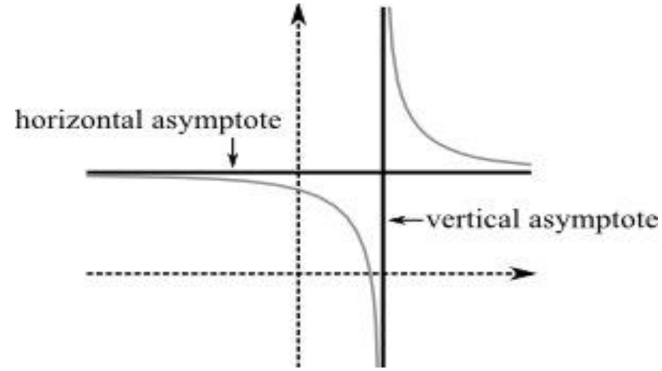
ASYMPTOTES

An asymptote of a curve is a straight line such that the distance between the curve and the line appears to be zero as one or both of the x or y coordinates tend to infinity. There are two types of asymptotes viz. Rectangular asymptotes and Oblique asymptotes

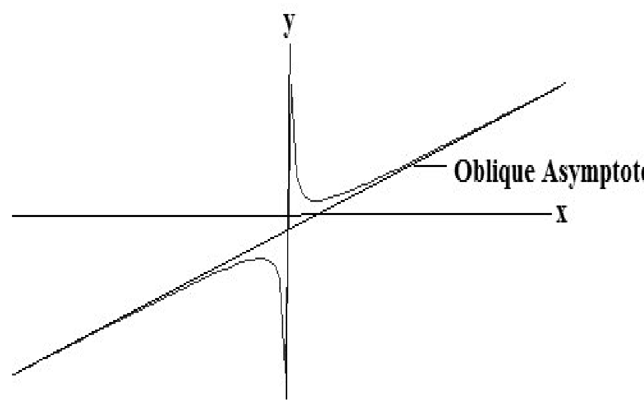
Rectangular Asymptote:

If an asymptote is parallel to x-axis or to y-axis, then it is called rectangular asymptote. An asymptote parallel to x-axis is called horizontal asymptote and the asymptote parallel to y-axis is called vertical asymptote.



Oblique Asymptote:

If an asymptote is neither parallel to x-axis nor to y-axis then it is called an oblique asymptote. An oblique asymptote occurs when the degree of polynomial in the numerator is greater than that of polynomial in the denominator. To find the oblique asymptote, numerator must be divided by the denominator by using either long division or synthetic division.



Results

1. Any curve of degree n cannot have more than n asymptotes.
2. Closed curves such as circle or ellipse do not have any associated asymptotes.
3. Hyperbolas are the only conic sections with asymptotes. Parabolas do not have any asymptote as a function $f(x)$ can have asymptote, if it looks like a straight line when x or y tend to $\pm\infty$.
4. A curve can have maximum two horizontal asymptotes.

Working Rule for Finding Asymptotes

1. Check the degree of the curve, a curve of degree n can have at the maximum n asymptotes.
2. Find rectangular asymptotes if they exist,
   i. To find an asymptote parallel to x-axis (horizontal asymptote), equate to zero the coefficient of highest power of x in the equation of the curve. If the coefficient of highest power of x is a constant, a horizontal asymptote does not exist.
   ii. To find an asymptote parallel to y-axis (vertical asymptote), equate to zero the coefficient of highest power of y in the equation of the curve. If the coefficient of highest power of y is a constant, a vertical asymptote does not exist.
   iii. Go to step 3 if no rectangular asymptotes are found or the degree of the curve exceeds the number of rectangular asymptotes.
3. Find oblique asymptotes $(y = mx + c)$ using the procedure given below
   i. Arrange the equation in the form $\phi_n(x,y) + \phi_{n-1}(x,y) + \cdots + \phi_1(x,y) + k = 0$, where $\phi_n(x,y)$ is the function containing n^{th} degree terms, $\phi_{n-1}(x,y)$ is the function of $(n-1)^{th}$ degree terms etc.
   ii. Put $x = 1$ and $y = m$ in all the functions $\phi_n(x,y), \phi_{n-1}(x,y), \ldots, \phi_1(x,y)$ to get the respective functions as $\phi_n(m), \phi_{n-1}(m), \ldots, \phi_1(m)$.
   iii. Find all the real roots of the equation $\phi_n(m) = 0$ and name them as $m_1, m_2, m_3, \ldots, m_n$.
4. For all non - repeated roots $m_1, m_2, \ldots$, compute the corresponding values of $c_1, c_2, \ldots$, using the formula $c = -\frac{\phi_{n-1}(m)}{\phi'_n(m)}$. If $\phi'_n(m) = 0$, then there is no asymptote corresponding to this value of m.
5. For any two repeated roots m and m, the corresponding values of c are given by the quadratic equation: $\frac{c^2}{2!}\phi''_n(m) + \frac{c}{1!}\phi'_{n-1}(m) + \phi_{n-2}(m) = 0$.

