

# Z-TRANSFORMS

## 4.1 Introduction

Z-Transform plays an important role in discrete analysis and may be seen as discrete analogue of Laplace transform. Role of Z-Transforms in discrete analysis is the same as that of Laplace and Fourier transforms in continuous systems.

**Definition:** The Z-Transform of a sequence  $u_n$  defined for discrete values  $n = 0, 1, 2, 3, \dots$  and ( $u_n = 0$  for  $n < 0$ ) is defined as  $Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$ . Z-Transform of the sequence  $u_n$  i.e.  $Z\{u_n\}$  is a function of  $z$  and may be denoted by  $U(z)$

**Remark:**

- Z-Transform exists only when the infinite series  $\sum_{n=0}^{\infty} u_n z^{-n}$  is convergent.
- $Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$  is termed as one-sided transform and for two sided Z-transform  $Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$

### Results on Z-Transforms of standard sequences

1. 
$$Z\{a^n\} = \frac{z}{z-a}$$

$$\begin{aligned} Z\{a^n\} &= \sum_{n=0}^{\infty} a^n z^{-n} \\ &= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots + \frac{a^n}{z^n} + \dots \\ &= \frac{1}{1-\frac{a}{z}}, \quad \left| \frac{a}{z} \right| < 1 \end{aligned}$$

$$\therefore Z\{a^n\} = \frac{z}{z-a}, \quad \left| \frac{a}{z} \right| < 1$$

2. 
$$Z\{1\} = \frac{z}{z-1}$$

$$Z\{1\} = \frac{z}{z-1} \quad \text{Putting } a = 1 \text{ in Result 1}$$

3. 
$$Z\{(-1)^n\} = \frac{z}{z+1}$$

$$Z\{(-1)^n\} = \frac{z}{z+1} \quad \text{Putting } a = -1 \text{ in Result 1}$$

4. 
$$Z\{k\} = \frac{kz}{z-1}$$

$$\begin{aligned} Z\{k\} &= \sum_{n=0}^{\infty} k z^{-n} = k \sum_{n=0}^{\infty} z^{-n} \\ &= k \left[ 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n} + \dots \right] \\ \therefore Z\{k\} &= \frac{kz}{z-1} \end{aligned}$$

5. Recurrence formula for  $n^p$ :

$$Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\}$$

$$Z\{n^p\} = \sum_{n=0}^{\infty} n^p z^{-n}, p \text{ is a positive integer} \quad \dots \textcircled{1}$$

$$Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} z^{-n} \quad \dots \textcircled{2}$$

Differentiating  $\textcircled{2}$  w.r.t.  $z$ , we get

$$\begin{aligned} \frac{d}{dz} Z\{n^{p-1}\} &= \sum_{n=0}^{\infty} n^{p-1} (-n) z^{-n-1} \\ &= -z^{-1} \sum_{n=0}^{\infty} n^p z^{-n} \end{aligned}$$

$$\Rightarrow \frac{d}{dz} Z\{n^{p-1}\} = -z^{-1} Z\{n^p\} \quad \text{using } \textcircled{1}$$

$$\Rightarrow Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\}$$

6. **Multiplication by  $n$ :**

$$Z\{nu_n\} = -z \frac{d}{dz} Z\{u_n\}$$

$$\begin{aligned} Z\{nu_n\} &= \sum_{n=0}^{\infty} nu_n z^{-n} \\ &= -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1} \\ &= -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} z^{-n} \\ &= -z \sum_{n=0}^{\infty} \frac{d}{dz} (u_n z^{-n}) \\ &= -z \frac{d}{dz} (\sum_{n=0}^{\infty} u_n z^{-n}) \\ &= -z \frac{d}{dz} Z\{u_n\} \end{aligned}$$

7.

$$Z\{n\} = \frac{z}{(z-1)^2}$$

$$Z\{n\} = -z \frac{d}{dz} Z\{n^0\} \quad \text{using Recurrence Result 5 or 6}$$

$$= -z \frac{d}{dz} Z\{1\}$$

$$= -z \frac{d}{dz} \frac{z}{z-1} \quad \text{using result 2}$$

$$\Rightarrow Z\{n\} = \frac{z}{(z-1)^2}$$

8.

$$Z\{n^2\} = \frac{z^2 + z}{(z-1)^3}$$

$$Z\{n^2\} = -z \frac{d}{dz} Z\{n\} \quad \text{using Recurrence Result 5 or 6}$$

$$= -z \frac{d}{dz} \frac{z}{(z-1)^2} \quad \text{using Result 7}$$

$$\Rightarrow Z\{n^2\} = \frac{z^2 + z}{(z-1)^3}$$

9.

$$Z\{u(n) = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}\} = \frac{z}{z-1}, u(n) = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases} \text{ is unit step sequence}$$

$$\begin{aligned} Z\{u(n)\} &= \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} 1 z^{-n} \\ &= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \dots + \frac{1}{z^n} + \dots \end{aligned}$$

$$\Rightarrow Z\{u(n)\} = \frac{z}{z-1}$$

$$10. \quad \boxed{Z\{\delta(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases} = 1} \quad \delta(n) = \begin{cases} 1, n = 0 \\ 0, n \neq 0 \end{cases} \text{ is unit impulse sequence}$$

$$\begin{aligned} Z\{\delta(n)\} &= \sum_{n=0}^{\infty} \delta(n)z^{-n} \\ &= 1 + 0 + 0 + \dots \\ \Rightarrow Z\{\delta(n)\} &= 1 \end{aligned}$$

## 4.2 Properties of Z-Transforms

**1. Linearity:**  $Z\{au_n + bv_n\} = aZ\{u_n\} + bZ\{v_n\}$

$$\begin{aligned} \text{Proof: } Z\{au_n + bv_n\} &= \sum_{n=0}^{\infty} (au_n + bv_n)z^{-n} \\ &= a \sum_{n=0}^{\infty} u_n z^{-n} + b \sum_{n=0}^{\infty} v_n z^{-n} \\ &= aZ\{u_n\} + bZ\{v_n\} \end{aligned}$$

**2. Change of scale (or Damping rule):**

If  $Z\{u_n\} \equiv U(z)$ , then  $Z\{a^{-n}u_n\} \equiv U(az)$  and  $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

$$\begin{aligned} \text{Proof: } Z\{a^{-n}u_n\} &= \sum_{n=0}^{\infty} a^{-n}u_n z^{-n} \\ &= \sum_{n=0}^{\infty} u_n (az)^{-n} \equiv U(az) \end{aligned}$$

Similarly,  $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

### Results from application of Damping rule

i.  $\boxed{Z\{a^n n\} = \frac{az}{(z-a)^2}}$

$$\begin{aligned} \text{Proof: } Z\{n\} &= \frac{z}{(z-1)^2} \equiv U(z) \text{ say} \\ \therefore Z\{a^n n\} &\equiv U\left(\frac{z}{a}\right) = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2} \end{aligned}$$

ii.  $\boxed{Z\{a^n n^2\} = \frac{az^2 + a^2 z}{(z-a)^3}}$

$$\begin{aligned} \text{Proof: } Z\{n^2\} &= \frac{z^2+z}{(z-1)^3} \equiv U(z) \text{ say} \\ \therefore Z\{a^n n^2\} &\equiv U\left(\frac{z}{a}\right) = \frac{\left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)}{\left(\left(\frac{z}{a}\right)-1\right)^3} = \frac{a(z^2+az)}{(z-a)^3} \end{aligned}$$

iii.  $\boxed{Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \quad Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}}$

