Radius of Curvature

Curvature (k) is the numerical measure of bending of a curve; i.e. it is the magnitude by which a curve deviates from being a straight line. The radius of curvature (ρ) at a point on a curve is the reciprocal of curvature, i.e. $\rho = \frac{1}{k}$

- For a cartesian curve given by $y = f(x)$, the radius of curvature is given by

  $\rho = \frac{\left[1+(y')^2\right]^{3/2}}{|y''|}$, when tangent is parallel to x-axis, here $y' = \frac{dy}{dx}$ and $y'' = \frac{d^2y}{dx^2}$

  Also $\rho = \frac{\left[1+(x')^2\right]^{3/2}}{|x''|}$, when tangent is parallel to y-axis, here $x' = \frac{dx}{dy}$ and $x'' = \frac{d^2x}{dy^2}$

- For parametric curves given in the form $x = x(t)$, $y = y(t)$

  $\rho = \frac{\left[(x')^2+(y')^2\right]^{3/2}}{|x'y''-y'x''|}$, where $x' = \frac{dx}{dt}$ and $y' = \frac{dy}{dt}$

- For polar curves in the form $r = r(\theta)$

  $\rho = \frac{\left[r^2+(r')^2\right]^{3/2}}{|r^2+2(r')^2-rr''|}$, where $r' = \frac{dr}{d\theta}$

For sign convention, the second derivative (y'') corresponds to the curvature or concavity of the graph. For upward concave curves, the second derivative is positive, therefore curvature is also positive; while for downward concave curves, the second derivative and curvature are negative.

Example1 Find the radius of curvature at the point $x = \frac{\pi}{2}$ of the curve $y = 4\sin x - \sin 2x$

Solution: Given $y = 4\sin x - \sin 2x$

$$\Rightarrow y' = 4\cos x - 2\cos 2x \qquad \therefore \; y']_{\left(x=\frac{\pi}{2}\right)} = 0 - 2(-1) = 2$$

Also $y'' = -4\sin x + 4\sin 2x \quad \therefore y'']_{\left(x=\frac{\pi}{2}\right)} = -4 + 0 = -4$

The radius of curvature at any point of a cartesian curve is given by

$$\rho = \frac{\left[1+(y\prime)^2\right]^{3/2}}{|y\prime\prime|} = \frac{\left[1+(2)^2\right]^{3/2}}{|-4|} = \frac{5^{3/2}}{4} = 2.795$$

$\therefore$ the radius of curvature at the point $x = \frac{\pi}{2}$ of the curve is given by: $\rho = 2.795$

Example2 Find the radius of curvature at the origin for the parabola $x^2 = 4ay$

Solution: The radius of curvature at any point of a cartesian curve is given by $\rho = \frac{\left[1+(y\prime)^2\right]^{3/2}}{|y\prime\prime|}$

Given $y = \frac{x^2}{4a}$, $\therefore y' = \frac{x}{2a}$ and $y'' = \frac{1}{2a}$

$$\Rightarrow \rho = \frac{\left[1+\left(\frac{x}{2a}\right)^2\right]^{3/2}}{\frac{1}{2a}}$$

$\therefore$ Radius of curvature at origin, i.e., at point $(0,0)$ is given by $\rho = 2a$

Example3 Find the radius of curvature of the curve $y^2 + x^3 = 0$ at $(-1,1)$.

Solution: Given $y^2 + x^3 = 0$

$$\Rightarrow (y^2 + x^3)' = 0$$

$$\Rightarrow 2yy' + 3x^2 = 0 \qquad \Rightarrow y' = -\frac{3x^2}{2y} \qquad \therefore y']_{(-1,1)} = -\frac{3}{2}$$

Also $(2yy' + 3x^2)' = 0$

$$\Rightarrow 2yy'' + 2(y')^2 + 6x = 0 \Rightarrow y'' = -\frac{(y\prime)^2+3x}{y} \quad \therefore y'']_{(-1,1)} = -\frac{\frac{9}{4}-3}{1} = \frac{3}{4}$$

The radius of curvature at any point of a cartesian curve is given by

$$\rho = \frac{[1+(y\prime)^2]^{3/2}}{|y\prime\prime|} = \frac{\left[1+\frac{9}{4}\right]^{3/2}}{\frac{3}{4}} = \frac{\left[\frac{13}{4}\right]^{3/2}}{\frac{3}{4}} = \frac{13^{3/2}}{6} = 7.812$$

∴ the radius of curvature at the point $(-1,1)$ is given by: $\rho = 7.812$

Example 4 Find the points on the parabola $y^2 = 8x$ at which the radius of curvature is $\frac{125}{16}$.