Similarly, for three repeated roots m , m and m , the corresponding values of c are given by the cubic equation: $\frac{c^3}{3!}\phi_n'''(m) + \frac{c^2}{2!}\phi_{n-1}''(m) + \frac{c}{1!}\phi_{n-2}'(m) + \phi_{n-3}(m) = 0$ etc.

6. Find the respective asymptotes as $y = m_1x + c_1$, $y = m_2x + c_2$, $y = m_3x + c_3$ etc.

Example1 Find the asymptotes parallel to coordinate axes of the curve $4x^2 + 9y^2 = x^2y^2$

Solution: The degree of curve is 4, therefore the curve can have maximum 4 asymptotes.

To find the asymptotes parallel to x-axis, rewriting the equation of the curve as

$x^2(4 - y^2) + 9y^2 = 0$ and equating the coefficient of x^2 as zero, we get

$4 - y^2 = 0 \Rightarrow y = \pm 2$

$\therefore y = 2, y = -2$ are the two asymptotes parallel to x -axis.

To find the asymptotes parallel to y-axis, rewriting the equation of the curve as

$y^2(9 - x^2) + 4x^2 = 0$ and equating the coefficient of y^2 as zero, we get

$9 - x^2 = 0 \Rightarrow x = \pm 3$

$\therefore x = 3, x = -3$ are the two asymptotes parallel to y-axis.

Example2 Find all the asymptotes parallel to coordinate axes for the following curves:

$i.\quad y = \frac{1}{x-2}\quad ii.\quad y = \frac{3x}{x-2}\quad iii.\ y = \frac{x^2-1}{8-2x^2}\quad iv.\ y = \frac{x^2}{x-3}\quad v.\ y = \frac{x-1}{x^2-3x+2}$

Solution: $i.$ Rewriting the equation of the curve as: $xy - 2y = 1$

The degree of curve is 2, therefore the curve can have maximum 2 asymptotes.

To find an asymptote parallel to x-axis, equating the coefficient of x as zero, we get $y = 0$.

$\therefore x$ -axis itself is an asymptote to the curve.

Again, to find the asymptotes parallel to y-axis, rewriting the equation of the curve as

$y(x - 2) - 1 = 0$ and equating the coefficient of y as zero, we get $x - 2 = 0$

$\therefore x = 2$ is an asymptotes parallel to y-axis.

ii. Rewriting the equation of the curve as: $xy - 3x - 2y = 0$

The degree of curve is 2, therefore the curve can have maximum 2 asymptotes.

To find an asymptote parallel to x-axis, rewriting the equation of the curve as $x(y-3) - 2y = 0$, and equating the coefficient of x as zero, we get $y = 3$ is the horizontal asymptote.

Again, to find the asymptote parallel to y-axis, rewriting the equation of the curve as $y(x-2) - 3x = 0$ and equating the coefficient of y as zero, we get $x - 2 = 0$

$\therefore x = 2$ is an asymptotes parallel to y-axis.

iii. Rewriting the equation of the curve as: $-2x^2y + 8y - x^2 + 1 = 0$

The degree of curve is 3, therefore the curve can have maximum 3 asymptotes.