Proof: We have  $Z\{e^{-in\theta}\} = Z\{(e^{i\theta})^{-n}\} = Z\{(e^{i\theta})^{-n} \cdot 1\}$

$$\text{Now } Z\{1\} = \frac{z}{z-1}$$

$$\begin{aligned} \therefore Z\{(e^{i\theta})^{-n} \cdot 1\} &= \frac{ze^{i\theta}}{ze^{i\theta}-1} & \therefore Z\{a^{-n}u_n\} &\equiv U(az) \\ &= \frac{z}{z-e^{-i\theta}} \\ &= \frac{z(z-e^{i\theta})}{(z-e^{-i\theta})(z-e^{i\theta})} \end{aligned}$$

$$= \frac{z(z - \cos\theta - i\sin\theta)}{z^2 - z(e^{i\theta} + e^{-i\theta}) + 1} \quad \because e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \frac{z(z - \cos\theta - i\sin\theta)}{z^2 - 2z\cos\theta + 1} \quad \because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\therefore Z\{e^{-in\theta}\} = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} - i \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\Rightarrow Z\{\cos n\theta - i\sin n\theta\} = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} - i \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1}$$

$$\therefore Z\{\cos n\theta\} = \frac{z(z - \cos\theta)}{z^2 - 2z\cos\theta + 1} \quad \dots \textcircled{3}$$

$$\text{and } Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2 - 2z\cos\theta + 1} \quad \dots \textcircled{4}$$

iv.  $Z\{a^n \cos n\theta\} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2}, \quad Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2 - 2z\cos\theta + a^2}$

By Damping rule, replacing  $z$  by  $\frac{z}{a}$  in  $\textcircled{3}$  and  $\textcircled{4}$ , we get

$$Z\{a^n \cos n\theta\} = \frac{z(z - a\cos\theta)}{z^2 - 2az\cos\theta + a^2} \quad \text{and} \quad Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2 - 2z\cos\theta + a^2}$$

### 3. Right Shifting Property

For  $n \geq k$ ,  $Z\{u_{n-k}\} = z^{-k}Z\{u_n\}$ ,  $k$  is positive integer

Proof:  $Z\{u_{n-k}\} = \sum_{n=0}^{\infty} u_{n-k}z^{-n}$

$$= u_{-k}z^0 + u_{1-k}z^{-1} + \dots + u_{-1}z^{-k+1} + u_0z^{-k} + u_1z^{-(k+1)} + u_2z^{-(k+2)} + \dots$$

$$= 0 + u_0z^{-k} + u_1z^{-(k+1)} + u_2z^{-(k+2)} + \dots \quad \because u_n = 0 \text{ for } n < 0$$

$$= \sum_{n-k=0}^{\infty} u_{n-k}z^{-n}$$

$$= \sum_{m=0}^{\infty} u_mz^{-k-m}$$

$$= z^{-k} \sum_{m=0}^{\infty} u_mz^{-m}$$

$$= z^{-k} \sum_{n=0}^{\infty} u_nz^{-n}$$

$$= z^{-k}Z\{u_n\}$$

### 4. Left Shifting Property

If  $k$  is a positive integer  $Z\{u_{n+k}\} = z^k \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \dots - \frac{u_{k-1}}{z^{k-1}} \right]$

Proof:  $Z\{u_{n+k}\} = \sum_{n=0}^{\infty} u_{n+k}z^{-n}$

$$= z^k \sum_{n=0}^{\infty} u_{n+k}z^{-(n+k)}$$

$$= z^k [u_kz^{-k} + u_{1+k}z^{-(1+k)} + u_{2+k}z^{-(2+k)} + \dots]$$

$$= z^k [u_0 + u_1z^{-1} + u_2z^{-2} + \dots + u_{k-1}z^{-(k-1)} + u_kz^{-k} + \dots]$$

$$= z^k [\sum_{n=0}^{\infty} u_nz^{-n} - \sum_{n=0}^{k-1} u_nz^{-n}]$$

$$= z^k \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \dots - \frac{u_{k-1}}{z^{k-1}} \right]$$

In particular for  $k = 1, 2, 3$

$$Z\{u_{n+1}\} = z[Z\{u_n\} - u_0]$$

$$Z\{u_{n+2}\} = z^2 \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} \right]$$

$$Z\{u_{n+3}\} = z^3 \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right]$$

## 5. Initial Value theorem:

If  $Z\{u_n\} = U(z)$ , then  $u_0 = \lim_{z \rightarrow \infty} U(z)$

$$u_1 = \lim_{z \rightarrow \infty} z[U(z) - u_0]$$

$$u_2 = \lim_{z \rightarrow \infty} z^2 \left[ U(z) - u_0 - \frac{u_1}{z} \right]$$

⋮

Proof: By definition  $U(z) = Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$

$$\Rightarrow U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots \quad \dots \textcircled{5}$$

$$\begin{aligned} \therefore \lim_{z \rightarrow \infty} U(z) &= \lim_{z \rightarrow \infty} \left[ u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots \right] \\ &= u_0 + 0 + 0 + 0 + \dots = u_0 \end{aligned}$$

Again, from  $\textcircled{5}$ , we get

$$U(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots$$

$$\Rightarrow z[U(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \dots$$

$$\Rightarrow \lim_{z \rightarrow \infty} z[U(z) - u_0] = \lim_{z \rightarrow \infty} \left[ u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \dots \right] = u_1$$

$$\text{Similarly, } u_2 = \lim_{z \rightarrow \infty} z^2 \left[ U(z) - u_0 - \frac{u_1}{z} \right]$$

**Note:** Initial value theorem may be used to determine the sequence  $u_n$  from the given function  $U(z)$

## 6. Final Value theorem:

If  $Z\{u_n\} = U(z)$ , then  $\lim_{n \rightarrow \infty} u_n = \lim_{z \rightarrow 1} (z - 1)U(z)$

$$\begin{aligned} \text{Proof: } Z\{u_{n+1} - u_n\} &= \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n} \\ &\Rightarrow Z\{u_{n+1}\} - Z\{u_n\} = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n} \\ &\Rightarrow z[Z\{u_n\} - u_0] - Z\{u_n\} = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n} \end{aligned}$$

By using left shifting property for  $k = 1$

$$\Rightarrow (z - 1)Z\{u_n\} - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n}$$

$$\text{or } (z - 1)U(z) - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)z^{-n} \quad \because Z\{u_n\} = U(z)$$

Taking limits  $z \rightarrow 1$  on both sides

$$\lim_{z \rightarrow 1} (z - 1)U(z) - u_0 = \sum_{n=0}^{\infty} (u_{n+1} - u_n)$$

$$\begin{aligned} \text{or } \lim_{z \rightarrow 1} (z - 1)U(z) - u_0 &= \lim_{n \rightarrow \infty} [(u_1 - u_0) + (u_2 - u_1) + \dots + (u_{n+1} - u_n)] \\ &= \lim_{n \rightarrow \infty} [u_{n+1}] - u_0 \end{aligned}$$

$$\Rightarrow \lim_{z \rightarrow 1} (z - 1)U(z) = u_{\infty}$$

$$\text{or } \lim_{z \rightarrow 1} (z - 1)U(z) = \lim_{n \rightarrow \infty} u_n$$

**Note:** Initial value and final value theorems determine the value of  $u_n$  for  $n = 0$  and for  $n \rightarrow \infty$  from the given function  $U(z)$ .