Solution: We have: $\rho = \frac{[1+(x\prime)^2]^{3/2}}{|x\prime\prime|}$, when tangent is parallel to $y -$ axis ...①

Here $x = \frac{y^2}{8}$ $\Rightarrow$ $x' = \frac{y}{4}$ and $x'' = \frac{1}{4}$

Substituting the values in ① and putting $\rho = \frac{125}{16}$, we get

$$\rho = \frac{\left[1+\left(\frac{y}{4}\right)^2\right]^{3/2}}{\left|\frac{1}{4}\right|} = \frac{125}{16}$$

$$\Rightarrow \left[1 + \left(\frac{y}{4}\right)^2\right]^{3/2} = \frac{125}{64}$$

$$\Rightarrow 1 + \left(\frac{y}{4}\right)^2 = \left[\frac{125}{64}\right]^{2/3} = \frac{25}{16}$$

$$\Rightarrow \left(\frac{y}{4}\right)^2 = \frac{9}{16}$$

$$\Rightarrow \frac{y}{4} = \pm\frac{3}{4} \text{ or } y = \pm 3$$

Putting the value of y in the given curve $y^2 = 8x$, we get $x = \frac{9}{8}$

Hence the points at which the radius of curvature is $\frac{125}{16}$ are $\left(\frac{9}{8}, \pm 3\right)$

Example 5 Find the radius of curvature of the cardioid $r = a(1 + \cos\theta)$ at $\theta = 0$. Also prove that $\frac{\rho^2}{r}$ is a constant.

Solution: Given $r = a(1 + \cos\theta)$ ∴ $r' = -a\sin\theta,\ r'' = -a\cos\theta$

The radius of curvature of polar curves is given by $\rho = \frac{\left[r^2+(r\prime)^2\right]^{3/2}}{|r^2+2(r\prime)^2-rr\prime\prime|} \cdots ①$

Substituting r' and r'' in ①, we get

$$\rho = \frac{\left[a^2(1+2\cos\theta+cos^2\theta)+a^2sin^2\theta\right]^{3/2}}{|a^2(1+2\cos\theta+cos^2\theta)+2a^2sin^2\theta+a^2(\cos\theta+cos^2\theta)|} = \frac{\left[2a^2(1+\cos\theta)\right]^{3/2}}{|3a^2(1+\cos\theta|} = \frac{2\sqrt{2}a(1+\cos\theta)^{1/2}}{3} \; ...②$$

$\therefore \rho]_{\theta=0} = \frac{4a}{3}$

Again, $\rho^2 = \frac{8a^2}{9}(1+\cos\theta)$ using ②

$\Rightarrow \rho^2 = \frac{8a}{9}r$

$\therefore \frac{\rho^2}{r} = \frac{8a}{9}$, which is a constant.

Example 6 Find the radius of curvature at any point of the cycloid $x = a\,(\theta + Sin\,\theta),\;\; y = a\,(1- cos\,\theta)$

Solution: We have $x' = \frac{dx}{d\theta} = a\,(1+cos\,\theta)$ and $y' = \frac{dy}{d\theta} = a\,sin\,\theta$

Also, $x'' = \frac{d^2x}{d\theta^2} = -\,a\,sin\,\theta$ and $y'' = \frac{d^2y}{d\theta^2} = \text{a}\cos\theta$

Now $\rho = \frac{\left(x'^2+y'^2\right)^{3/2}}{|x'y''-y'x''|} = \frac{\left(a^2(1+Cos\,\theta)^2+a^2sin^2\theta\right)^{3/2}}{a^2(1+Cos\,\theta)\,Cos\,\theta + a^2\,sin^2\theta} = \frac{a\left(1+cos^2\theta+2cos\theta+sin^2\theta\right)^{3/2}}{cos\theta+cos^2\theta+sin^2\theta}$