To find an asymptote parallel to x-axis, rewriting the equation of the curve as

$x^2(-2y-1) + 8y + 1 = 0$

Equating the coefficient of x^2 as zero, we get $y = \frac{-1}{2}$ is the horizontal asymptote.

Again, to find the asymptote parallel to y-axis, rewriting the equation of the curve as $y(-2x^2+8) - x^2 + 1 = 0$ and equating the coefficient of y as zero, we get $x^2 = 4$

$\therefore x = \pm 2$ are the asymptotes parallel to y-axis.

iv. Rewriting the equation of the curve as: $x^2 - xy + 3y = 0$

The degree of curve is 2, therefore the curve can have maximum 2 asymptotes.

There is no asymptote parallel to x-axis as coefficient of x^2 is a constant number.

Again, to find the asymptote parallel to y-axis, rewriting the equation of the curve as $y(-x+3) + x^2 = 0$ and equating the coefficient of y as zero, we get $x = 3$

$\therefore x = 3$ is an asymptotes parallel to y-axis.

v. Rewriting the equation of the curve as: $y = \frac{x-1}{(x-1)(x-2)}$, we have $y = \frac{1}{(x-2)}$

The degree of curve is 2, therefore the curve can have maximum 2 asymptotes.

To find an asymptote parallel to x-axis, rewriting the equation of the curve as $xy - 2y = 1$, and equating the coefficient of x as zero, we get $y = 0$ is the horizontal asymptote. i.e. x-axis itself is an asymptote.

Again, to find the asymptotes parallel to y-axis, rewriting the equation of the curve as $y(x-2) - 1 = 0$ and equating the coefficient of y as zero, we get $x - 2 = 0$

$\therefore x = 2$ is an asymptotes parallel to y-axis.

Example3 Find all the asymptotes of the curve $x^2y^2 - a^2(x^2 + y^2) = 0$

Solution: The degree of curve is 4, therefore the curve can have maximum 4 asymptotes.

To find the asymptotes parallel to x-axis, rewriting the equation of the curve as $x^2(y^2 - a^2) - a^2y^2 = 0$ and equating the coefficient of x^2 as zero, we get

$y^2 - a^2 = 0 \Longrightarrow y = \pm a$

$\therefore y = a$, $y = -a$ are the two asymptotes parallel to x -axis.

To find the asymptotes parallel to y-axis, rewriting the equation of the curve as $y^2(x^2 - a^2) - a^2x^2 = 0$ and equating the coefficient of y^2 as zero, we get

$x^2 - a^2 = 0 \Longrightarrow x = \pm a$

$\therefore x = a$, $x = -a$ are the two asymptotes parallel to y-axis.

There are no oblique asymptotes as the curve cannot have more than 4 asymptotes.

Example4 Find all the asymptotes of the curve $y = \frac{x^2}{x-3}$

Solution: Rewriting the equation of the curve as: $x^2 - xy + 3y = 0$

$x = 3$ is an asymptotes parallel to y-axis (see example 2 (iv))

Now finding the oblique asymptote, putting $x = 1$ and $y = m$ in the functions $\phi_2(x, y)$and $\phi_1(x, y)$.

$\phi_2(x,y) = x^2 - xy \qquad \Rightarrow \phi_2(m) = 1 - m$

$\phi_1(x,y) = 3y \qquad \Rightarrow \phi_1(m) = 3\text{m}$

Solving the equation $\phi_2(m) = 0$ to find m

$\Rightarrow 1 - m = 0$ or $m = 1$

Now, for the non-repeated root $m = 1$, $c = -\left.\frac{\phi_1(m)}{\phi_2'(m)}\right]_{m=1} \qquad \Rightarrow c = -\left.\frac{3m}{-1}\right]_{m=1} \Rightarrow c = 3$

$\therefore$ The required asymptote is: $y = mx + c$, i.e., $y = x + 3$

Example5 Find all asymptotes of the curve $y^3 - 3xy^2 - x^2y + 3x^3 - 3x^2 + 10xy - 3y^2 - 10x - 10y + 7 = 0$

Solution: Here coefficient of x^3 is 1(a constant) and also the coefficient of y^3 is a constant. Therefore, rectangular asymptotes do not exist. Now finding the oblique asymptotes, putting $x = 1$ and $y = m$ in the functions $\phi_3(x,y)$, $\phi_2(x,y)$ and $\phi_1(x,y)$.

