in (1) , we get $-kX_1 - kX_2 + kX_3 = 0$

$$\Rightarrow -k(X_1 + X_2 - X_3) = 0$$

$$\Rightarrow X_1 + X_2 - X_3 = 0$$

which is the required relation between them.

(i) Consider $C_1X_1 + C_2X_2 = 0$ ………..(1)

$\Rightarrow C_1[1 \quad 2 \quad 3] + C_2[2 \quad -2 \quad 6]$

$\Rightarrow C_1 + 2C_2 = 0$

$2C_1 - 2C_2 = 0$

$3C_1 + 6C_2 = 0$

$\Rightarrow\ C_1 = 0$ and $C_2 = 0$

Hence the given vectors are linearly independent.

4.8 Consistency and Inconsistency of Linear System of Equations

Consider $a_{11}x_1 + a_{12}x_2 + \ldots.. + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \ldots.. + a_{2n}x_n = b_2$

……

$a_{m1}x_1 + a_{m2}x_2 + \ldots.. + a_{mn}x_n = b_m$

This is the system of *m* equations in *n* unknowns and it can be written in the form $AX = B$ where

$$A = \begin{bmatrix} a_{11} & a_{12} \ldots \ldots & a_{1n} \\ a_{21} & a_{22} \ldots \ldots & a_{2n} \\ a_{31} & a_{32} \ldots \ldots & a_{3n} \\ & .. & \\ & .. & \\ a_{m1} & a_{m2} \ldots \ldots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ .. \\ .. \\ x_n \end{bmatrix}, B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ .. \\ .. \\ b_m \end{bmatrix}$$

Here if $b_i = 0\ \forall i$ then system of equations is said to be homogeneous otherwise it is non-homogeneous.

The matrix $C = [A:B] = \begin{bmatrix} a_{11} & a_{12} \dots \dots & a_{1n} : b_1 \\ a_{21} & a_{22} \dots \dots & a_{2n} : b_2 \\ a_{31} & a_{32} \dots \dots & a_{3n} : b_3 \\ & \vdots & \\ a_{m1} & a_{m2} \dots \dots & a_{mn} : b_m \end{bmatrix}$ is called augmented matrix.

4.8.1 Working Rule to find solution of Non-Homogeneous System of Equations

1. For the system of equations, $AX = B$, form an augmented matrix $C = [A:B]$.
2. Find the ranks of matrix A and matrix C

i. If Rank of $A \neq$ Rank of C, then the given system is inconsistent i.e., it has no solution.

ii. If Rank of $A =$ Rank of $C =$ Number of variables in the given system of equations, then the system has a unique solution.

iii. If Rank of $A =$ Rank of $C <$ Number of variables in the given system of equations, then the system has infinitely many solutions.

4.8.2 Working Rule to find solution of Homogeneous System of Equations i.e. *AX = 0*

In case of homogeneous equations, $b_i = 0\ \forall i$, therefore augmented matrix is not required. Here we find the ranks of matrix A

i. If Rank of $A =$ Number of variables in the given system of equations, then the system has the trivial solution, i.e., $x_1 = x_2 = \cdots x_n = 0$

ii. If Rank of A is less than the number of variables in the given system of equations, then the system has infinitely many solutions.

Example Show that the following system of equations is inconsistent.

$$x + 2y + z = 2$$
$$3x + y - 2z = 1$$
$$4x - 3y - z = 3$$
$$2x + 4y + 2z = 5$$

Solution: Let the system of equations be represented as $AX = B$

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 1 & -2 \\ 4 & -3 & -1 \\ 2 & 4 & 2 \end{pmatrix},\ X = \begin{pmatrix} x \\ y \\ z \end{pmatrix},\ B = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 5 \end{pmatrix}$$

Transforming elements at (2,1) , (3,1) and (4,1) positions to zeros

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$A \sim \begin{pmatrix} 1 & 2 & 1 \\ 0 & -5 & -5 \\ 0 & -11 & -5 \\ 0 & 0 & 0 \end{pmatrix}$$

Now matrix A has a non-zero minor $\begin{vmatrix} 1 & 2 & 3 \\ 0 & -5 & -5 \\ 0 & -11 & -5 \end{vmatrix} = -30$

$\therefore$ Rank of matrix A is 3

Again $C = [A:B] = \begin{pmatrix} 1 & 2 & 1 & : & 2 \\ 3 & 1 & -2 & : & 1 \\ 4 & -3 & -1 & : & 3 \\ 2 & 4 & 2 & : & 5 \end{pmatrix}$

Applying $R_2 \to R_2 - 3R_1$, $R_3 \to R_3 - 4R_1$, $R_4 \to R_4 - 2R_1$, we get

$$C = [A:B] \sim \begin{pmatrix} 1 & 2 & 1 & : & 2 \\ 0 & -5 & -5 & : & -5 \\ 0 & -11 & -5 & : & -5 \\ 0 & 0 & 0 & : & 1 \end{pmatrix}$$

$$\lceil C \rceil = 1 \begin{vmatrix} 2 & 1 & 2 \\ -5 & -5 & -5 \\ -11 & -5 & -5 \end{vmatrix} + 0 = 2(0) + 5(-5 + 10) - 11(-5 - 10) = 190$$

$\therefore$ Rank of matrix $C = [A:B]$ is 4

Hence the given system of equations is inconsistent.