## 7. Convolution theorem

Convolution of two sequences  $u_n$  and  $v_n$  is defined as  $u_n * v_n = \sum_{m=0}^n u_m v_{n-m}$

Convolution theorem for Z-transforms states that

If  $U(z) = Z\{u_n\}$  and  $V(z) = Z\{v_n\}$ , then  $Z\{u_n * v_n\} = U(z) \cdot V(z)$

$$\begin{aligned} \text{Proof: } U(z) \cdot V(z) &= Z\{u_n\} \cdot Z\{v_n\} \\ &= \left[ \sum_{n=0}^{\infty} u_n z^{-n} \right] \cdot \left[ \sum_{n=0}^{\infty} v_n z^{-n} \right] \end{aligned}$$

$$\begin{aligned}
&= \left[ u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \dots + \frac{u_n}{z^n} + \dots \right] \cdot \left[ v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \dots + \frac{v_n}{z^n} + \dots \right] \\
&= (u_0 v_0) + (u_0 v_1 + u_1 v_0) z^{-1} + (u_0 v_2 + u_1 v_1 + u_2 v_0) z^{-2} + \dots \\
&= \sum_{n=0}^{\infty} (u_0 v_n + u_1 v_{n-1} + u_2 v_{n-2} + \dots + u_n v_0) z^{-n} \\
&= \sum_{n=0}^{\infty} (\sum_{m=0}^n u_m v_{n-m}) z^{-n} \\
\Rightarrow U(z) \cdot V(z) &= Z\{\sum_{m=0}^n u_m v_{n-m}\} \quad \because \sum_{n=0}^{\infty} u_n z^{-n} = Z\{u_n\} \\
\Rightarrow U(z) \cdot V(z) &= Z\{u_n * v_n\} \quad \because u_n * v_n = \sum_{m=0}^n u_m v_{n-m}
\end{aligned}$$

**Example1** Find the Z–transform of  $2n + 3 \sin \frac{n\pi}{4} - 5a^4$

**Solution:** By linearity property

$$\begin{aligned}
Z\left\{2n + 3 \sin \frac{n\pi}{4} - 5a^4\right\} &= 2Z\{n\} + 3Z\left\{\sin \frac{n\pi}{4}\right\} - 5Z\{a^4\} \\
&= 2Z\{n\} + 3Z\left\{\sin \frac{n\pi}{4}\right\} - 5a^4 Z\{1\} \\
&= \frac{2z}{(z-1)^2} + \frac{3z \sin \frac{\pi}{4}}{z^2 - 2z \cos \frac{\pi}{4} + 1} - \frac{5a^4 z}{z-1} \\
\because Z\{n\} &= \frac{z}{(z-1)^2}, Z\{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}, Z\{1\} = \frac{z}{z-1} \\
\therefore Z\left\{2n + 3 \sin \frac{n\pi}{4} - 5a^4\right\} &= \frac{2z}{(z-1)^2} + \frac{\frac{3z}{\sqrt{2}}}{z^2 - \sqrt{2}z + 1} - \frac{5a^4 z}{z-1}
\end{aligned}$$

**Example2** Find the Z–transform of the sequence  $\{4, 8, 16, 32, \dots\}$

**Solution:**  $u_n = 2^{n+2}, n = 0, 1, 2, 3, \dots$

$$\begin{aligned}
Z\{2^{n+2}\} &= Z\{2^2 2^n\} \\
&= 4 Z\{2^n\} = \frac{4z}{z-2}, \left|\frac{2}{z}\right| < 1 \because Z\{a^n\} = \frac{z}{z-a}, \left|\frac{a}{z}\right| < 1
\end{aligned}$$

**Example3** Find the Z–transform of  $(n + 1)^2$

$$\begin{aligned}
\text{Solution: } Z\{(n + 1)^2\} &= Z\{n^2 + 2n + 1\} \\
&= Z\{n^2\} + 2Z\{n\} + Z\{1\} \\
&= \frac{z^2 + z}{(z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1} \\
\because Z\{n^2\} &= \frac{z^2 + z}{(z-1)^3}, Z\{n\} = \frac{z}{(z-1)^2}, Z\{1\} = \frac{z}{z-1}
\end{aligned}$$

**Example4** Find the Z–transform of

$$\text{i. } n \cos n\theta \quad \text{ii. } \sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) \quad \text{iii. } e^{-2n} \quad \text{iv. } a^{|n|}$$

$$\begin{aligned}
\text{Solution: i. } Z\{\cos n\theta\} &= \frac{z(z - \cos\theta)}{z^2 - 2z \cos\theta + 1} \\
\therefore Z\{n \cos n\theta\} &= -Z \frac{d}{dz} \left[ \frac{z(z - \cos\theta)}{z^2 - 2z \cos\theta + 1} \right] \\
&\because Z\{nu_n\} = -z \frac{d}{dz} Z\{u_n\} \\
&= -Z \left[ \frac{-z^2 \cos\theta + 2z - \cos\theta}{(z^2 - 2z \cos\theta + 1)^2} \right] \\
&= \frac{z^3 \cos\theta - 2z^2 + z \cos\theta}{(z^2 - 2z \cos\theta + 1)^2}
\end{aligned}$$

$$\begin{aligned}
\text{ii. } Z\left\{\sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right\} &= Z\left\{\sin \frac{n\pi}{2} \cos \frac{\pi}{4} + \cos \frac{n\pi}{2} \sin \frac{\pi}{4}\right\} \\
&= \cos \frac{\pi}{4} Z\left\{\sin \frac{n\pi}{2}\right\} + \sin \frac{\pi}{4} Z\left\{\cos \frac{n\pi}{2}\right\}
\end{aligned}$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{z \sin \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} + \frac{z \left( \frac{z - \cos \frac{\pi}{2}}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right)}{z^2 - 2z \cos \frac{\pi}{2} + 1} \right]$$

$$\therefore Z\{\sin n\theta\} = \frac{z \sin \theta}{z^2 - 2z \cos \theta + 1}, Z\{\cos n\theta\} = \frac{z(z - \cos \theta)}{z^2 - 2z \cos \theta + 1}$$