$\phi_3(x,y) = y^3 - 3xy^2 - x^2y + 3x^3 \qquad \Rightarrow \phi_3(m) = m^3 - 3m^2 - m + 3 = 0$

$\phi_2(x,y) = -3x^2 + 10xy - 3y^2 \qquad \Rightarrow \phi_2(m) = -3 + 10m - 3m^2$

$\phi_1(x,y) = -10x - 10y \qquad \Rightarrow \phi_1(m) = -10 - 10m$

Solving the equation $\phi_3(m) = 0$ to find m_1, m_2 and m_3

$\Rightarrow m^3 - 3m^2 - m + 3 = 0$

$\Rightarrow m^3 - m^2 - 2m^2 + 2m - 3m + 3 = 0$

$\Rightarrow m^2(m-1) - 2m(m-1) - 3(m-1) = 0 \qquad \Rightarrow (m^2 - 2m - 3)(m-1) = 0$

$\Rightarrow (m-3)(m+1)(m-1) = 0$

$\Rightarrow m = 3, -1, 1$

Now, for the non-repeated root $m = 3$, $c = -\left.\frac{\phi_2(m)}{\phi_3'(m)}\right]_{m=3}$

$$\Rightarrow c = -\frac{-3+10m-3m^2}{3m^2-6m-1}\Bigg]_{m=3} \Rightarrow c = -\frac{-3+10(3)-3(3)^2}{3(3)^2-6(3)-1} = -\frac{-3+30-27}{27-18-1} = 0$$

$\therefore$ The required asymptote is: $y = mx + c$, i.e., $y = 3x$

Again, for the non-repeated root $m = -1$, $c = -\frac{\phi_2(m)}{\phi_3'(m)}\Big]_{m=-1}$

$$\Rightarrow c = -\frac{-3+10m-3m^2}{3m^2-6m-1}\Bigg]_{m=-1} \Rightarrow c = -\frac{-3+10(-1)-3(-1)^2}{3(-1)^2-6(-1)-1} = -\frac{-3-10-3}{3+6-1} = 2$$

$\therefore$ The required asymptote is: $y = mx + c$, i.e., $y = -x + 2$

Also, for the non-repeated root $m = 1$, $c = -\frac{\phi_2(m)}{\phi_3'(m)}\Big]_{m=1}$

$$\Rightarrow c = -\frac{-3+10m-3m^2}{3m^2-6m-1}\Bigg]_{m=1} \Rightarrow c = -\frac{-3+10(1)-3(1)^2}{3(1)^2-6(1)-1} = -\frac{-3+10-3}{3-6-1} = 1$$

$\therefore$ The required asymptote is: $y = mx + c$, i.e., $y = x + 1$

Hence, the three asymptotes to the curve are given by:

$y - 3x = 0$, $y + x - 2 = 0$ and $y - x - 1 = 0$

Example6 Find all asymptotes of the curve $x^3 - x^2y - xy^2 + y^3 + 2x^2 - 4y^2 + 2xy + x + y + 1 = 0$

Solution: Here coefficient of x^3 is 1(a constant) and also the coefficient of y^3 is a constant. Therefore, rectangular asymptotes do not exist. Now finding the oblique asymptotes, putting $x = 1$ and $y = m$ in the functions $\phi_3(x, y)$, $\phi_2(x, y)$ and $\phi_1(x, y)$.

