Expansion of Functions & Error Approximation

Taylor Series

Consider the expansion of a function $f(x)$ in terms the power series about any given point 'a'as:

$f(x) = \sum_{n=0}^{\infty} c_n h^n$, where $h = x - a$.

If $f(x)$ is infinitely differentiable about the point 'a', then $f(x)$ can be represented as a special type of series known as Taylor series. It can be shown by repeated differentiation that $c_n = \frac{f^{(n)}(a)}{n!}$, where $f^{(n)}(x)$ is the n^{th} derivative.

Thus $f(x) = f(a+h) \approx f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots$ ...①

or $f(x) \approx f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \cdots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \cdots$, if $h = (x-a)$ ...②

Typically, Taylor series is used to evaluate a function, if the functional value and all it's derivatives can be computed at the given point 'a'.

Example 1 Expand $f(x) = 2x^3 + 7x^2 + x - 6$ in powers of $(x-2)$

Solution: Let $h = x - 2$**,** Then using Taylor's series expansion as given by ②

$$f(x) \approx f(2) + (x-2)f'(2) + \frac{(x-2)^2}{2!}f''(2) + \frac{(x-2)^3}{3!}f'''(2) + \cdots + \frac{(x-2)^n}{n!}f^{(n)}(2) + \cdots \quad ③$$

Now $f(x) = 2x^3 + 7x^2 + x - 6 \quad \Rightarrow f(2) = 40$

$f'(x) = 6x^2 + 14x + 1 \quad \Rightarrow f'(2) = 53$

$f''(x) = 12x + 14 \quad \Rightarrow f''(2) = 38$

$f'''(x) = 12 \quad \Rightarrow f'''(2) = 12$

$f^{iv}(x) = 0$

Using these values in ③ , we get

$$f(x) = 40 + 53(x-2) + \frac{(x-2)^2}{2!}38 + \frac{(x-2)^3}{3!}12$$

$$\Rightarrow f(x) = 40 + 53(x-2) + 19(x-2)^2 + 2(x-2)^3$$

Example2 Expand $tan\ x$ in powers of $\left(x - \frac{\pi}{4}\right)$ upto first four terms.

Solution: Let $h = x - \frac{\pi}{4}$**,** then using Taylor's series expansion as given in ②

$$f(x) \approx f\left(\frac{\pi}{4}\right) + \left(x - \frac{\pi}{4}\right) f'\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^2}{2!} f''\left(\frac{\pi}{4}\right) + \frac{\left(x-\frac{\pi}{4}\right)^3}{3!} f'''\left(\frac{\pi}{4}\right) + \cdots + \frac{\left(x-\frac{\pi}{4}\right)^n}{n!} f^{(n)}\left(\frac{\pi}{4}\right) + \cdots \quad ④$$

Now $f(x) = tan\ x \qquad \Rightarrow f\left(\frac{\pi}{4}\right) = 1$

$f'(x) = sec^2 x \qquad \Rightarrow f'\left(\frac{\pi}{4}\right) = 2$

$f''(x) = 2sec^2 x\ tan\ x \qquad \Rightarrow f''\left(\frac{\pi}{4}\right) = 4$

$f'''(x) = 2sec^4 x + 4\ tan^2 x sec^2 x \qquad \Rightarrow f'''\left(\frac{\pi}{4}\right) = 16$

Using these values in ④, we get

$$f(x) = 1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3$$

Example 3 Estimate the value of $\sqrt{10}$ correct to four places of decimal.

Solution: Here $f(x) = \sqrt{10} = \sqrt{9+1}$, taking $a = 9$ and $h = 1$, $\because x = a + h$

Using Taylor's series expansion as given by ①, we have

$$f(x) \approx f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \cdots + \frac{h^n}{n!} f^{(n)}(a) + \cdots$$

$$\Rightarrow (10)^{\frac{1}{2}} \approx (9+1)^{\frac{1}{2}} = 9^{\frac{1}{2}} + 1.f'(9) + \frac{1^2}{2!} f''(9) + \frac{1^3}{3!} f'''(9) + \frac{1^4}{4!} f^{iv}(9) + \cdots \quad ⑤$$

Now $f(x) = x^{\frac{1}{2}} \qquad \Rightarrow f(9) = 3$

$f'(x) = \frac{1}{2} x^{\frac{-1}{2}} \qquad \Rightarrow f'(9) = \frac{1}{6} = 0.1667$

$$f''(x) = -\frac{1}{4}x^{\frac{-3}{2}} \qquad \Rightarrow f''(9) = -0.0093$$

$$f'''(x) = \frac{3}{8}x^{\frac{-5}{2}} \qquad \Rightarrow f'''(2) = 0.0015$$

$\vdots$

Using these values in ⑤ , we get

$$(10)^{\frac{1}{2}} = 3 + 0.1667 - \frac{0.0093}{2} + \frac{0.0015}{6} + \cdots$$

$$\Rightarrow (10)^{\frac{1}{2}} = 3 + 0.1667 - .0047 + .0003 + \cdots$$

$$= 3.1623 \text{ approx}$$

Maclaurin Series

It is the special case of Taylor series about the point zero. Putting $a = 0$ in ② ,