$$= \frac{1}{\sqrt{2}} \left[ \frac{z}{z^2 + 1} + \frac{z^2}{z^2 + 1} \right] = \frac{1}{\sqrt{2}} \left[ \frac{z + z^2}{z^2 + 1} \right]$$

iii.  $Z\{e^{-2n}\} = Z\{(e^{-2})^n\}$

$$= \frac{z}{z - e^{-2}} \quad \therefore Z\{a^n\} = \frac{z}{z - a}$$

$$= \frac{ze^2}{ze^2 - 1}$$

iv.  $a^{|n|} = \begin{cases} a^{-n}, & n < 0 \\ a^n, & n \geq 0 \end{cases}$

Taking two sided Z-transform:  $Z\{a^{|n|}\} = \sum_{n=-\infty}^{\infty} a^{|n|} z^{-n}$

$$\therefore Z\{a^{|n|}\} = \sum_{-\infty}^{-1} a^{-n} z^{-n} + \sum_0^{\infty} a^n z^{-n}$$

$$= [\dots + a^3 z^3 + a^2 z^2 + az] + \left[ 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \dots \right]$$

$$= \frac{az}{1 - az} + \frac{1}{1 - \frac{a}{z}}, \quad |az| < 1 \text{ and } \left| \frac{a}{z} \right| < 1$$

$$= \frac{az}{1 - az} + \frac{z}{z - a}, \quad |z| < \frac{1}{|a|} \text{ and } |a| < |z|$$

$$= \frac{z(1 - az)}{(1 - az)(z - a)}, \quad |a| < |z| < \frac{1}{|a|}$$

$$\Rightarrow Z\{a^{|n|}\} = \frac{z}{z - a}, \quad |a| < |z| < \frac{1}{|a|}$$

**Example5** Find the Z-transform of  $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$

Taking two sided Z-transform:  $Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$

$$\therefore Z\{u_n\} = \sum_{-\infty}^{-1} 2^n z^{-n} + \sum_0^{\infty} 3^n z^{-n}$$

$$= \left[ \dots + \frac{z^3}{2^3} + \frac{z^2}{2^2} + \frac{z}{2} \right] + \left[ 1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \dots \right]$$

$$= \frac{\frac{z}{2}}{1 - \frac{z}{2}} + \frac{1}{1 - \frac{3}{z}}, \quad \left| \frac{z}{2} \right| < 1 \text{ and } \left| \frac{3}{z} \right| < 1$$

$$= \frac{z}{2 - z} + \frac{z}{z - 3}, \quad |z| < |2| \text{ and } |3| < |z|$$

$$= \frac{z(z - 3) + z(2 - z)}{(2 - z)(z - 3)}, \quad |3| < |z| < |2|$$

$$\Rightarrow Z\{u_n\} = \frac{z}{z^2 - 5z + 6}, \quad 3 < |z| < 2$$

$\therefore$  Z-transform does not exist for  $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$  as the set  $3 < |z| < 2$  is infeasible.

**Example6** Find the Z-transform of

i.  ${}^n C_r$   $0 \leq r \leq n$     ii.  ${}^{n+r} C_r$     iii.  $\frac{1}{(n+r)!}$     iv.  $\frac{1}{(n-r)!}$

**Solution:** i.  $Z\{{}^n C_r\} = \sum_{r=0}^n {}^n C_r z^{-r}$

$$= 1 + {}^n C_1 z^{-1} + {}^n C_2 z^{-2} + \dots + {}^n C_n z^{-n}$$

$$= (1 + z^{-1})^n$$

ii.  $Z\{{}^{n+r} C_r\} = \sum_{r=0}^{\infty} {}^{n+r} C_r z^{-r}$

$$= 1 + {}^{n+1} C_1 z^{-1} + {}^{n+2} C_2 z^{-2} + {}^{n+3} C_3 z^{-3} + \dots$$

$$= 1 + (n + 1) z^{-1} + \frac{(n+2)(n+1)}{2!} z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!} z^{-3} + \dots$$

$$\begin{aligned}
&= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!} (-z^{-1})^{-2} \\
&\quad + \frac{(-n-1)(-n-2)(-n-3)}{3!} (-z^{-1})^{-3} + \dots \\
&= (1 - z^{-1})^{-n-1}
\end{aligned}$$

**Example 7** Find the Z-transform of

i.  $\frac{1}{n!}$     ii.  $\frac{1}{(n+r)!}$     iii.  $\frac{1}{(n-r)!}$

i.  $Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \dots$   
 $= 1 + \frac{1}{1!} \frac{1}{z} + \frac{1}{2!} \frac{1}{z^2} + \frac{1}{3!} \frac{1}{z^3} + \dots = e^{\frac{1}{z}}$

$$\Rightarrow Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$$

ii.  $Z\left\{\frac{1}{(n+r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+r)!} z^{-n}$

Now  $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from left shifting property  $Z\{u_{n+k}\} = z^k \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \dots - \frac{u_{k-1}}{z^{k-1}} \right]$

$$\therefore Z\left\{\frac{1}{(n+r)!}\right\} = z^r \left[ e^{\frac{1}{z}} - 1 - \frac{1}{z} - \frac{1}{2!z^2} - \dots - \frac{1}{(r-1)!z^{r-1}} \right]$$

In particular  $\left\{\frac{1}{(n+1)!}\right\} = z^1 \left[ e^{\frac{1}{z}} - 1 \right]$

$$\left\{\frac{1}{(n+2)!}\right\} = z^2 \left[ e^{\frac{1}{z}} - 1 - \frac{1}{z} \right]$$

⋮

iii.  $Z\left\{\frac{1}{(n-r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n-r)!} z^{-n}$

Now  $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from right shifting property,  $Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$ ,  $k$  is positive integer

$$\therefore Z\left\{\frac{1}{(n-r)!}\right\} = z^{-r} e^{\frac{1}{z}}$$

In particular  $\left\{\frac{1}{(n-1)!}\right\} = z^{-1} e^{\frac{1}{z}}$

$$\left\{\frac{1}{(n+2)!}\right\} = z^{-2} e^{\frac{1}{z}}$$

⋮

**Example 8** Find  $Z\{u_{n+2}\}$  if  $Z\{u_n\} = \frac{z}{z-1} + \frac{z}{z^2+1}$

**Solution:** Given  $Z\{u_n\} = U(z) = \frac{z}{z-1} + \frac{z}{z^2+1}$

From left shifting property  $Z\{u_{n+2}\} = z^2 \left[ Z\{u_n\} - u_0 - \frac{u_1}{z} \right]$  ...①