$= \frac{a(2+2cos\theta)^{3/2}}{1+cos\theta} = 2\sqrt{2}\,a\sqrt{1+cos\theta} = 2\sqrt{2}\,a\sqrt{2cos^2\frac{\theta}{2}} = 4a\,cos\,\frac{\theta}{2}$

Example 7 Find the radius of curvature at any point of the curve $y = c\cosh\frac{x}{c}$

Solution: Here $y' = \sinh\frac{x}{c}$, $y'' = \frac{1}{c}\cosh\frac{x}{c}$

Now, $\rho = \frac{\left[1+(y\prime)^2\right]^{3/2}}{|y\prime\prime|} = \frac{\left(1+Sinh^2\frac{x}{c}\right)^{3/2}}{\frac{1}{c}\cos h\frac{x}{c}} = \frac{c\left(cosh^2\frac{x}{c}\right)^{3/2}}{\cos h\frac{x}{c}} \quad \because cosh^2\frac{x}{c} - Sinh^2\frac{x}{c} = 1$

$$\Rightarrow \rho = c.cosh^2 \frac{x}{c}$$

$$\therefore \frac{\rho}{c} = cosh^2 \frac{x}{c} = \frac{y^2}{c^2}$$

or $\rho = \frac{1}{c} y^2$

Example8 Show that the radius of curvature at any point of the curve $x = a\ cos^3\theta,\ y = a\ sin^3\theta$ is equal to three times the length of the perpendicular from the origin to the tangent.

Solution: Given $x = a\ cos^3\theta,\ y = a\ sin^3\theta$

$$\therefore\ x' = -\ 3a\ cos^2\theta\ sin\theta\ ,\quad x'' = -\ 3a(-2\cos\theta\sin^2\theta + \cos^3\theta)$$

$$y' = \ 3a\ sin^2\theta\ cos\theta\ ,\quad y'' = 3a(2\sin\theta\ cos^2\theta - \ sin^3\theta)$$

$$\Rightarrow \rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|} = \frac{(9a^2cos^4\theta sin^2\theta + 9a^2sin^4\theta cos^2\theta)^{3/2}}{|(-3acos^2\theta\ sin\theta)(6a\ sin\theta cos^2\theta - 3asin^3\theta) - 3asin^2\theta\cos\theta(6a\ cos\theta\ sin^2\theta - 3a\ cos^3\theta)|}$$

$$= \frac{\left[9a^2cos^2sin^2\theta\left(cos^2\theta + sin^2\theta\right)\right]^{3/2}}{|-18a^2\ sin^2\ \theta cos^4\ \theta + 9a^2\ sin^4\theta cos^2\theta - 18a^2\ sin^4\ \theta cos^2\theta + 9a^2\ sin^2\ \theta cos^4\theta|}$$

$$= \frac{9^{3/2}(a\ cos\theta\ sin\theta)^3}{|-9a^2sin^2\theta cos^4\theta - 9a^2\ cos^2\theta\ sin^4\theta|} = \frac{(9)^{3/2}(a\ cos\theta\sin\theta)^3}{9a^2\ cos^2\theta sin^2\theta\ (cos^2\theta + sin^2\theta)}$$

$$\Rightarrow \rho = \ 3a\sin\theta\ \cos\theta \quad ...\ (1)$$

Again, the equation of the tangent at any point on the curve is given by: $y - y_1 = m(x - x_1)$

Here $(x_1, y_1) \equiv (a\ cos^3\theta, a\ sin^3\theta)$ and $m = \frac{y'}{x'} = -\frac{\sin\theta}{\cos\theta}$

$$y - a\ sin^3\theta = -\frac{\sin\theta}{\cos\theta}\ (x - a\ cos^3\theta)$$

$$\Rightarrow x\sin\theta + ycos\ \theta - a\sin\theta\cos\theta\ (sin^2\theta + cos^2\theta) = 0$$

∴ The equation of the tangent at any point on the curve is given by

$$x\sin\theta + ycos\ \theta - a\sin\theta\cos\theta = 0$$

Hence the length of the perpendicular from the origin to the tangent is:

$$p = \frac{|0.sin\theta + 0.cos\theta - asin\theta\ cos\theta|}{\sqrt{sin^2\theta + cos^2\theta}} = a\sin\theta\cos\theta \quad \ldots ②$$

