Gamma and Beta Functions

Introduction

As introduced by the Swiss mathematician Leonhard Euler in18th century, gamma function is the extension of factorial function to real numbers. Beta function **(**also known as Euler's integral of the first kind**)** is closely connected to gamma function; which itself is a generalization of the factorial function. Both Beta and Gamma functions are very important in calculus as complex integrals can be moderated into simpler form using and Beta and Gamma function.

I Gamma Function

We define Gamma function as: $\Gamma n = \int_0^\infty e^{-x}\, x^{n-1} dx$

Important results

1. $\boldsymbol{i.\ \Gamma 1 = 1}$

Proof: $\Gamma 1 = \int_0^\infty e^{-x}\, x^0 dx \ = -[e^{-x}]_0^\infty = 1$

$\boldsymbol{ii.\ \Gamma \frac{1}{2} = \sqrt{\pi}}$

Proof: $\Gamma \frac{1}{2} = \int_0^\infty e^{-x}\, x^{-\frac{1}{2}} dx = \int_0^\infty e^{-t^2}\, t^{-1} 2t\, dt$, by putting $x = t^2$

$$= 2\int_0^\infty e^{-t^2} dt = \sqrt{\pi}\,, \qquad \because \int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$$

$$\therefore \Gamma \frac{1}{2} = \Gamma(0.5) = \sqrt{\pi} = 1.772$$

2. Reduction formula for $\boldsymbol{\Gamma n}$ **:** $\boldsymbol{\Gamma(n+1) = n\Gamma n}$

We have $\Gamma(n+1) = \int_0^\infty e^{-x}\, x^n dx$

$$= -[x^n e^{-x}]_0^\infty + n\int_0^\infty x^{n-1} e^{-x}\,dx = 0 + n\Gamma n$$

$$\therefore \Gamma(n+1) = n\Gamma n$$

3. $\boldsymbol{\int_0^\infty e^{-kx}\, x^{n-1} dx = \frac{\Gamma n}{k^n}}$

Proof: We have $\Gamma n = \int_0^\infty e^{-t}\, t^{n-1} dt$

Putting $t = kx \quad \Rightarrow dt = kdx$

$$\therefore \Gamma n = \int_0^\infty e^{-kx}\,(kx)^{n-1} kdx = \ k^n \int_0^\infty e^{-kx}\, x^{n-1} dx$$

$$\Rightarrow \int_0^\infty e^{-kx}\, x^{n-1} dx = \frac{\Gamma n}{k^n}$$

Extension of Gamma function from factorial notation

Case $\boldsymbol{i}$. When $\boldsymbol{n}$ is a positive integer

We have $\Gamma(n+1) = n\Gamma n$

$$= n(n-1)\Gamma(n-1)$$

$$= n(n-1)(n-2)\Gamma(n-2)$$

$$\vdots$$

$$= n(n-1)(n-2)\cdots 3.2.1\Gamma 1 = n!$$

$\therefore \Gamma 2 = 1!\,, \Gamma 3 = 2!\,, \Gamma 4 = 3!$ etc.

case $\boldsymbol{ii}$. When $\boldsymbol{n}$ is a positive rational number

$\Gamma n = (n-1)(n-2)\cdots$ upto a positive number in Γ function

Illustration: $\Gamma\frac{7}{2} = \frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\frac{1}{2} = \frac{15\sqrt{\pi}}{8}$

Also $\Gamma\frac{11}{4} = \frac{7}{4}.\frac{3}{4}\ \Gamma\frac{3}{4}$

Now value of $\Gamma\frac{3}{4}$ can be obtained from table of gamma function.

case iii. When n is a negative rational number

Using $\Gamma(n+1) = n\Gamma n$

$$\Rightarrow \Gamma n = \frac{\Gamma(n+1)}{n} = \frac{(n+1)\Gamma(n+1)}{n(n+1)}$$

$$= \frac{\Gamma(n+2)}{n(n+1)}$$

$$= \frac{\Gamma(n+3)}{n(n+1)(n+2)}$$

$$\vdots$$

Continuing in this manner, we get $\Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\ldots(n+k)}$, where k is the least positive integer such that $(n+k+1) > 0$

Illustration: $\Gamma(-3.4) = \frac{\Gamma(-3.4+k+1)}{(-3.4)(-2.4)\ldots(-3.4+k)}$, $(-3.4+k+1) > 0$

$$\Rightarrow k > 2.4 \Rightarrow k = 3$$

$$\therefore \Gamma(-3.4) = \frac{\Gamma(-3.4+4)}{(-3.4)(-2.4)(-1.4)(-0.4)} = \frac{\Gamma 0.6}{(-3.4)(-2.4)(-1.4)(-0.4)}$$

$\Gamma 0.6$ can be found using tables.