Now from initial value theorem  $u_0 = \lim_{z \rightarrow \infty} U(z)$

$$= \lim_{z \rightarrow \infty} \left[ \frac{z}{z-1} + \frac{z}{z^2+1} \right]$$



$$= \lim_{z \rightarrow \infty} \left[ \frac{1}{1 - \frac{1}{z}} + \frac{\frac{1}{z}}{1 + \frac{1}{z^2}} \right] = 1 + 0$$

$$\therefore u_0 = 1 \quad \dots \textcircled{2}$$

Also from initial value theorem  $u_1 = \lim_{z \rightarrow \infty} z[U(z) - u_0]$

$$= \lim_{z \rightarrow \infty} z \left[ \frac{z}{z-1} + \frac{z}{z^2+1} - 1 \right]$$

$$= \lim_{z \rightarrow \infty} z \left[ \frac{2z^2 - z + 1}{(z^2+1)(z-1)} \right] = 2$$

$$\therefore u_1 = 2 \quad \dots \textcircled{3}$$

Using  $\textcircled{2}$  and  $\textcircled{3}$  in  $\textcircled{1}$ , we get  $Z\{u_{n+2}\} = z^2 \left[ \frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z} \right]$

$$\Rightarrow Z\{u_{n+2}\} = \frac{z(z^2 - z + 2)}{(z-1)(z^2+1)}$$

**Example9** Verify convolution theorem for  $u_n = n$  and  $v_n = 1$

**Solution:** Convolution theorem states that  $Z\{u_n * v_n\} = U(z).V(z)$

We know that  $u_n * v_n = \sum_{m=0}^n u_m v_{n-m}$

$$= \sum_{m=0}^n m \cdot 1$$

$$= 0 + 1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

$$\Rightarrow Z\{u_n * v_n\} = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} z^{-n} = \frac{1}{2} [\sum_{n=0}^{\infty} n^2 z^{-n} + \sum_{n=0}^{\infty} n z^{-n}]$$

$$= \frac{1}{2} [Z\{n^2\} + Z\{n\}]$$

$$= \frac{1}{2} \left[ \frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2} \right] = \frac{1}{2} \left[ \frac{z(z+1) + z(z-1)}{(z-1)^3} \right] = \frac{1}{2} \left[ \frac{2z^2}{(z-1)^3} \right]$$

$$\therefore Z\{u_n * v_n\} = \frac{z^2}{(z-1)^3} \quad \dots \textcircled{1}$$

$$\text{Also } U(z) = Z\{n\} = \frac{z}{(z-1)^2} \quad \text{and } V(z) = Z\{1\} = \frac{z}{z-1}$$

$$\Rightarrow U(z).V(z) = \frac{z^2}{(z-1)^3} \quad \dots \textcircled{2}$$

From  $\textcircled{1}$  and  $\textcircled{2}$   $Z\{u_n * v_n\} = U(z).V(z)$

**Example10** If  $u_n = \delta(n) - \delta(n-1)$ ,  $v_n = 2\delta(n) + \delta(n-1)$ , Find the Z-transform of their convolution.

**Solution:**  $U(z) = Z\{\delta(n) - \delta(n-1)\}$ ,  $V(z) = Z\{2\delta(n) + \delta(n-1)\}$

$$\text{Now } Z\{\delta(n)\} = 1 \text{ and } Z\{\delta(n-1)\} = z^{-1} \quad \therefore Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$$

$$\therefore U(z) = 1 - z^{-1} \text{ and } V(z) = 2 + z^{-1}$$

Also  $Z\{u_n * v_n\} = U(z).V(z)$

$$\Rightarrow Z\{u_n * v_n\} = (1 - z^{-1})(2 + z^{-1}) = 2 - z^{-1} - z^{-2}$$

### 4.3 Inverse Z-Transform

Given a function  $U(z)$ , we can find the sequence  $u_n$  by one of the following methods

- Inspection (Direct inversion)
- Direct division
- Partial fractions

- Residues (Inverse integral)
- Power-Series
- Convolution theorem

### 4.3.1 Inspection (Direct inversion) method

Sometimes by observing the coefficients in the given series  $U(z)$ , it is possible to find the sequence  $u_n$  as illustrated in the given examples.

**Example 11** Find  $u_n$  if  $U(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \dots$

**Solution:** Given that  $U(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$  ... ①

Also, by the definition of Z-transform  $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$  ... ②

Comparing ① and ②, we get  $u_n = \left(\frac{1}{2}\right)^n$

**Example 12** Find  $u_n$  if  $U(z) = \frac{z^3}{(z-1)^3}$

**Solution:**  $U(z) = \frac{z^3}{(z-1)^3} = \left(\frac{z-1}{z}\right)^{-3} = \left(1 - \frac{1}{z}\right)^{-3}$

$$\therefore U(z) = 1 + \frac{3}{z} + \frac{6}{z^2} + \frac{10}{z^3} + \dots$$

$$\therefore (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \dots$$

$$\Rightarrow U(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^{-n}$$

Comparing with  $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ , we get  $u_n = \frac{(n+1)(n+2)}{2}$

**Example 13** Find inverse Z-transform of  $3 + \frac{2z}{z-1} - \frac{z}{2z-1}$

**Solution:** Given that  $U(z) = 3 + \frac{2z}{z-1} - \frac{z}{2z-1}$

$$\therefore u_n = 3Z^{-1}[1] + 2Z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = 3\delta(n) + 2u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n = 3\delta(n) + 2u(n) - \left(\frac{1}{2}\right)^{n+1}$$

$$\therefore Z^{-1}[1] = \delta(n), Z^{-1}\left[\frac{z}{z-1}\right] = u(n), Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

where  $\delta(n)$  and  $u(n)$  are unit impulse and unit step sequences respectively.

**Example 14** Find inverse Z-transform of  $2z^{-2} - \frac{z^{-3}}{z-1}$

**Solution:** Given that  $U(z) = 2z^{-2} - \frac{z^{-3}}{z-1}$

$$\therefore u_n = 2Z^{-1}[z^{-2} \cdot 1] - Z^{-1}\left[z^{-4} \cdot \frac{z}{z-1}\right]$$

$$\Rightarrow u_n = 2\delta(n-2) - u(n-4)$$

$$\therefore \text{From Right shifting property } Z^{-1}[z^{-k}U(z)] = u_{n-k}$$

### 4.3.2 Direct division method

Direct division is one of the simplest methods for finding inverse Z-transform and can be used for almost every type of expression given in fractional form.