From ① and ②, we get $\rho = 3p$

Example 9 If ρ & ρ' are the radii of curvature at the extremities of two conjugate diameters of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ prove that $\left(\rho^{2/3} + \rho'^{2/3}\right)(ab)^{2/3} = a^2 + b^2$

Solution: Parametric equation of the ellipse is given by: $x = a\cos\theta$, $y = b\sin\theta$

$$x' = -a\sin\theta, \quad y' = b\cos\theta$$

$$x'' = -a\cos\theta, \ y'' = -b\sin\theta$$

The radius of curvature at any point of the ellipse is given by

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{|x'y'' - y'x''|} = \frac{(a^2 sin^2\theta + b^2\ cos^2\ b)^{3/2}}{|(-a\ sin\theta)\ (-bsin\theta) - (bcos\theta)\ (-acos\theta)|} = \frac{(a^2 sin^2\theta + b^2\ cos^2\theta)^{3/2}}{ab} \quad \ldots ①$$

If ρ' be the radius of curvature of conjugate diameter at the other extremity,

Then $\rho' = \frac{(a^2 cos^2\theta + b^2 sin^2\ \theta)^{3/2}}{ab}$ ...② by replacing θ by $\theta + \frac{\pi}{2}$ in ①

$$\therefore \rho^{2/3} + \rho'^{2/3} = \frac{a^2 sin^2\theta + b^2\ cos^2\theta}{(ab)^{2/3}} + \frac{a^2 cos^2\theta + b^2 sin^2\ \theta}{(ab)^{2/3}} = \frac{a^2 + b^2}{(ab)^{2/3}}$$

$$\Rightarrow \left(\rho^{2/3} + \rho'^{2/3}\right)(ab)^{2/3} = a^2 + b^2$$

Example 10 For the curve $y = \frac{ax}{a+x}$, prove that $\left(\frac{2\rho}{a}\right)^{2/3} = \left(\frac{y}{x}\right)^2 + \left(\frac{x}{y}\right)^2$

where ρ is the radius of curvature of the curve at its point (x, y)

Solution: Here $y' = \frac{(a+x)a - ax\ (1)}{(a+x)^2} = \frac{a^2}{(a+x)^2}$ and $y'' = \frac{-2a^2}{(a+x)^3}$

Now, $\rho = \frac{[1+(y')^2]^{3/2}}{|y''|} = \left[1 + \frac{a^4}{(a+x)^4}\right]^{3/2} \cdot \frac{(a+x)^3}{2a^2}$

$$\therefore \rho^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \cdot \frac{(a+x)^2}{2^{2/3}\ a^{4/3}}$$

$$\left(\frac{2\rho}{a}\right)^{2/3} = \left[1 + \frac{a^4}{(a+x)^4}\right] \cdot \frac{(a+x)^2}{2^{2/3}\, a^{4/3}} \cdot \frac{2^{2/3}}{a^{2/3}} = \frac{1}{a^2}\left[1 + \frac{a^4}{(a+x)^4}\right](a+x)^2$$

$$= \frac{1}{a^2}\left[(a+x)^2 + \frac{a^4}{(a+x)^2}\right] = \left(\frac{a+x}{a}\right)^2 + \left(\frac{a}{a+x}\right)^2 = \left(\frac{x}{y}\right)^2 + \left(\frac{y}{x}\right)^2$$

Example 11 Find the curvature of the curve $x = 4\cos t$, $y = 3\sin t$. At what points on this ellipse, the curvature has the greatest and the least values? What are the magnitudes?

Solution: For parametric curves, $\rho = \frac{\left(x'^2 + y'^2\right)^{3/2}}{|x'y'' - y'x''|}$

Here, $x' = -4\sin t$, $x'' = -4\cos t$

$y' = -3\cos t$, $y'' = -3\sin t$

$$\therefore\ \rho = \frac{\left(16\sin^2 t + 9\cos^2 t\right)^{3/2}}{-4\sin t(-3\sin t) - 3\cos t\,(-4\cos t)} = \frac{\left(16\sin^2 t + 9\cos^2 t\right)^{3/2}}{12\sin^2 t + 12\cos^2 t} = \frac{1}{12}\left(16\sin^2 t + 9\cos^2 t\right)^{3/2}$$

$\Rightarrow (12\rho)^{2/3} = 16\sin^2 t + 9\cos^2 t$ ... ①

For maximum or minimum values of ρ, $\frac{d}{dt}\left(16\sin^2 t + 9\cos^2 t\right) = 0$