Also, to evaluate $\Gamma(-2.5)$,

$$\Gamma(-2.5) = \frac{\Gamma(-2.5+k+1)}{(-2.5)(-1.5)\ldots(-2.5+k)}, \ (-2.5+k+1) > 0$$

$$\Rightarrow k > 1.5 \Rightarrow k = 2$$

$$\therefore \Gamma(-2.5) = \frac{\Gamma(-2.5+3)}{(-2.5)(-1.5)(-0.5)} = \frac{\Gamma 0.5}{(-2.5)(-1.5)(-0.5)} = -\frac{1.772}{1.875} = -0.945$$

case iv. Γn is not defined when $n = 0$ or a negative integer

We know $\Gamma n = \frac{\Gamma(n+k+1)}{n(n+1)\ldots(n+k)}$, $n = 0, -1, -2, \ldots$

For all $n = 0, -1, -2, \ldots$, we will have a zero in the denominator

For instance, $\Gamma 0 = \frac{\Gamma(0+k+1)}{0(1)\ldots(0+k)}$, $\Gamma(-1) = \frac{\Gamma(-1+k+1)}{(-1)(0)\ldots(-1+k)}$, ...

Hence, we can conclude that gamma function cannot be defined for zero or negative integers.

Example 1 If n is a positive integer, show that

$$2^n \Gamma\left(n + \frac{1}{2}\right) = 1.3.5 \ldots (2n-1)\sqrt{\pi}$$

Solution: $\Gamma\left(n + \frac{1}{2}\right) = \Gamma\left(n - \frac{1}{2} + 1\right)$

$$= \left(n - \frac{1}{2}\right)\Gamma\left(n - \frac{1}{2}\right) \qquad \because \Gamma(n+1) = n\Gamma n$$

$$= \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right)\Gamma\left(n - \frac{3}{2}\right)$$

$$\vdots$$

$$= \left(n-\frac{1}{2}\right)\left(n-\frac{3}{2}\right)\left(n-\frac{5}{2}\right)\ldots\frac{3}{2}.\frac{1}{2}.\Gamma\frac{1}{2}$$

$$= \left(\frac{2n-1}{2}\right)\left(\frac{2n-3}{2}\right)\left(\frac{2n-5}{2}\right)\ldots\frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

$$\Rightarrow 2^n\Gamma\left(n+\frac{1}{2}\right) = 1.3.5\ldots(2n-1)\sqrt{\pi}$$

Example 2 Evaluate the following integrals

$i.\ \int_0^\infty e^{-x^2}x^{2n-1}dx\ ,\ n>1$

$ii.\ \int_0^\infty e^{-\sqrt{x}}x^{\frac{1}{4}}dx$

$iii.\ \int_0^\infty \frac{x^a}{a^x}\ dx$

$iv.\ \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx\ ,\ n>0$

Solution: $i.$ We have $\Gamma n = \int_0^\infty e^{-t}\, t^{n-1}dt,\quad \ldots①$

Putting $t = x^2$ in ① , we get

$$\Gamma n = \int_0^\infty e^{-x^2}\, x^{2n-2}.\,2xdx$$

$$\Rightarrow \int_0^\infty e^{-x^2}x^{2n-1}dx = \frac{\Gamma n}{2}$$

$ii.$ Putting $t = \sqrt{x}$ in ① , we get

$$\Gamma n = \int_0^\infty e^{-\sqrt{x}}\ x^{\frac{n}{2}-\frac{1}{2}}.\frac{1}{2}x^{-\frac{1}{2}}dx = \frac{1}{2}\int_0^\infty e^{-\sqrt{x}}\ x^{\frac{n}{2}-1}dx$$

Substituting $\frac{n}{2}-1=\frac{1}{4}$, i.e. $n=\frac{5}{2}$, we get

$$\Gamma\left(\frac{5}{2}\right) = \frac{1}{2}\int_0^\infty e^{-\sqrt{x}}\ x^{\frac{1}{4}}dx$$

$$\therefore \int_0^\infty e^{-\sqrt{x}} x^{\frac{1}{4}}dx = 2\Gamma\left(\frac{5}{2}\right) = 2.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3\sqrt{\pi}}{2}$$