**Example 15** Find the inverse Z-transform of  $\frac{z}{z^2-3z+2}$

**Solution:** Given that  $U(z) = \frac{z}{z^2-3z+2}$

By actual division, we get

$$\begin{array}{r}
 z^2 - 3z + 2 \overline{) \begin{array}{l} z^{-1} + 3z^{-2} + 7z^{-3} \\ z - 3 + 2z^{-1} \\ \hline 3 - 2z^{-1} \\ 3 - 9z^{-1} + 6z^{-2} \\ \hline 7z^{-1} - 6z^{-2} \\ 7z^{-1} - 21z^{-2} + 14z^{-3} \\ \hline 15z^{-2} - 14z^{-3} \\ \vdots \end{array} \\
 \Rightarrow U(z) = z^{-1} + 3z^{-2} + 7z^{-3} + \dots \\
 = \sum_{n=0}^{\infty} (2^n - 1)z^{-n} \\
 \therefore u_n = 2^n - 1
 \end{array}$$

**Example16** Find the inverse  $Z$  -transform of  $\frac{4z^2+2z}{2z^2-3z+1}$

**Solution:** Given that  $U(z) = \frac{4z^2+2z}{2z^2-3z+1}$ , by actual division, we get

$$\begin{array}{r}
 2z^2 - 3z + 1 \overline{) \begin{array}{l} 2 + 4z^{-1} + 5z^{-2} \\ 4z^2 + 2z \\ \hline 4z^2 - 6z + 2 \\ \hline 8z - 2 \\ 8z - 12 + 4z^{-1} \\ \hline 10 - 4z^{-1} \\ 10 - 15z^{-1} + 5z^{-2} \\ \hline 11z^{-1} - 5z^{-2} \\ \vdots \end{array} \\
 \Rightarrow U(z) = 2 + 4z^{-1} + 5z^{-2} + \dots \\
 = \sum_{n=0}^{\infty} (6 - 2^{2-n})z^{-n} \\
 \therefore u_n = 6 - 2^{2-n} \\
 \therefore u_n = 6 - 4\left(\frac{1}{2}\right)^n
 \end{array}$$

### 4.3.3 Partial fractions method

Partial fractions method can be used only if order of expression in the numerator is less than or equal to that in the denominator. If order of expression in the numerator is greater, then the fraction may be brought to desired form by direct division. Partial fractions are formed of the expression  $\frac{U(z)}{z}$  as demonstrated in the examples below.

**Example17** Find the inverse  $Z$  -transform of  $\frac{z}{6z^2-5z+1}$

**Solution:** Given that  $U(z) = \frac{z}{6z^2-5z+1}$

$$\begin{aligned}
 \therefore \frac{U(z)}{z} &= \frac{1}{6z^2-5z+1} = \frac{1}{(2z-1)(3z-1)} \\
 \Rightarrow \frac{U(z)}{z} &= \frac{2}{2z-1} - \frac{3}{3z-1} \\
 \Rightarrow U(z) &= \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}} \\
 \therefore u_n &= \left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n \quad \because Z\{a^n\} = \frac{z}{z-a} \text{ or } Z^{-1}\left[\frac{z}{z-a}\right] = a^n \\
 \text{i.e. } u_n &= 2^{-n} - 3^{-n}
 \end{aligned}$$

**Example18** Find the inverse  $Z$  -transform of  $\frac{4z^2+2z}{2z^2-3z+1}$

**Solution:** Given that  $U(z) = \frac{4z^2-2z}{2z^2-3z+1} = \frac{2z(2z-1)}{(2z-1)(z-1)}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z+1)}{(2z-1)(z-1)}$$

By partial fractions, we get

$$\begin{aligned} \frac{U(z)}{z} &= \frac{-8}{2z-1} + \frac{6}{z-1} \\ \Rightarrow U(z) &= \frac{-8z}{2z-1} + \frac{6z}{z-1} \end{aligned}$$

$$\therefore u_n = -4Z^{-1} \left[ \frac{z}{z-\frac{1}{2}} \right] + 6Z^{-1} \left[ \frac{z}{z-1} \right]$$

$$\Rightarrow u_n = -4 \left( \frac{1}{2} \right)^n + 6(1)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n$$

$$\text{i.e. } u_n = -4 \left( \frac{1}{2} \right)^n + 6$$

**Example19** Find the inverse  $Z$  -transform of  $\frac{1}{(1-z^{-1})(2-z^{-1})}$

**Solution:** Given that  $U(z) = \frac{1}{(1-z^{-1})(2-z^{-1})}$

Multiplying and dividing by  $z^2$ , we get

$$U(z) = \frac{z^2}{z(1-z^{-1})z(2-z^{-1})} = \frac{z^2}{(z-1)(2z-1)}$$

$$\therefore \frac{U(z)}{z} = \frac{z}{(z-1)(2z-1)}$$

By partial fractions, we get

$$\begin{aligned} \frac{U(z)}{z} &= \frac{1}{(z-1)} - \frac{1}{(2z-1)} \\ \Rightarrow U(z) &= \frac{z}{(z-1)} - \frac{z}{(2z-1)} \end{aligned}$$

$$\therefore u_n = Z^{-1} \left[ \frac{z}{(z-1)} \right] - \frac{1}{2} Z^{-1} \left[ \frac{z}{z-\frac{1}{2}} \right]$$

$$\Rightarrow u_n = (1)^n - \frac{1}{2} \left( \frac{1}{2} \right)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n$$

$$\text{i.e. } u_n = 1 - \left( \frac{1}{2} \right)^{n+1}$$

**Example20** Find the inverse  $Z$  -transform of  $\frac{4z^2-2z}{z^3-5z^2+8z-4}$

**Solution:** Given that  $U(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4} = \frac{2z(2z-1)}{(z-1)(z-2)^2}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z-1)}{(z-1)(z-2)^2}$$

By partial fractions, we get

$$\begin{aligned} \frac{U(z)}{z} &= \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2} \\ \Rightarrow U(z) &= \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{6z}{(z-2)^2} \end{aligned}$$

$$\begin{aligned} \therefore u_n &= 2Z^{-1} \left[ \frac{z}{z-1} \right] - 2Z^{-1} \left[ \frac{z}{z-2} \right] + 3Z^{-1} \left[ \frac{2z}{(z-2)^2} \right] \\ \Rightarrow u_n &= 2(1)^n - 2(2)^n + 3n(2)^n \quad \because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n \text{ and } Z^{-1} \left[ \frac{az}{(z-a)^2} \right] = na^n \\ \text{i.e. } u_n &= 2 - 2^{n+1} + 3n \cdot 2^n \end{aligned}$$

#### 4.3.4 Method of residues (Inverse integral)

By using the theory of complex variables, it can be shown that the inverse Z-transforms is given by  $u_n = \frac{1}{2\pi i} \oint_C U(z)z^{n-1} dz = \text{sum of residues of } U(z)$  where  $c$  is the closed contour which contains all the insulated singularities of  $U(z)$  in the region of convergence.

Method of residues is one of the most efficient methods and can be used to find the inverse Z-transform where partial fractions are tedious to find.