$\Rightarrow 32\sin t\cos t - 18\cos t\sin t = 0$

$\Rightarrow 14\sin t\cos t = 0$

$\Rightarrow \sin 2t = 0$

$\Rightarrow\ t = 0, \frac{\pi}{2}$

At $t = 0$, i.e. $(x, y) \equiv (4,0)$

$(12\,\rho)^{2/3} = 9$, by putting $t = 0$ in ①

$\Rightarrow\ 12\,\rho = 9^{3/2}$

$\Rightarrow\ \rho = \frac{9}{4}$

Similarly, at $t = \frac{\pi}{2}$ i.e. $(x, y) \equiv (0,3)$

$(12\,\rho)^{2/3} = 16$, by putting $t = \frac{\pi}{2}$ in ①

$$\Rightarrow \quad 12\,\rho = 16^{3/2}\,, \quad \therefore \rho = \frac{16}{3}$$

Hence, the least value of radius of curvature is $\frac{9}{4}$ and the greatest value is $\frac{16}{3}$.

Now the curvature $k = \frac{1}{\rho}$, therefore least value of radius of curvature corresponds to greatest value of curvature and vice versa. Hence least value of curvature is $\frac{3}{16}$, corresponding to point $(0,3)$ on the given curve and the greatest value is $\frac{4}{9}$, corresponding to point $(4,0)$.

Example12 Find the radius of curvature for the curve $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$, at the points where it touches the coordinate axes.

Solution: Given curve is $\sqrt{\frac{x}{a}} - \sqrt{\frac{y}{b}} = 1$

$$\Rightarrow \frac{1}{2\sqrt{ax}} - \frac{1}{2\sqrt{by}}\, y' = 0$$

$$\Rightarrow y' = \sqrt{\frac{by}{ax}} \qquad \ldots (1)$$

$$\therefore y'' = \sqrt{\frac{b}{a}}\left[\sqrt{y}\left(-\frac{1}{2}x^{-\frac{3}{2}}\right) + \frac{1}{\sqrt{x}}\left(\frac{1}{2}y^{-\frac{1}{2}}\right)y'\right] = \sqrt{\frac{b}{a}}\left[-\frac{1}{2x}\sqrt{\frac{y}{x}} + \frac{1}{2x}\sqrt{\frac{b}{a}}\right]$$

The given curve touches x-axis at $y = 0$, i.e., at the point $(a, 0)$

$$y']_{(a,0)} = 0\;, \quad y'']_{(a,0)} = \frac{b}{2a^2}$$

$$\therefore \rho]_{(a,0)} = \left.\frac{[1+(y')^2]^{3/2}}{|y''|}\right]_{(a,0)} = \frac{2a^2}{b}$$

Also, the curve touches y-axis at $x = 0$, i.e., at the point $(0, b)$, here the tangent is parallel to y-axis;

$$\therefore \rho = \frac{[1+(x')^2]^{3/2}}{|x''|}$$

We have, $x' = \sqrt{\frac{ax}{by}}$ using (1) Also, $x'' = \sqrt{\frac{a}{b}}\left[-\frac{1}{2y}\sqrt{\frac{x}{y}} + \frac{1}{2y}\sqrt{\frac{a}{b}}\right]$

$\therefore x']_{(0.b)} = 0 \, , x'']_{(0,b)} = \frac{a}{2b^2}$

$\Rightarrow \rho]_{(0,b)} = \left.\frac{[1+(x')^2]^{3/2}}{|x''|}\right]_{(0,b)} = \frac{2b^2}{a}$

Example 13 Show that at the point of intersection of the curves $r = a\theta$ and $r\theta = a$, the curvatures are in the ratio 3:1, $(0 < \theta < 2\pi)$

Solution: The points of intersection of curves $r = a\theta$ and $r\theta = a$ are given by $\theta = \pm 1$

For the curve $r = a\theta$, we have $r' = a$ and $r'' = 0$

$$\therefore \rho]_{\theta=\pm1} = \left.\frac{[r^2+(r')^2]^{3/2}}{|r^2+2(r')^2-rr''|}\right]_{\theta=\pm1} = \left.\frac{[(a\theta)^2+a^2]^{3/2}}{|(a\theta)^2+2a^2-a^2\theta|}\right]_{\theta=\pm1} = \frac{a\,(2\sqrt{2})}{3} = \rho_1 \text{ say}$$