$iii.$ Putting $a^x = e^t$ or $x \log a = t \Rightarrow dx = \frac{dt}{\log a}$

$$\therefore \int_0^\infty \frac{x^a}{a^x}\ dx = \int_0^\infty e^{-t}\left(\frac{t}{\log a}\right)^a \frac{dt}{\log a}$$

$$= \frac{1}{(\log a)^{a+1}}\int_0^\infty e^{-t} t^{(a+1)-1}\ dt = \frac{\Gamma(a+1)}{(\log a)^{a+1}}$$

$iv.$ We have $\Gamma n = \int_0^\infty e^{-t}\ t^{n-1}dt$

Putting $t = \log\frac{1}{x} \Rightarrow -t = \log x \ \Rightarrow e^{-t} = x$

Also $\ \ dt = -\frac{1}{x}dx$

as $t = 0 \Rightarrow x = 1$, $\ t = \infty \Rightarrow x = 0$

$$\therefore \Gamma n = \int_1^0 x\left(\log\frac{1}{x}\right)^{n-1}\left(-\frac{1}{x}\right)dx = \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx$$

$$\Rightarrow \int_0^1 \left(\log\frac{1}{x}\right)^{n-1} dx = \Gamma n$$

II Beta Function

Beta function is defined as:

$$\boldsymbol{\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx\ , \qquad m,n > 0}$$

Important Results

4. **Beta function is symmetric i.e. $\boldsymbol{\beta(m,n) = \beta(n,m)}$**

Proof: $\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx\,, \quad m,n>0$

$$= \int_0^1 (1-x)^{m-1}(1-(1-x)^{n-1}dx\,, m,n>0$$

$$\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^1 x^{n-1}(1-x)^{m-1}dx\,, \quad n,m>0$$

$$= \beta(n,m)$$

5. **Another definition of Beta function:**

$$\boldsymbol{\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx\,, \quad m,n>0}$$

Proof: $\beta(m,n) = \int_0^1 y^{m-1}(1-y)^{n-1}dy\,, \; m,n>0$

Putting $y = \frac{1}{1+x}\,, \; dy = -\frac{1}{(1+x)^2}dx$

$$\Rightarrow \beta(m,n) = -\int_\infty^0 \left(\frac{1}{1+x}\right)^{m-1}\left(1-\frac{1}{1+x}\right)^{n-1}\frac{1}{(1+x)^2}dx\,, \; m,n>0$$

$$= \int_0^\infty \left(\frac{1}{1+x}\right)^{m-1}\left(\frac{x}{1+x}\right)^{n-1}\left(\frac{1}{1+x}\right)^2 dx$$

$$= \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}dx$$

$$= \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx \qquad \because \beta(m,n) = \beta(n,m)$$

6. Another form of Beta function is given by:

$$\boldsymbol{\beta(m,n) = 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\,\theta\, cos^{2n-1}\,\theta d\theta}$$

Proof: we have $\beta(m,n) = \int_0^1 x^{m-1}\,(1-x)^{n-1}dx$

Let $x = sin^2\,\theta \Rightarrow dx = 2\sin\theta\cos\theta\; d\theta$

$$\therefore \beta(m,n) = \int_0^{\frac{\pi}{2}}(sin^2\,\theta)^{m-1}\,(cos^2\,\theta)^{n-1}2\sin\theta\cos\theta$$

$$= 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\,\theta\, cos^{2n-1}\,\theta d\theta$$

7. Relation between Beta Gamma functions:

$$\boldsymbol{\beta(m,n) = \frac{\Gamma m\Gamma n}{\Gamma(m+n)}}, \quad \boldsymbol{m,n > 0}$$

Proof: Using result **3**, $\int_0^\infty e^{-kx}\,x^{m-1}dx = \frac{\Gamma m}{k^m}$ ...①

Replacing k by y , we get $\frac{\Gamma m}{y^m} = \int_0^\infty e^{-yx}\,x^{m-1}dx$

$$\Rightarrow \Gamma m = \int_0^\infty e^{-yx}\,y^m x^{m-1}dx$$

$$\Rightarrow e^{-y}y^{n-1}\Gamma m = \int_0^\infty e^{-y(1+x)}\,y^{m+n-1}x^{m-1}dx$$