**Example21** Find the inverse z-transform of  $\frac{z}{z^2+7z+10}$

**Solution:**  $U(z) = \frac{z}{z^2+7z+10}$

$$\begin{aligned} \text{Now } u_n &= \frac{1}{2\pi i} \oint_C U(z)z^{n-1} dz \\ \Rightarrow u_n &= \frac{1}{2\pi i} \oint_C \frac{z}{z^2+7z+10} z^{n-1} dz \\ &= \frac{1}{2\pi i} \oint_C \frac{z^n}{z^2+7z+10} dz \\ &= \frac{1}{2\pi i} \oint_C \frac{z^n}{(z+2)(z+5)} dz \end{aligned}$$

There are two simple poles at  $z = -2$  and  $z = -5$

$$\text{Residue at } z = -2 \text{ is given by } \lim_{z \rightarrow -2} (z+2) \frac{z^n}{(z+2)(z+5)} = \frac{(-2)^n}{3}$$

$$\text{Residue at } z = -5 \text{ is given by } \lim_{z \rightarrow -5} (z+5) \frac{z^n}{(z+2)(z+5)} = \frac{(-5)^n}{-3}$$

$$\therefore u_n = \text{sum of residues} = \frac{(-2)^n}{3} + \frac{(-5)^n}{-3} = \frac{1}{3} \{(-2)^n - (-5)^n\}$$

**Example22** Find the inverse z-transform of  $\frac{z^2+z}{(z-1)(z^2+1)}$

**Solution:**  $U(z) = \frac{z^2+z}{(z-1)(z^2+1)}$

$$\begin{aligned} \text{Now } u_n &= \frac{1}{2\pi i} \oint_C U(z)z^{n-1} dz \\ \Rightarrow u_n &= \frac{1}{2\pi i} \oint_C \frac{z^2+z}{(z-1)(z^2+1)} z^{n-1} dz \\ &= \frac{1}{2\pi i} \oint_C \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} dz \end{aligned}$$

There are three simple poles at  $z = 1$ ,  $z = -i$  and  $z = i$

$$\text{Residue at } z = 1 \text{ is given by } \lim_{z \rightarrow 1} (z-1) \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = 1$$

$$\text{Residue at } z = -i \text{ is given by } \lim_{z \rightarrow -i} (z+i) \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}(-i)^n$$

$$\text{Residue at } z = i \text{ is given by } \lim_{z \rightarrow i} (z-i) \frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}i^n$$

$$\therefore u_n = \text{sum of residues} = 1 - \frac{1}{2}(-i)^n - \frac{1}{2}i^n = 1 - \frac{1}{2} \{(-i)^n + i^n\}$$

**Example23** Find the inverse z-transform of  $\frac{z(z+1)}{(z-1)^3}$

**Solution:**  $U(z) = \frac{z(z+1)}{(z-1)^3}$

Now  $u_n = \frac{1}{2\pi i} \oint_C U(z)z^{n-1} dz$   
 $\Rightarrow u_n = \frac{1}{2\pi i} \oint_C \frac{z^n(z+1)}{(z-1)^3} dz$

Here  $z = 1$  is a pole of order 3

Residue at  $z = 1$  is given by  $\frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} \left[ \frac{(z-1)^3 z^n (z+1)}{(z-1)^3} \right]$   
 $= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d^2}{dz^2} [z^n(z+1)]$   
 $= \frac{1}{2!} \lim_{z \rightarrow 1} \frac{d}{dz} [(n+1)z^n + nz^{n-1}]$   
 $= \frac{1}{2!} \lim_{z \rightarrow 1} [(n+1)nz^{n-1} + n(n-1)z^{n-2}]$   
 $= [n^2 + n + n^2 - n] = n^2$   
 $\therefore u_n = \text{Sum of residues} = n^2$

### 4.3.5 Power series method

In this method, we find the inverse  $Z$  - transform by expanding  $U(z)$  in power series.

**Example 24** Find  $u_n$  if  $U(z) = \log \frac{z}{z+1}$

**Solution:** Given  $U(z) = \log \frac{z}{z+1} = \log \left( \frac{z+1}{z} \right)^{-1} = -\log \frac{z+1}{z} = -\log \left( 1 + \frac{1}{z} \right)$

$\therefore U(z) = -\log(1 + y)$       Putting  $\frac{1}{z} = y$   
 $= -y + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \dots$   
 $\because \log(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$   
 $\Rightarrow U(z) = -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \dots$        $\because y = \frac{1}{z}$

$\Rightarrow U(z) = 0 + \sum_{n=1}^{\infty} \frac{(-1)^n}{n} z^{-n}$

Comparing with  $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ , we get

$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & \text{otherwise} \end{cases}$$

### 4.3.6 Convolution theorem method

Convolution theorem for  $Z$ -transforms states that:

If  $U(z) = Z\{u_n\}$  and  $V(z) = Z\{v_n\}$ , then  $Z\{u_n * v_n\} = U(z).V(z)$   
 $\Rightarrow Z^{-1}[U(z).V(z)] = u_n * v_n$

**Example 25** Find the inverse  $z$ -transform of  $\frac{z^2}{(z-1)(2z-1)}$  using convolution theorem.

**Solution:** Let  $U(z) = Z\{u_n\} = \frac{z}{(z-1)}$  and  $V(z) = Z\{v_n\} = \frac{z}{(2z-1)} = \frac{1}{2} \left( \frac{z}{z-\frac{1}{2}} \right)$

Clearly  $u_n = (1)^n$  and  $v_n = \frac{1}{2} \left( \frac{1}{2} \right)^n$        $\because Z^{-1} \left[ \frac{z}{z-a} \right] = a^n$

Now by convolution theorem  $Z^{-1}[U(z).V(z)] = u_n * v_n$

$\Rightarrow Z^{-1} \left[ \frac{z^2}{(z-1)(2z-1)} \right] = (1)^n * \left( \frac{1}{2} \right)^{n+1}$

$$\begin{aligned}
\text{We know that } u_n * v_n &= \sum_{m=0}^n u_m v_{n-m} \\
&= \sum_{m=0}^n (1)^m \left(\frac{1}{2}\right)^{n+1-m} \\
&= \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^n + \left(\frac{1}{2}\right)^{n-1} + \dots + \frac{1}{2} \\
&= \frac{1}{2} \left[ 1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \dots + \left(\frac{1}{2}\right)^n \right] \\
&= \frac{1}{2} \left[ \frac{1}{1-\frac{1}{2}} \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) \right] \\
&\qquad\qquad\qquad \because S_n = \frac{a}{1-r} (1 - r^n) \\
&= \frac{1}{2} \left[ 2 \left( 1 - \left(\frac{1}{2}\right)^{n+1} \right) \right] \\
&= 1 - \left(\frac{1}{2}\right)^{n+1}
\end{aligned}$$

#### 4.4 Applications of Z-Transforms in Solving Difference Equations

Difference equations may be solved using usual methods of calculating Complimentary Function (C.F.) and Particular Integral. Z- Transforms can also be used to solve a difference equation particularly when the particular integral is difficult to evaluate.

##### Algorithm to solve a difference equation using Z- Transforms

1. Apply Z-transform on both sides of the given difference equation.
2. Use left shifting property to expand  $Z\{u_{n+1}\}$ ,  $Z\{u_{n+2}\}$ ,  $Z\{u_{n+3}\}$  etc
3. Keeping  $\frac{U(z)}{z}$  on one side, resolve the terms on other side into partial fractions.
4. Find  $u_n$  by applying inverse Z-Transform using partial fractions or any other suitable method.