Also, for the curve $r\theta = a$, we have $r' = -\frac{a}{\theta^2}$ and $r'' = \frac{2a}{\theta^3}$

$$\therefore \rho]_{\theta=\pm1} = \left.\frac{[r^2+(r')^2]^{3/2}}{|r^2+2(r')^2-rr''|}\right]_{\theta=\pm1} = \left.\frac{\left(\frac{a^2}{\theta^2}+\frac{a^2}{\theta^4}\right)^{3/2}}{\frac{a^2}{\theta^2}+\frac{2a^2}{\theta^4}-\frac{2a^2}{\theta^4}}\right]_{\theta=\pm1} = 2a\sqrt{2} = \rho_2 \text{ , say}$$

$\therefore \frac{\rho_2}{\rho_1} = \frac{2a\sqrt{2}}{a\,(2\sqrt{2})/3} = \frac{3}{1}$

$\therefore \rho_2 : \rho_1 \equiv 3:1$

Example14 Find the radius of curvature at any point (r, θ) of the curve $r^m = a^m \cos m\theta$

Solution: Given curve is $r^m = a^m \cos m\theta$

$\Rightarrow m \log r = m \log a + \log \cos m\theta$

Differentiating with respect to θ, we get

$\frac{m}{r} r' = -m \frac{\sin m\theta}{\cos m\theta}$

$\Rightarrow r' = -r \tan m\theta \quad ...(1)$

Also, $r'' = -(r' \tan m\theta + rm\, sec^2\, m\theta)$

$= r\, tan^2\, m\theta - rm\, sec^2\, m\theta$

Now, for polar curves, $\rho = \frac{\left[r^2+(r\prime)^2\right]^{3/2}}{|r^2+2(r\prime)^2-rr\prime\prime|}$

$$\therefore \rho = \frac{\left(r^2+r^2tan^2m\theta\right)^{3/2}}{r^2+2r^2tan^2m\theta-r^2tan^2m\theta+r^2msec^2m\theta}$$

$$= \frac{r^3sec^3m\theta}{r^2sec^2m\theta+r^2msec^2m\theta} = \frac{r\sec m\theta}{m+1}$$

Example15 Find the radius of curvature at any point (r, θ) of the curve $r^2 \cos 2\theta = a^2$

Solution: Given curve is $r^2 = a^2 \sec 2\theta$

Differentiating with respect to θ, we get

$$2rr' = 2a^2 \sec 2\theta\ tan2\theta$$

$$\Rightarrow \quad r' = r\ tan2\theta\ , \qquad \because a^2 \sec 2\theta = r^2$$

$$\Rightarrow \quad r'' = 2r\ sec^2\theta + r'tan2\theta$$

$$= 2r\ sec^2\theta + rtan^2 2\theta \qquad \because r' = r\ tan2\theta$$

Now, for polar curves, $\rho = \frac{\left[r^2+(r\prime)^2\right]^{3/2}}{|r^2+2(r\prime)^2-rr\prime\prime|}$

$$\therefore \rho = \frac{\left(r^2+r^2tan^2 2\theta\right)^{3/2}}{2r^2\ tan^2 2\ \theta+r^2-r^2\ (2sec^2 2\theta+tan^2 2\theta)}$$

$$= \frac{r^3sec^3 2\theta}{r^2(2\ tan^2 2\theta+1-2sec^2 2\theta-tan^2 2\theta)} = \frac{r^3sec^3 2\theta}{r^2sec^2 2\theta} = r\sec 2\theta = r\frac{r^2}{a^2} = \frac{r^3}{a^2}$$

Radius of curvature at the origin by Newton's method

Newton’s method is applicable only when the curve passes through the origin and has x-axis or y-axis as the tangent at origin.

- if x-axis is the tangent at origin, then $\rho = \lim_{x \to 0} \frac{x^2}{2y}$
- if y-axis is the tangent at origin, then $\rho = \lim_{x \to 0} \frac{y^2}{2x}$

Example16 Find the radius of curvature at the origin of the curve $x^3 + y^3 - 2x^2 + 6y = 0$

Solution: The equation of tangent at origin can be found by equating to zero, the lowest degree term in the equation.

