Chapter 5

Numerical Solutions of Algebraic and Transcendental Equations

5.1 Introduction

An expression of the form $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$, $a_0 \neq 0$ is called a polynomial of degree 'n'and the polynomial $f(x) = 0$ is called an algebraic equation of n^{th} degree.If $f(x)$ contains trigonometric, logarithmic or exponential functions, then $f(x) = 0$ is called a transcendental equation. For example $x^2 + 2\sin x + e^x = 0$ is a transcendental equation.

If $f(x)$is an algebraic polynomial of degree less than or equal to 4, direct methods for finding the roots of such equation are available. But if $f(x)$is of higher degree or it involves transcendental functions, direct methods do not exist and we need to apply numerical methods to find the roots of the equation $f(x) = 0$.

Some useful results

- If α is root of the equation $f(x) = 0$, then $f(\alpha) = 0$
- Every equation of n^{th}degree has exactly n roots (real or imaginary)
- **Intermediate Value Theorem**: If $f(x)$is a continuous function in a closed interval $[a, b]$and $f(a)$ & $f(b)$ are having opposite signs, then the equation $f(x) = 0$ has atleast one real root or odd number of roots between $a$and b.
- If $f(x)$is a continuous function in the closed interval $[a, b]$and $f(a)$ & $f(b)$ are of same signs, then the equation $f(x) = 0$ has no root or even number of roots between $a$and b.

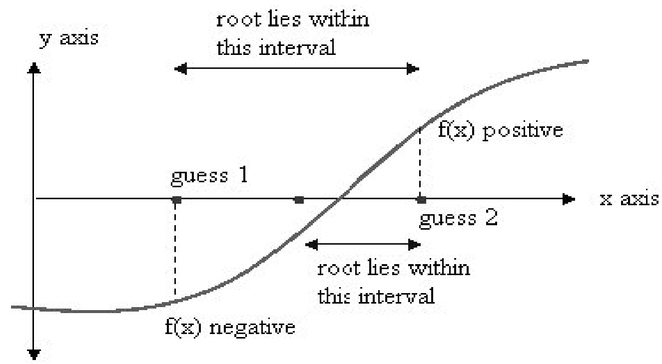
5.2Numerical methods to find roots of algebraic and transcendental equations

Most numerical methods use iterative procedures to find an approximate root of an equation $f(x) = 0$. They require an initial guess of the root as starting value and each subsequent iteration leads closer to the actual root.

Order of convergence: For any iterative numerical method, each successive iteration gives an approximation that moves progressively closer to actual solution. This is known as convergence. Any numerical method is said have order of convergence ρ, if ρ is the largest positive number such that $|\epsilon_{n+1}| \leq k|\epsilon_n|^\rho$, where ϵ_n and ϵ_{n+1} are errors in $n^{th}$and $(n+1)^{th}$ iterations, k is a finite positive constant.

5.2.1 Bisection Method (or Bolzano Method)

Bisection method is used to find an approximate root in an interval by repeatedly bisecting into subintervals. It is a very simple and robust method but it is also relatively slow. Because of this it is often used to obtain a rough approximation to a solution which is then used as a starting point for more rapidly converging methods. This method is based on the intermediate value theorem for continuous functions.



Algorithm:

Let $f(x)$ be a continuous function in the interval $[a, b]$, such that $f(a)$and $f(b)$ are of opposite signs, i.e. $f(a).f(b) < 0$.

Step1. Take the initial approximation given by $x_0 = \frac{a+b}{2}$, one of the three conditions arises for finding the 1st approximation x_1

i. $f(x_0) = 0$, we have a root at x_0.

ii. If $f(a).f(x_0) < 0$, the root lies between a and x_0 $\therefore$ $x_1 = \frac{a+x_0}{2}$ and repeat the procedure by halving the interval again.

iii. If $f(b).f(x_0) < 0$, the root lies between x_0 and b $\therefore$ $x_1 = \frac{x_0+b}{2}$ and repeat the procedure by halving the interval again.

iv. Continue the process until root is found to be of desired accuracy.

Remarks:

- Convergence is not unidirectional as none of the end points is fixed. As a result convergence of Bisection method is very slow.
- Repeating the procedure n times, the new interval will be exactly half the length of the previous one, until the root is found of desired accuracy (error less than ∈). ∴ and at the end ofn^{th} iteration, the interval containing the root will be of length $\frac{|b-a|}{2^n}$, such that $\frac{|b-a|}{2^n} < \in$

  $\Rightarrow \log \frac{|b-a|}{2^n} < \log \in$

  $\Rightarrow \log|b-a| - \log 2^n < \log \in$

  $\Rightarrow \log|b-a| - \log \in < n \log 2$

  $\Rightarrow n > \frac{\log|b-a| - \log \in}{\log 2}$

∴In bisection method, the minimum number of iterations required to achieve the desired accuracy (error less than ∈)are $\frac{\log\frac{|b-a|}{\epsilon}}{\log 2}$.