Thus, Maclaurin series of $f(x)$ is given by: $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$ ⑥

Maclaurin series of some standard functions:

1. $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$
2. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$
3. $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
4. $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, |x| < 1$
5. $\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots, |x| < 1$
6. $(1+x)^m = 1 + mx + \frac{m(m-1)}{2!}x^2 + \frac{m(m-1)(m-2)}{3!}x^3 + \cdots$

Example 4 Expand $e^x cos\ x$ by Maclaurin series.

Solution: We have, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$ ⑥

Here $f(x) = e^x cos\ x \Rightarrow f(0) = e^0 cos\ 0 = 1$

$f'(x) = e^x cos\ x - e^x sin\ x \Rightarrow f'(0) = e^0 cos\ 0 - e^0 sin\ 0 = 1$

$f''(x) = -2e^x sin\ x \Rightarrow f''(0) = -2e^0 sin\ 0 = 0$

$f'''(x) = -2e^x cos\ x - 2e^x sin\ x \Rightarrow f'''(0) = -2e^0 cos\ 0 - 2e^0 sin\ 0 = -2$

$f^{iv}(x) = -4e^x cos\ x \Rightarrow f^{iv}(0) = -4e^0 cos\ 0 = -4$

Putting these values in ⑥, we get

$$\mathrm{e}^x \cos x = 1 + x - \frac{2x^3}{3!} - \frac{4x^4}{4!} - \cdots$$

or $\mathrm{e}^x \cos x = 1 + x - \frac{x^3}{3} - \frac{x^4}{6} + \cdots$

Example5 Show that $log\ sec\ x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$

Solution: We have, $f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \cdots + \frac{x^n}{n!}f^{(n)}(0) + \cdots$ ⑥

Here $f(x) = log\ sec\ x \Rightarrow f(0) = log\ 1 = 0$

$f'(x) = tan\ x \Rightarrow f'(0) = tan\ 0 = 0$

$f''(x) = sec^2 x \Rightarrow f''(0) = sec^2 0 = 1$

$f'''(x) = 2sec^2 x\ tan\ x \Rightarrow f'''(0) = 2sec^2 0\ tan\ 0 = 0$

$f^{iv}(x) = 2sec^4 x + 4\ tan^2 x sec^2 x \Rightarrow f^{iv}(0) = 2$

Putting these values in ⑥, we get $log\ sec\ x = \frac{x^2}{2} + \frac{x^4}{12} + \cdots$

Example 6 Show that $sin\left(\frac{\pi}{4} + \theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots\right)$

Solution: By Taylor's expansion, we have

$$f(x) = f(a+h) \approx f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \frac{h^3}{3!}f'''(a) + \cdots + \frac{h^n}{n!}f^{(n)}(a) + \cdots \quad \ldots①$$

$$\therefore\ sin\ x = sin(a+h) \approx sin\ a + h\ cos\ a + \frac{h^2}{2!}(-sina) + \frac{h^3}{3!}(-cosa) + \cdots$$

Taking $a = \frac{\pi}{4}$ and $h = \theta$

$$sin\left(\frac{\pi}{4}+\theta\right) = sin\frac{\pi}{4} + \theta cos\ \frac{\pi}{4} + \frac{\theta^2}{2!}\left(-sin\frac{\pi}{4}\right) + \frac{\theta^3}{3!}\left(-cos\frac{\pi}{4}\right) + \cdots$$

or $sin\left(\frac{\pi}{4}+\theta\right) = \frac{1}{\sqrt{2}}\left(1 + \theta - \frac{\theta^2}{2!} - \frac{\theta^3}{3!} + \cdots\right)$

Approximation of Errors

Consider a function $y = f(x)$, then if δx be a small change in x and δy be the resulting change in y, then $\delta y = \frac{dy}{dx}\delta x$ approximately.

Example 7 Find the change in the total surface area of a right circular cone when

(i) the radius is constant but there is a small change in the altitude

(ii) the altitude is constant but there is a small change in the radius.

Solution: Let S be the total surface area of the cone , then S $= \pi r^2 + \pi r\sqrt{r^2+h^2}$

(i) The radius r is constant, and altitude h changes, then $\delta S = \frac{dS}{dh}\delta h$

Now $\frac{dS}{dh} = 0 + \frac{\pi r}{2\sqrt{r^2+h^2}}.2h = \frac{\pi rh}{\sqrt{r^2+h^2}}$

$\therefore\ \delta S = \frac{dS}{dh}\delta h = \frac{\pi rh}{\sqrt{r^2+h^2}}\delta h$ approximately.