$\therefore y = 0$, i.e. x-axis is tangent to given curve at origin.

Hence $\rho = \lim_{x \to 0} \frac{x^2}{2y}$

Dividing the given equation by $2y$, we get

$$\frac{x^3}{2y} + \frac{y^3}{2y} - \frac{2x^2}{2y} + \frac{6y}{2y} = 0$$

$$\Rightarrow x\left(\frac{x^2}{2y}\right) + \frac{y^2}{2} - 2\left(\frac{x^2}{2y}\right) + 3 = 0$$

$$\Rightarrow \lim_{x \to 0}\left[x\left(\frac{x^2}{2y}\right) + \frac{y^2}{2} - 2\left(\frac{x^2}{2y}\right) + 3\right] = 0$$

$\Rightarrow 0 + 0 - 2\rho + 3 = 0$ $\because$ at origin, $x = 0$ and $y = 0$ and $\rho = \lim_{x \to 0} \frac{x^2}{2y}$

$$\Rightarrow \rho = \frac{3}{2}$$

Example17 Find the radius of curvature at the origin of the curve $x^3y - xy^3 + 2x^2y + xy - y^2 + 2x = 0$

Solution: The equation of tangent at origin can be found by equating to zero, the lowest degree term in the equation.

$\therefore 2x = 0$, i.e. y-axis is tangent to given curve at origin.

Hence $\rho = \lim_{y \to 0} \frac{y^2}{2x}$

Dividing the given equation by $2x$, we get

$$\frac{x^3y}{2x} - \frac{xy^3}{2x} + \frac{2x^2y}{2x} + \frac{xy}{2x} - \frac{y^2}{2x} + \frac{2x}{2x} = 0$$

$$\Rightarrow x^3\left(\frac{y}{2x}\right) - xy^2\left(\frac{y}{2x}\right) + 2x^2\left(\frac{y}{2x}\right) + x\left(\frac{y}{2x}\right) - y\left(\frac{y}{2x}\right) + 1 = 0$$

$$\Rightarrow \lim_{y \to 0}\left[x^3\left(\frac{y}{2x}\right) - xy^2\left(\frac{y}{2x}\right) + 2x^2\left(\frac{y}{2x}\right) + x\left(\frac{y}{2x}\right) - \left(\frac{y^2}{2x}\right) + 1\right] = 0$$

$\Rightarrow \quad 0 - 0 + 0 + 0 + 0 - \rho + 1 = 0$ $\because$ at origin, $x = 0$ and $y = 0$ and $\rho = \lim_{y \to 0} \frac{y^2}{2x}$

$$\Rightarrow \rho = 1$$

Exercise A

1. Prove that If ρ_1, ρ_2 are the radii of curvature at the extremes of any chord of the cardioid $r = a(1 + \cos\theta)$ which passes through the pole, then $\rho_1^2 + \rho_2^2 = \frac{16a^2}{9}$
2. Find the radius of curvature of $y^2 = x^2(a + x)(a - x)$ at the origin. **Ans. $a\sqrt{2}$**
3. Find the radius of curvature at any point 't' of the curve $x = a\left(\cos t + \log\tan\frac{t}{2}\right)$, $y = a\sin t$ **Ans. $a\cos t$**
4. Find the radius of curvature at the origin for the curve $2x^3 - 3x^2y + 4y^3 + y^2 - 3x = 0$ **Ans. ρ = 3/2**
5. Find the radius of curvature of the curve $y^2 = \frac{4a^2(2a-x)}{x}$ at a point where the curve meets x- axis **Ans. $\rho = a$**
6. Prove the if ρ_1, ρ_2 are the radii of curvature at the extremities of a focal chord of a parabola whose semi latus rectum is l then $(\rho_1)^{-\frac{2}{3}} + (\rho_2)^{-\frac{2}{3}} = (l)^{-\frac{2}{3}}$
7. Find the radius of curvature to the curve $r = a(1 + \cos\theta)$ at the point where the tangent is parallel to the initial line.

   Ans. $\rho = \frac{2}{\sqrt{3}}a$
8. For the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, prove that $\rho = \frac{a^2b^2}{p^3}$ where p is the perpendicular distance from the centre on the tangent at (x, y).