Example1 Apply bisection method to find a root of the equation $x^4 + 2x^3 - x - 1 = 0$

Solution:$f(x) = x^4 + 2x^3 - x - 1$

Here $f(0) = -1$and $f(1) = 1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$is continuous in $[0,1]$, ∴ atleast one root exists in $[0,1]$

Initial approximation: $a = 0, b = 1$

$x_0 = \frac{0+1}{2} = .5,\ f(0.5) = -1.1875,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$x_1 = \frac{0.5+1}{2} = 0.75,\ f(0.75) = -0.5898,\ f(0.75).f(1) < 0$

Second approximation: $a = 0.75, b = 1$

$x_2 = \frac{0.75+1}{2} = 0.875,\ f(0.875) = 0.051,\ f(0.75).f(0.875) < 0$

Third approximation: $a = 0.75, b = 0.875$

$x_3 = \frac{0.75+0.875}{2} = 0.8125,\ f(0.8125) = -0.30394, f(0.8125).f(0.875) < 0$

Fourth approximation: $a = 0.8125, b = 0.875$

$x_4 = \frac{0.8125+0.875}{2} = 0.84375, f(0.84375) = -0.135, f(0.84375).\ f(0.875) < 0$

Fifth approximation: $a = 0.84375, b = 0.875$

$x_5 = \frac{0.84375+0.875}{2} = 0.8594,\ f(0.8594) = -0.0445, .f(0.8594).f(0.875) < 0$

Sixth approximation: $a = 0.8594, b = 0.875$

$x_6 = \frac{0.8594 + 0.875}{2} = 0.8672, f(0.8672) = 0.0027,\ f(0.8594).f(0.8672). < 0$

Seventh approximation: $a = 0.8594, b = 0.8672$

$x_7 = \frac{0.8594+0.8672}{2} = 0.8633$

First 2 decimal places have been stabilized; hence 0.8633 is the real root correct to two decimal places.

Example2 Apply bisection method to find a root of the equation $x^3 - 2x^2 - 4 = 0$ correct to three decimal places.

Solution:$f(x) = x^3 - 2x^2 - 4$

Here $f(2) = -4$ and $f(3) = 5 \Rightarrow f(2).f(3) < 0$

Also $f(x)$is continuous in [2,3], ∴ atleast one root exists in [2,3]

Initial approximation: $a = 2, b = 3$

$x_0 = \frac{2+3}{2} = 2.5,\ f(2.5) = -1.8750,\ f(2.5).f(3) < 0$

First approximation: $a = 2.5, b = 3$

$x_1 = \frac{2.5+3}{2} = 2.75,\ f(2.75) = 1.6719,\ f(2.5).f(2.75) < 0$

Second approximation: $a = 2.5, b = 2.75$

$x_2 = \frac{2.5+2.75}{2} = 2.625,\ f(2.625) = 0.3066,\ f(2.5).f(2.625) < 0$

Third approximation: $a = 2.5, b = 2.625$

$x_3 = \frac{2.5+2.625}{2} = 2.5625,\ f(2.5625) = -.3640,\ f(2.5625).f(2.625) < 0$

Fourth approximation: $a = 2.5625, b = 2.625$

$x_4 = \frac{2.5625+2.625}{2} = 2.59375, f(2.59375) = -.0055, f(2.59375).f(2.625) < 0$

Fifth approximation: $a = 2.59375, b = 2.625$

$x_5 = \frac{2.59375+2.625}{2} = 2.60938, f(2.60938) = .1488, f(2.59375).f(2.60938) < 0$

Sixth approximation: $a = 2.59375, b = 2.60938$

$x_6 = \frac{2.59375+2.60938}{2} = 2.60157, f(2.60157) = .0719, f(2.59375).f(2.60157) < 0$

Seventh approximation: $a = 2.59375, b = 2.60157$

$x_7 = \frac{2.59375+2.60157}{2} = 2.59765, f(2.59765) = .0329, f(2.59375).f(2.59765) < 0$

Eighth approximation: $a = 2.59375, b = 2.59765$

$x_8 = \frac{2.59375+2.59765}{2} = 2.5957, f(2.5957) = .0136, f(2.59375).f(2.5957) < 0$

Ninth approximation: $a = 2.59375, b = 2.5957$

$x_9 = \frac{2.59375+2.5957}{2} = 2.5947, f(2.5947) = -.004, f(2.5947).f(2.5957) < 0$

Tenth approximation: $a = 2.5947, b = 2.5957$

$x_{10} = \frac{2.5947+2.5957}{2} = 2.5952$

Hence 2.5952 is the real root correct to three decimal places.