**Example 26** Using Z-transforms, solve the difference equation  $y_{n+2} - 4y_{n+1} + 4y_n = 0$ ,  
given that  $y_0 = 1$  and  $y_1 = 0$

**Solution:** Taking Z-transforms on both sides such that  $Z\{y_n\}$  is denoted by  $Y(z)$

$$Z\{y_{n+2}\} - 4Z\{y_{n+1}\} + 4Z\{y_n\} = 0 \quad \dots \textcircled{1}$$

Using left shifting property

$$Z\{y_{n+1}\} = z[Y(z) - y_0] \quad \dots \textcircled{2}$$

$$Z\{y_{n+2}\} = z^2 \left[ Y(z) - y_0 - \frac{y_1}{z} \right] \quad \dots \textcircled{3}$$

Using and  $\textcircled{2}$  and  $\textcircled{3}$  in  $\textcircled{1}$

$$z^2 \left[ Y(z) - y_0 - \frac{y_1}{z} \right] - 4z[Y(z) - y_0] + 4Y(z) = 0$$

$$\Rightarrow Y(z)[z^2 - 4z + 4] - y_0[z^2 - 4z] - y_1 z = 0$$

$$\Rightarrow Y(z)[z^2 - 4z + 4] - [z^2 - 4z] = 0 \quad \because y_0 = 1 \text{ and } y_1 = 0$$

$$\Rightarrow Y(z)(z - 2)^2 = z(z - 4)$$

$$\Rightarrow \frac{Y(z)}{z} = \frac{(z-4)}{(z-2)^2}$$

Resolving into Partial fractions, we get

$$\Rightarrow \frac{Y(z)}{z} = \frac{(z-2)}{(z-2)^2} - \frac{2}{(z-2)^2} = \frac{1}{(z-2)} - \frac{2}{(z-2)^2}$$

$$Y(z) = \frac{z}{(z-2)} - \frac{2z}{(z-2)^2}$$

Taking inverse Z-transform

$$y_n = 2^n - 2^n n \quad \because Z^{-1}\left(\frac{z}{z-a}\right) = a^n \text{ and } Z^{-1}\left(\frac{az}{(z-a)^2}\right) = a^n n$$

$$\Rightarrow y_n = 2^n(1 - n)$$

**Example 27** Using Z-transforms, solve the difference equation  $u_{n+2} - 4u_{n+1} + 3u_n = 5^n$

**Solution:** Taking Z-transforms on both sides such that  $Z\{u_n\}$  is denoted by  $U(z)$

$$Z\{u_{n+2}\} - 4Z\{u_{n+1}\} + 3Z\{u_n\} = Z\{5^n\} \quad \dots \textcircled{1}$$

Using left shifting property

$$Z\{u_{n+1}\} = z[U(z) - u_0] \quad \dots \textcircled{2}$$

$$Z\{u_{n+2}\} = z^2 \left[ U(z) - u_0 - \frac{u_1}{z} \right] \quad \dots \textcircled{3}$$

Using and  $\textcircled{2}$  and  $\textcircled{3}$  in  $\textcircled{1}$

$$z^2 \left[ U(z) - u_0 - \frac{u_1}{z} \right] - 4z[U(z) - u_0] + 3U(z) = \frac{z}{z-5}$$

$$\Rightarrow U(z)[z^2 - 4z + 3] - u_0[z^2 - 4z] - u_1 z = \frac{z}{z-5}$$

$$\Rightarrow U(z)(z-1)(z-3) = u_0 z(z-4) + u_1 z + \frac{z}{z-5}$$

$$\Rightarrow \frac{U(z)}{z} = \frac{u_0(z-4)+u_1}{(z-1)(z-3)} + \frac{1}{(z-1)(z-3)(z-5)}$$

Resolving into Partial fractions, we get

$$\Rightarrow \frac{U(z)}{z} = \left[ \frac{A}{z-1} + \frac{B}{z-3} \right] + \left[ \frac{C}{z-1} + \frac{D}{z-3} + \frac{E}{z-5} \right] \quad \dots \textcircled{4}$$

$$\text{Here } \frac{u_0(z-4)+u_1}{(z-1)(z-3)} = \left[ \frac{A}{z-1} + \frac{B}{z-3} \right]$$

$$\Rightarrow u_0(z-4) + u_1 = A(z-3) + B(z-1)$$

Putting  $z = 1$  and  $z = 3$  respectively on both sides, we get

$$-3u_0 + u_1 = -2A \text{ and } -u_0 + u_1 = 2B$$

$$\Rightarrow A = \frac{1}{2}(3u_0 - u_1) \text{ and } B = \frac{1}{2}(-u_0 + u_1) \quad \dots \textcircled{5}$$

$$\text{Also } \frac{1}{(z-1)(z-3)(z-5)} = \frac{C}{z-1} + \frac{D}{z-3} + \frac{E}{z-5}$$

$$\Rightarrow 1 = C(z-3)(z-5) + D(z-1)(z-5) + E(z-1)(z-3)$$

Putting  $z = 1, 3$  and  $5$  respectively, we get

$$\text{we get } C = \frac{1}{8}, D = -\frac{1}{4} \text{ and } E = \frac{1}{8} \quad \dots \textcircled{6}$$

Using  $\textcircled{5}$  and  $\textcircled{6}$  in  $\textcircled{4}$ , we get

$$U(z) = \left[ \frac{(3u_0 - u_1)z}{2(z-1)} + \frac{(-u_0 + u_1)z}{2(z-3)} \right] + \left[ \frac{1}{8} \left( \frac{z}{z-1} \right) - \frac{1}{4} \left( \frac{z}{z-3} \right) + \frac{1}{8} \left( \frac{z}{z-5} \right) \right]$$

Taking inverse Z-transform

$$u_n = \left( \frac{3u_0 - u_1}{2} \right) 1^n + \left( \frac{-u_0 + u_1}{2} \right) 3^n + \frac{1}{8} 1^n - \frac{1}{4} 3^n + \frac{1}{8} 5^n$$



$$\Rightarrow u_n = \left(\frac{3u_0 - u_1}{2} + \frac{1}{8}\right) 1^n + \left(\frac{-u_0 + u_1}{2} - \frac{1}{4}\right) 3^n + \frac{1}{8} 5^n$$

$$\Rightarrow u_n = c_1 + c_2 3^n + \frac{1}{8} 5^n$$

$$\text{where } c_1 = \frac{3u_0 - u_1}{2} + \frac{1}{8}, \quad c_2 = \frac{-u_0 + u_1}{2} - \frac{1}{4}$$

### Exercise 4A

1. Find the Z-transform of  $u_n = \begin{cases} 5^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$
2. Find the Z-transform of  $u_n = \frac{1}{(n-p)!}$
3. Find the inverse Z-transform of  $\frac{2z}{(z-1)(z^2+1)}$
4. Solve the difference equation  $y_{x+2} + 4y_{x+1} + 3y_x = 3^x$ ,  $y_0 = 0, y_1 = 1$  using Z-transforms

### Answers

1.  $\frac{2z}{z^2 - 8z + 15}, \quad 3 < |z| < 5$
2.  $z^{-p} e^{\frac{1}{z}}$
3.  $1 + \frac{(-i)^n}{-1+i} - \frac{i^n}{1+i}$
4.  $y_x = \frac{1}{24} 3^x - \frac{5}{12} (-3)^x + \frac{3}{8} (-1)^x$