Integrating both sides with respect to y within limits 0 to ∞

$$\Gamma m\int_0^\infty e^{-y}y^{n-1}dy = \int_0^\infty\int_0^\infty e^{-y(1+x)}\,y^{m+n-1}x^{m-1}dx\,dy$$

$$\Rightarrow \Gamma m\Gamma n = \int_0^\infty\left[\int_0^\infty e^{-(1+x)y}\,y^{(m+n-1)}dy\right]x^{m-1}dx$$

$$\Rightarrow \Gamma m \Gamma n = \int_0^\infty \frac{\Gamma(m+n)}{(1+x)^{m+n}} x^{m-1} dx \text{ , comparing with } ①$$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}} \, dx$$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \beta(m,n), \text{ using result 5}$$

8. $\boldsymbol{\int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}}$

Proof: we have $\beta(m,n) = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = 2\int_0^{\frac{\pi}{2}} \sin^{2m-1}\theta \cos^{2n-1}\theta d\theta \quad \because \beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)}$$

Replacing $2m-1$ by p and $2n-1$ by q

i.e $m = \frac{p+1}{2}$ and $n = \frac{q+1}{2}$

$$\Rightarrow \int_0^{\frac{\pi}{2}} \sin^p \theta \cos^q \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)} \quad \ldots ①$$

Putting $q = 0$ in ①, we get $\int_0^{\frac{\pi}{2}} \sin^p \theta d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{p+2}{2}\right)}$

Putting $p = 0$ in ①, we get $\int_0^{\frac{\pi}{2}} \sin^p \theta d\theta = \frac{\Gamma\left(\frac{q+1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(\frac{q+2}{2}\right)}$

9. Duplication formula is given by:

$$\boldsymbol{\Gamma m \Gamma\left(m+\frac{1}{2}\right)=\frac{\sqrt{\pi}\,.\Gamma(2m)}{2^{2m-1}},\ m>0}$$

Proof: We have $\beta(m,n)=\frac{\Gamma m\Gamma n}{\Gamma(m+n)}=2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2n-1}\theta d\theta$

$$\therefore 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2n-1}\theta d\theta=\frac{\Gamma m\Gamma n}{\Gamma(m+n)} \qquad \ldots (1)$$

Putting $n=\frac{1}{2}$ on both sides, we get

$$2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta d\theta=\frac{\Gamma m\sqrt{\pi}}{\Gamma(m+\frac{1}{2})} \qquad \ldots (2)$$

Again Putting $n=m$ in (1), we get

$$\frac{(\Gamma m)^2}{\Gamma(2m)}=2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2m-1}\theta d\theta$$

$$=2\int_0^{\frac{\pi}{2}}\left(\frac{2\sin\theta\cos\theta}{2}\right)^{2m-1} d\theta$$

$$=\frac{1}{2^{2m-2}}\int_0^{\frac{\pi}{2}} sin^{2m-1} 2\theta d\theta$$

$$=\frac{1}{2^{2m-1}}\int_0^{\pi} sin^{2m-1} tdt \text{ , Putting } 2\theta=t$$

$$=\frac{2}{2^{2m-1}}\int_0^{\frac{\pi}{2}} sin^{2m-1} tdt$$

$$\because \int_0^{2a} f(x)dx=2\int_0^{a} f(x)dx \text{ , if } f(2a-x)=f(x)$$

$$\Rightarrow \frac{(\Gamma m)^2}{\Gamma(2m)} = \frac{2}{2^{2m-1}} \int_0^{\frac{\pi}{2}} sin^{2m-1}\theta d\theta$$

$$\Rightarrow 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta d\theta = \frac{2^{2m-1}(\Gamma m)^2}{\Gamma(2m)} \qquad \ldots ③$$

Comparing ② and ③, we get $\frac{\Gamma m\sqrt{\pi}}{\Gamma(m+\frac{1}{2})} = \frac{2^{2m-1}(\Gamma m)^2}{\Gamma(2m)}$

$$\Rightarrow \Gamma m \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\ \Gamma(2m)}{2^{2m-1}}$$

10. $\boldsymbol{\Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi},\ 0 < n < 1}$

Proof: we have $\beta(m,n) = \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx,\ m,n > 0$

$$\Rightarrow \frac{\Gamma m \Gamma n}{\Gamma(m+n)} = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