(ii) The altitude h is constant, and radius r changes, then $\delta S = \frac{dS}{dr}\delta r$

Now $\frac{dS}{dr} = 2\pi r + \pi\sqrt{r^2+h^2} + \frac{2\pi r^2}{2\sqrt{r^2+h^2}} = 2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}$

$\therefore\ \delta S = \frac{dS}{dr}\delta r = \left(2\pi r + \frac{\pi(2r^2+h^2)}{\sqrt{r^2+h^2}}\right)\delta r$ approximately.

Example 8 If a , b , c are the sides of the triangle ABC and S is the semi- perimeter, show that if there is a small error δc in the measurement of side c then the error $\delta\Delta$ in the area Δ of the triangle is given by

$$\delta\Delta = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{S-a} + \frac{1}{S-b} - \frac{1}{S-c}\right)\delta c$$

Solution: We know that $\Delta^2 = S(S-a)(S-b)(S-c)$

Taking log of both sides, we get $2log\Delta = log\ S + \log(S-a) + \log(S-b) + \log\ (S-c)$

Differentiating both the sides w.r.t. c , we get $\frac{2}{\Delta}\frac{d\Delta}{dc} = \frac{1}{S}\frac{dS}{dc} + \frac{1}{S-a}\frac{d(S-a)}{dc} + \frac{1}{S-b}\frac{d(S-b)}{dc} + \frac{1}{S-c}\frac{d(S-c)}{dc}$

Also, $S = \frac{(a+b+c)}{2}$,

$$\therefore \frac{2}{\Delta}\frac{d\Delta}{dc} = \frac{1}{S}\frac{1}{2} + \frac{1}{2(S-a)} + \frac{1}{2(S-b)} - \frac{1}{2(S-c)}$$

$$\Rightarrow \frac{d\Delta}{dc} = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} - \frac{1}{(S-c)}\right)$$

$$\Rightarrow \delta\Delta = \frac{d\Delta}{dc}\delta c = \frac{\Delta}{4}\left(\frac{1}{S} + \frac{1}{(S-a)} + \frac{1}{(S-b)} - \frac{1}{(S-c)}\right)\delta c$$

Example 9 If $T = 2\pi\sqrt{\left(l/g\right)}$, find the error in T corresponding to 2% error in l where g is constant.

Solution: Error in T is given by $\delta T = \frac{dT}{dl}\delta l$

$$\text{Now } \frac{dT}{dl} = \frac{2\pi}{\sqrt{g}}\frac{1}{2\sqrt{l}}$$

$$\therefore\ \delta T = \frac{\pi}{\sqrt{g}}\frac{1}{\sqrt{l}}\delta l$$

$$\Rightarrow\ \frac{\delta T}{T} = \frac{\pi}{\sqrt{g}}\frac{\delta l}{\sqrt{l}}\frac{\sqrt{g}}{2\pi\sqrt{l}} = \frac{1}{2}\frac{\delta l}{l}$$

$$\Rightarrow\ \left(\frac{\delta T}{T}.100\right) = \frac{1}{2}\left(\frac{\delta T}{T}.100\right)$$

i.e. percentage error in T is half of the percentage error in l

$\therefore$ corresponding to 2% error in l , percentage error in T is 1%.

Exercise

1. Expand $tan^{-1}x$ in powers of $(x-1)$.

   Ans. $tan^{-1}x = \frac{\pi}{4} + \frac{1}{2}(x-1) - \frac{1}{4}(x-1)^2 + \frac{1}{12}(x-1)^3 + \cdots$

2. Using Taylor's theorem find the approximate value of $f\left(\frac{11}{10}\right)$ where $f(x) = x^3 + 3^2 + 15x - 10$

   Ans. 11.461

3. Show that $\log(1 + sinx) = x - \frac{x^2}{2} + \frac{x^3}{6} - \frac{x^4}{12} + \cdots$

4. Show that $tan\left(\frac{\pi}{4} + x\right) = 1 + 2x + 2x^2 + \frac{8}{3}x^3 + \frac{10}{3}x^4 + \cdots$ and hence find tan 46^o

   Ans. 1.0355

5. A soap bubble of radius 2cm shrinks to radius 1.9 cm. Find the decrease in volume and surface area.

   Ans. -5.024 cm^3 and -5.024cm^2

6. If $\log_{10}4$ = 0.6021, find the approximate value of $\log_{10}404$.

   Ans. 2.61205

7. Let A, B and C be the angles of a triangle opposite to the sides a, b and c respectively. If small errors $\delta a, \delta b$ and $\delta c$are made in the sides then show that $\delta A = \frac{a}{2\Delta}(\delta a - \delta b\ cosC - \delta c\ cosB)$ where Δ is the area of the triangle.