$$\phi_3(x, y) = x^3 - x^2y - xy^2 + y^3 \Rightarrow \phi_3(m) = 1 - m - m^2 + m^3$$
$$\phi_2(x, y) = 2x^2 - 4y^2 + 2xy \Rightarrow \phi_2(m) = 2 - 4m^2 + 2m$$
$$\phi_1(x, y) = x + y \Rightarrow \phi_1(m) = 1 + m$$

Solving the equation $\phi_3(m) = 0$ to find m_1, m_2 and m_3

$\Rightarrow m^3 - m^2 - m + 1 = 0$

$\Rightarrow m^2(m-1)-(m-1)=0 \quad \Rightarrow (m^2-1)(m-1)=0$

$\Rightarrow m = 1\,,1\,,-1$

Now, for the non-repeated root $m=-1\,,\, c=-\frac{\phi_2(m)}{\phi_3'\,(m)}\Big]_{m=-1}$

$$\Rightarrow c=-\frac{2-4m^2+2m}{-1-2m+3m^2}\Big]_{m=-1} \Rightarrow\ c=-\frac{2-4(-1)^2+2(-1)}{-1-2(-1)+3(-1)^2}=-\frac{2-4-2}{-1+2+3}=1$$

∴ The required asymptote is: $y=mx+c$, i.e., $y=-x+1$

Again, for the repeated roots $m=1$, 1, the corresponding values of c are given by the quadratic equation:

$$\frac{c^2}{2!}\phi_3''(m)+\frac{c}{1!}\ \phi_2'(m)+\phi_1(m)=0$$

$$\Rightarrow\ \frac{c^2}{2!}(-2+6m)+\frac{c}{1!}\ (-8m+2)+(1+m)=0$$

Putting $m=1$, we get

$$\frac{c^2}{2}(4)+\frac{c}{1}\ (-6)+(2)=0$$

$$\Rightarrow\ c^2-3c+1=0$$

$$\therefore c=\frac{3\pm\sqrt{5}}{2}\ \text{, i.e., } c=\frac{3+\sqrt{5}}{2},\ \frac{3-\sqrt{5}}{2}$$

∴ The required asymptote for $m=1$ and $c=\frac{3+\sqrt{5}}{2}$ is given by: $y=x+\frac{3+\sqrt{5}}{2}$

The required asymptote for $m=1$ and $c=\frac{3-\sqrt{5}}{2}$ is given by: $y=x+\frac{3-\sqrt{5}}{2}$

Hence, the three asymptotes to the curve are given by:

$$x+y=1\,,\ 2(y-x)=3+\sqrt{5}\ \text{and}\ 2(y-x)=3-\sqrt{5}$$

Example 7 Find the asymptotes of the curve:

$$y^4-2xy^3+2x^3y-x^4-3x^3+3x^2y+3xy^2-3y^3-2x^2+2y^2-1=0$$

Solution: Here coefficient of x^4 is -1 (a constant) and also the coefficient of y^4 is a constant. Therefore, rectangular asymptotes do not exist. Now finding the oblique asymptotes, putting $x = 1$ and $y = m$ in the functions $\phi_4(x, y)$, $\phi_3(x, y)$, $\phi_2(x, y)$ and $\phi_1(x, y)$.

$\phi_4(x, y) = y^4 - 2xy^3 + 2x^3y - x^4 \quad \Rightarrow \phi_4(m) = m^4 - 2m^3 + 2m - 1$

$\phi_3(x, y) = -3x^3 + 3x^2y + 3xy^2 - 3y^3 \quad \Rightarrow \phi_3(m) = -3 + 3m + 3m^2 - 3m^3$

$\phi_2(x, y) = -2x^2 + 2y^2 \quad \Rightarrow \phi_2(m) = -2 + 2m^2$

$\phi_1(x, y) = 0 \quad \Rightarrow \phi_1(m) = 0$

Solving the equation $\phi_4(m) = 0$ to find m_1, m_2, m_3 and m_4

$\Rightarrow m^4 - 2m^3 + 2m - 1 = 0$

$m = -1$ satisfies the given equation, $\therefore (m + 1)$ is a factor

Rewriting the equation as:

$m^4 + m^3 - 3m^3 - 3m^2 + 3m^2 + 3m - m - 1 = 0$

$\Rightarrow m^3(m + 1) - 3m^2(m + 1) + 3m(m + 1) - 1(m + 1) = 0$

$\Rightarrow (m^3 - 3m^2 + 3m - 1)(m + 1) = 0$

$\Rightarrow (m - 1)^3(m + 1) = 0$

$\Rightarrow m = 1, 1, 1 \; -1$

Now, for the non-repeated root $m = -1$, $c = -\left.\frac{\phi_3(m)}{\phi_4'(m)}\right]_{m=-1}$

$\Rightarrow c = -\left.\frac{-3+3m+3m^2-3m^3}{4m^3-6m^2+2}\right]_{m=-1} \Rightarrow c = -\frac{-3+3(-1)+3(-1)^2-3(-1)^3}{4(-1)^3-6(-1)^2+2} = -\frac{-3-3+3+3}{-4-6+2} = 0$

$\therefore$ The required asymptote is: $y = mx + c$, i.e., $y = -x$

Again, for the repeated roots $m = 1, 1, 1$, the corresponding values of c are given by the quadratic equation:

$$\frac{c^3}{3!}\phi_4'''(m) + \frac{c^2}{2!}\phi_3''(m) + \frac{c}{1!}\phi_2'(m) + \phi_1(m) = 0$$

$$\Rightarrow \frac{c^3}{3!}(24m - 12) + \frac{c^2}{2!}(6 - 18m) + \frac{c}{1!}(4m) + 0 = 0$$

Putting $m = 1$, we get

$$\frac{c^3}{3!}(12) + \frac{c^2}{2!}(-12) + \frac{c}{1!}(4) = 0$$

$$\Rightarrow 2c^3 - 6c^2 + 4c = 0$$

$$\Rightarrow 2c(c^2 - 3c + 2) = 0$$

$$\Rightarrow c(c-1)(c-2) = 0$$

$\therefore c = 0, 1, 2$

$\therefore$ The required asymptote for $m = 1$ and $c = 0$ is given by: $y = x$

The required asymptote for $m = 1$ and $c = 1$ is given by: $y = x + 1$

The required asymptote for $m = 1$ and $c = 2$ is given by: $y = x + 2$

Hence, the four asymptotes to the curve are given by:

$x + y = 0$, $y - x = 0$, $y - x - 1 = 0$ and $y - x - 2 = 0$

Exercise A

1. Find the asymptotes of the following curves:

(i) $\frac{a^2}{x^2} + \frac{b^2}{y^2} = 1$ Ans. $x = \pm b, y = \pm a$

(ii) $x^3 + 3x^2y - 4y^3 - x + y + 3 = 0$ Ans. $y = x$, $x + 2y - 1 = 0$ and $x + 2y + 1 = 0$

(iii) $y(x - y)^2 = x + y$ Ans. $y = 0$, $y = x + \sqrt{2}$ *and* $y = x - \sqrt{2}$

(iv) $x^2y + xy^2 - xy + y^2 + 3x = 0$ Ans. $y = 0$ and $y = -x$

(v) $y^3 - 2xy^2 - x^2y + 2x^3 + 2x^2 - 3xy + x - 2y + 1 = 0$

Ans. $2(x - y) = 1,\ 6(x + y) + 5 = 0$ and $6x - 3y + 4 = 0$

(vi) $(x^2 - y^2)^2 - 4x^2 + x = 0$ Ans. y $= x + 1,\ y = x - 1,\ y = -x + 1, y = -x - 1$

(vii) $x^3 + 4x^2y + 5xy^2 + 2y^2 + 2x^2 + 4xy + 2y^2 - x - 9y + 1 = 0$

Ans. $x + 2y + 2 = 0,\ x + y \pm 2\sqrt{2} = 0$

(viii) $(x - y)^2 (x - 2y)(x - 3y) - 2a(x^3 - y^3) - (x - 2y)(x + y) = 0$

Ans. $2y = x + 14a,\ 3y = x - 13a,\ y = x - a,\ y = x - 2a$