Putting $m = 1-n$ on both sides, we get

$$\Gamma n\, \Gamma(1-n) = \int_0^\infty \frac{x^{n-1}}{1+x} dx$$

Putting $x = e^t$, $dx = e^t dt$

As $x \to 0$, $t \to -\infty$ and as $x \to \infty$, $t \to \infty$

$$\therefore \Gamma n\, \Gamma(1-n) = \int_{-\infty}^\infty \frac{e^{nt}}{1+e^t} dt$$

Now by using complex integration, we have:

$$\int_{-\infty}^\infty \frac{e^{nt}}{1+e^t} dt = \frac{\pi}{\sin n\pi},\ 0 < n < 1$$

$\therefore \Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}\,, \;\; 0 < n < 1$

Example 3 Evaluate $i. \int_0^{\frac{\pi}{2}} sin^3\, x\, cos^{\frac{5}{2}}\, x dx \quad ii. \int_0^{\frac{\pi}{2}} sin^{10}\, x dx \quad iii. \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} + \sqrt{\sec\theta}\;\; d\theta$

$iv. \int_0^2 x^{m-1}\,(2-x)^{n-1} dx \quad v. \int_0^{\pi} sin^2\,\theta(1+\cos\theta)^4 d\theta \quad vi. \int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx$

Solution: $i. \int_0^{\frac{\pi}{2}} sin^3\, x\, cos^{\frac{5}{2}}\, x dx = \frac{\Gamma\left(\frac{3+1}{2}\right)\Gamma\left(\frac{\frac{5}{2}+1}{2}\right)}{2\Gamma\left(\frac{3+\frac{5}{2}+2}{2}\right)} = \frac{\Gamma 2\Gamma\left(\frac{7}{4}\right)}{2\Gamma\left(\frac{15}{4}\right)}$

$$= \frac{1.\Gamma\left(\frac{7}{4}\right)}{2.\frac{11}{4}.\frac{7}{4}\Gamma\left(\frac{7}{4}\right)} = \frac{8}{77} \qquad \because \Gamma 2 = 1! = 1, \text{ also } \Gamma(n+1) = n\Gamma n$$

$ii. \int_0^{\frac{\pi}{2}} sin^{10}\, x dx = \frac{\Gamma\left(\frac{11}{2}\right)\Gamma\left(\frac{1}{2}\right)}{2\Gamma 6} = \frac{\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{240} = \frac{945\pi}{7680}$

$$\because \Gamma 6 = 5! = 120, \text{ also } \Gamma(n+1) = n\Gamma n$$

$$= \frac{63\pi}{512} \qquad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$iii. \int_0^{\frac{\pi}{2}} \sqrt{\tan\theta} + \sqrt{\sec\theta}\;\; d\theta = \int_0^{\frac{\pi}{2}} (sin^{\frac{1}{2}}\,\theta\, cos^{-\frac{1}{2}}\,\theta + cos^{-\frac{1}{2}}\,\theta) d\theta$

$$= \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma 1} + \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{4}\right)}{2\Gamma\left(\frac{3}{4}\right)}$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{4}\right)\left\{\Gamma\left(\frac{3}{4}\right) + \frac{\sqrt{\pi}}{\Gamma\left(\frac{3}{4}\right)}\right\}$$

$iv.$ Let $I = \int_0^2 x^{m-1}(2-x)^{n-1}dx$

Putting $x = 2sin^2\theta$,

$$I = \int_0^{\frac{\pi}{2}} 2^{m-1} sin^{2m-2}\theta . 2^{n-1} cos^{2n-2}\theta . 2\sin\theta\cos\theta\, d\theta$$

$$\Rightarrow I = 2^{m+n-2}.2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2m-1}\theta d\theta = 2^{m+n-2}\beta(m,n)$$

$$\because 2\int_0^{\frac{\pi}{2}} sin^{2m-1}\theta\, cos^{2m-1}\theta d\theta = \beta(m,n)$$

$v.$ Let $I = \int_0^{\pi} sin^2\theta(1+\cos\theta)^4 d\theta$

$$= \int_0^{\pi}\left(2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right)^2\left(2cos^2\frac{\theta}{2}\right)^4 d\theta$$

$$= 64\int_0^{\pi} sin^2\frac{\theta}{2} cos^{10}\frac{\theta}{2} d\theta$$

Putting $\frac{\theta}{2} = x$, $d\theta = 2xdx$

$$= 128\int_0^{\frac{\pi}{2}} sin^2 x\, cos^{10} x\, dx$$

$$= \frac{64.\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{11}{2}\right)}{\Gamma 7} = \frac{64.\ \frac{1}{2}\Gamma\left(\frac{1}{2}\right).\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{720} = \frac{21\pi}{16}$$

$$\because \Gamma 7 = 6! = 720, \text{ also } \Gamma(n+1) = n\Gamma n$$

$vi.$ Let $I = \int_0^1 x^{m-1}\left(\log\frac{1}{x}\right)^{n-1} dx$

Putting $\log\frac{1}{x} = t$ or $x = e^{-t} \Rightarrow dx = -e^{-t}dt,$

$$I = -\int_\infty^0 e^{-(m-1)t}\, t^{n-1} e^{-t} dt$$

$$= \int_0^\infty e^{-mt}\, t^{n-1} dt$$

Putting $mt = y$

$$I = \frac{1}{m}\int_0^\infty e^{-y}\left(\frac{y}{m}\right)^{n-1} dy = \frac{1}{m^n}\int_0^\infty e^{-y}\, y^{n-1} dy = \frac{\Gamma n}{m^n}$$

Example 4 Prove that $i.\ \beta(m,n) = \beta(m+1,n) + \beta(m,n+1)$

$$ii.\ \frac{\beta(m+1,n)}{\beta(m,n)} = \frac{m}{m+n}$$

$$iii.\ \beta\left(m,\frac{1}{2}\right) = 2^{2m-1}\beta(m,m)$$

Solution: $i.$ R.H.S. $= \beta(m+1,n) + \beta(m,n+1)$

$$= \frac{\Gamma(m+1)\Gamma n}{\Gamma(m+n+1)} + \frac{\Gamma m\Gamma(n+1)}{\Gamma(m+n+1)}$$

$$= \frac{m\Gamma m.\Gamma n + \Gamma m.n\Gamma n}{\Gamma(m+n+1)}$$

$$= \frac{\Gamma m.\Gamma n(m+n)}{(m+n)\Gamma(m+n)}$$

$$= \frac{\Gamma m\Gamma n}{\Gamma(m+n)} = \beta(m,n) = \text{L.H.S.}$$

ii. L.H.S.$= \frac{\beta(m+1,n)}{\beta(m,n)} = \frac{\Gamma(m+1)\Gamma n}{\Gamma(m+n+1)} \cdot \frac{\Gamma(m+n)}{\Gamma m \Gamma n}$

$$= \frac{m\Gamma m\Gamma n}{(m+n)\Gamma(m+n)} \cdot \frac{\Gamma(m+n)}{\Gamma m\Gamma n} = \frac{m}{m+n} = \text{R.H.S.}$$

iii. We have $\beta\left(m,\frac{1}{2}\right) = \frac{\Gamma m\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(m+\frac{1}{2}\right)}$ ... ①

Again, by Duplication formula $\Gamma m\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\ \Gamma(2m)}{2^{2m-1}}$

$\therefore \Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}\ \Gamma(2m)}{2^{2m-1}\Gamma m}$ ... ②

Using ② in ①, we get $\beta\left(m,\frac{1}{2}\right) = \frac{\Gamma m\Gamma\left(\frac{1}{2}\right)2^{2m-1}\Gamma m}{\sqrt{\pi}\ \Gamma(2m)}$

$$= 2^{2m-1}\frac{\Gamma m\Gamma m}{\Gamma(2m)} \quad \because \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$= 2^{2m-1}\beta(m,m)$$

Example 5 Express the following integrals in terms of Beta function

i. $\int_0^1 x^m (1-x^2)^n dx$ ii. $\int_0^1 \frac{x^2}{\sqrt{1-x^5}} dx$

Solution: i. Let $I = \int_0^1 x^m (1-x^2)^n dx = \frac{1}{2}\int_0^1 x^{m-1} (1-x^2)^n 2x dx$

Putting $x^2 = y \quad \Rightarrow 2xdx = dy$

$$\Rightarrow I = \frac{1}{2}\int_0^1 y^{\frac{m-1}{2}} (1-y)^n dy$$

$$= \frac{1}{2}\int_0^1 y^{\frac{m+1}{2}-1}(1-y)^{(n+1)-1}dy$$

$$= \frac{1}{2}\beta\left(\frac{m+1}{2}, n+1\right)$$

ii. Let $I = \int_0^1 \frac{x^2}{\sqrt{1-x^5}}dx = \frac{1}{5}\int_0^1 x^{-2}(1-x^5)^{-\frac{1}{2}}5x^4dx$

Putting $x^5 = y \quad \Rightarrow 5x^4dx = dy$

$$\Rightarrow I = \frac{1}{5}\int_0^1 y^{-\frac{2}{5}}(1-y)^{-\frac{1}{2}}dy$$

$$= \frac{1}{5}\int_0^1 y^{\frac{3}{5}-1}(1-y)^{\frac{1}{2}-1}dy = \frac{1}{5}\beta\left(\frac{3}{5}, \frac{1}{2}\right)$$

Example 6 Prove that *i*. $\Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right) = \left(\frac{1}{4}-x^2\right)\pi.\sec\pi x$

ii. $\int_a^b (x-a)^m(b-x)^n\,dx = (b-a)^{m+n+1}\beta(m+1, n+1)$

Solution *i*. L.H.S.$= \Gamma\left(\frac{3}{2}-x\right)\Gamma\left(\frac{3}{2}+x\right)$

$$= \left(\frac{1}{2}-x\right)\Gamma\left(\frac{1}{2}-x\right).\left(\frac{1}{2}+x\right)\Gamma\left(\frac{1}{2}+x\right)$$

$$\because \Gamma(n+1) = n\Gamma n$$

$$= \left(\frac{1}{4}-x^2\right)\Gamma\left(\frac{1}{2}+x\right)\Gamma\left(1-\left(\frac{1}{2}+x\right)\right)$$

$$\because \Gamma\left(\frac{1}{2}-x\right) = \Gamma\left(1-\left(\frac{1}{2}+x\right)\right)$$

$$= \left(\frac{1}{4}-x^2\right)\frac{\pi}{\sin\left(\frac{1}{2}+x\right)\pi}, \; 0 < \frac{1}{2}+x < 1$$

$$\because \Gamma n\, \Gamma(1-n) = \frac{\pi}{\sin n\pi}, \; 0 < n < 1$$

$$= \left(\frac{1}{4} - x^2\right) \frac{\pi}{\cos \pi x}$$

$$= \left(\frac{1}{4} - x^2\right) \pi.\sec \pi x \;, \; -\frac{1}{2} < x < \frac{1}{2}$$

$ii.$ Let $I = \int_a^b (x-a)^m (b-x)^n \, dx$

$$= \int_0^{b-a} y^m (b-a-y)^n \, dy \qquad \text{By putting } x - a = y$$

$$= \int_0^1 (b-a)^m t^m (b-a-(b-a)t)^n \, (b-a)dt$$

By putting $y = (b-a)t$

$$I = \int_0^1 x^m \, (1-x^n)^p dx = \int_0^1 (b-a)^m t^m \, (b-a)^n (1-t)^n (b-a)dt$$

$$= (b-a)^{m+n-1} \int_0^1 t^m \, (1-t)^n dt$$

$$= (b-a)^{m+n-1} \int_0^1 t^{(m+1)-1} \, (1-t)^{(n+1)-1} dt$$

$$= (b-a)^{m+n+1} \beta(m+1, n+1)$$

Example 7 Express the integral $\int_0^1 x^m \, (1-x^n)^p dx$ in terms of gamma function and hence evaluate $\int_0^1 x^{\frac{3}{2}} \left(1 - x^{\frac{1}{2}}\right)^{\frac{1}{2}} dx$

Solution: Let $I = \int_0^1 x^m \, (1-x^n)^p dx$

Putting x^n =t, so that $nx^{n-1}dx = dt$, we get

$$I=\frac{1}{n}\int_0^1 t^{\frac{m}{n}}(1-t)^p t^{-\frac{n-1}{n}}dt=\frac{1}{n}\int_0^1 t^{\frac{m+1}{n}-1}(1-t)^{p+1-1}dt$$

$$\therefore I=\int_0^1 x^m(1-x^n)^p dx=\frac{1}{n}\beta\left(\frac{m+1}{n},p+1\right)=\frac{1}{n}\frac{\Gamma\left(\frac{m+1}{n}\right)\Gamma(p+1)}{\Gamma\left(\frac{m+1}{n}+p+1\right)}\quad \ldots①$$

Putting $m=\frac{3}{2}$, $n=\frac{1}{2}$, $p=\frac{1}{2}$ in ①, we get

$$\int_0^1 x^{\frac{3}{2}}\left(1-x^{\frac{1}{2}}\right)^{\frac{1}{2}}dx=2\beta\left(\frac{\frac{3}{2}+1}{\frac{1}{2}},\frac{1}{2}+1\right)=2\beta\left(5,\frac{3}{2}\right)$$

$$=\frac{2\Gamma(5)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(5+\frac{3}{2}\right)}=\frac{2.4!\,\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{13}{2}\right)}=\frac{48\Gamma\left(\frac{3}{2}\right)}{\frac{11}{2}.\frac{9}{2}.\frac{7}{2}.\frac{5}{2}.\frac{3}{2}.\Gamma\left(\frac{3}{2}\right)}=\frac{512}{3465}$$

Example 8 Evaluate $i.\int_0^\infty x^{n-1}\cos ax\,dx$ $\;ii.\int_0^\infty x^{n-1}\sin ax\,dx$

Solution: let $I=\int_0^\infty x^{n-1}e^{-iax}dx=\int_0^\infty x^{n-1}(\cos ax-i\sin ax)\,dx$

Putting $iax=t$, $dx=\frac{dt}{ia}$

$$\therefore I=\frac{1}{ia}\int_0^\infty e^{-t}\left(\frac{t}{ia}\right)^{n-1}dt=\frac{1}{i^n a^n}\int_0^\infty e^{-t}t^{n-1}\,dt=\frac{\Gamma n}{i^n a^n}$$

$$=\frac{\Gamma n}{a^n}(-i)^n=\frac{\Gamma n}{a^n}\left(\cos\frac{\pi}{2}-i\sin\frac{\pi}{2}\right)^n$$

$$=\frac{\Gamma n}{a^n}\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$$

$$\therefore\int_0^\infty x^{n-1}(\cos ax-i\sin ax)\,dx=\frac{\Gamma n}{a^n}\left(\cos\frac{n\pi}{2}-i\sin\frac{n\pi}{2}\right)$$

Comparing real and imaginary parts we get,

$i. \int_0^\infty x^{n-1} \cos ax\, dx = \frac{\Gamma n}{a^n} \cos \frac{n\pi}{2}$

$ii. \int_0^\infty x^{n-1} \sin ax\, dx = \frac{\Gamma n}{a^n} \sin \frac{n\pi}{2}$

Exercise

1. Show that $\int_0^{\frac{\pi}{2}} tan^n x\, dx = \frac{\pi}{2} \sec\left(\frac{n\pi}{2}\right)$, $0 < n < 1$
2. Given that $\Gamma\left(\frac{8}{5}\right) = 0.8935$, find the values of $\Gamma\left(-\frac{5}{2}\right)$ and $\Gamma\left(-\frac{12}{5}\right)$
3. Find $i. \beta\left(\frac{3}{2}, \frac{1}{2}\right)$ $\quad ii. \beta\left(\frac{4}{3}, \frac{5}{3}\right)$
4. Evaluate $\int_0^1 x^2 (1 - x^2)^4 dx$
5. Prove that $\Gamma n = 2 \int_0^\infty e^{-x^2} x^{2n-1}\, dx$
6. Prove that for $a, b > 0$
   $i. \int_0^\infty x^{n-1} e^{-ax} \cos bx\, dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \cos\left(n \tan^{-1} \frac{b}{a}\right)$
   $ii. \int_0^\infty x^{n-1} e^{-ax} \sin bx\, dx = \frac{\Gamma n}{(a^2+b^2)^{n/2}} \sin\left(n \tan^{-1} \frac{b}{a}\right)$
7. Show that $\frac{\beta(m+1,n)}{m} = \frac{\beta(m,n)}{m+n+1}$
8. Show that $\int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{\sin x}} dx. \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\, dx = \pi$
9. Show that $\int_0^1 \frac{x^{m-1}+x^{n-1}}{(1+x)^{m+n}} dx = \beta(m, n)$

10. Prove that $\beta\left(m,\frac{1}{2}\right) = 2^{2m-1}.\beta(m,m)$

Answers

2. $-\frac{8}{15}\sqrt{\pi}$, -1.108
3. $i.\frac{\pi}{2}$ $ii.\frac{2\pi}{9\sqrt{3}}$
4. $\frac{128}{3465}$