Example3 Apply bisection method to find a root of the equation $xe^x = 1$ correct to three decimal places.

Solution: $f(x) = xe^x - 1$

Here $f(0) = -1$ and $f(1) = e - 1 = 1.718 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in [0,1], ∴ atleast one root exists in [0,1]

Initial approximation: $a = 0, b = 1$

$x_0 = \frac{0+1}{2} = 0.5,\ f(0.5) = -0.1756,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$x_1 = \frac{0.5+1}{2} = 0.75,\ f(0.75) = 0.5877,\ f(0.5).f(0.75) < 0$

Second approximation: $a = 0.5, b = 0.625$

$x_2 = \frac{0.5+0.75}{2} = 0.625,\ f(0.625) = 0.8682,\ f(0.5).f(0.625) < 0$

Third approximation: $a = 0.5, b = 0.625$

$x_3 = \frac{0.5+0.625}{2} = 0.5625,\ f(0.5625) = -0.0128,\ f(0.5625).f(0.625) < 0$

Fourth approximation: $a = 0.5625, b = 0.625$

$x_4 = \frac{0.5625+0.625}{2} = 0.59375, f(0.59375) = 0.0751, f(0.5625).f(0.59375) < 0$

Fifth approximation: $a = 0.5625, b = 0.59375$

$x_5 = \frac{0.5625+0.59375}{2} = 0.5781, f(0.5781) = 0.0305, f(0.5625).f(0.5781) < 0$

Sixth approximation: $a = 0.5625, b = 0.5781$

$x_6 = \frac{0.5625+0.5781}{2} = 0.5703, f(0.5703) = .0087, f(0.5625).f(0.5703) < 0$

Seventh approximation: $a = 0.5625, b = 0.5703$

$x_7 = \frac{0.5625+0.5703}{2} = 0.5664, f(0.5664) = -.002, f(0.5664).f(0.5703) < 0$

Eighth approximation: $a = 0.5664, b = 0.5703$

$x_8 = \frac{0.5664+0.5703}{2} = 0.5684, f(0.5684) = 0.0035, f(0.5664).f(0.5684) < 0$

Ninth approximation: $a = 0.5664, b = 0.5684$

$x_9 = \frac{0.5664+0.5684}{2} = 0.5674, f(0.5674) = .0007, f(0.5664).f(0.5674) < 0$

Tenth approximation: $a = 0.5664, b = 0.5674$

$x_{10} = \frac{0.5664+0.5674}{2} = 0.5669, f(0.5669) = -.0007, f(0.5669).f(0.5674) < 0$

Eleventh approximation: $a = 0.5669, b = 0.5674$

$x_{11} = \frac{0.5669+0.5674}{2} = 0.56715, f(0.56715) = .00001 \sim 0$

Hence 0.56715 is the real root correct to three decimal places.

Example4 Using bisection method find an approximate root of the equation $\sin x = \frac{1}{x}$ correct to two decimal places.

Solution: $f(x) = x\sin x - 1$

Here $f(1) = \sin 1 - 1 = -0.1585$ and $f(2) = 2\sin 2 - 1 = 0.8186$

Also $f(x)$ is continuous in $[1,2]$, $\therefore$ atleast one root exists in $[1,2]$

Initial approximation: $a = 1, b = 2$

$x_0 = \frac{1+2}{2} = 1.5,\ f(1.5) = 0.4963,\ f(1).f(1.5) < 0$

First approximation: $a = 1, b = 1.5$

$x_1 = \frac{1+1.5}{2} = 1.25,\ f(1.25) = 0.1862,\ f(1).f(1.25) < 0$

Second approximation: $a = 1, b = 1.25$

$x_2 = \frac{1+1.25}{2} = 1.125,\ f(1.125) = 0.0151,\ f(1).f(1.125) < 0$

Third approximation: $a = 1, b = 1.125$

$x_3 = \frac{1+1.125}{2} = 1.0625,\ f(1.0625) = -0.0718,\ f(1.0625).f(1.125) < 0$

Fourth approximation: $a = 1.0625, b = 1.125$

$x_4 = \frac{1.0625+1.125}{2} = 1.09375, f(1.09375) = -0.0284, f(1.09375).f(1.125) < 0$

Fifth approximation: $a = 1.09375, b = 1.125$

$x_5 = \frac{1.09375+1.125}{2} = 1.10937, f(1.10937) = -0.0066, f(1.10937).f(1.125) < 0$

Sixth approximation: $a = 1.10937, b = 1.125$

$x_6 = \frac{1.10937+1.125}{2} = 1.11719, f(1.11719) = .0042, f(1.10937).f(1.11719) < 0$

Seventh approximation: $a = 1.10937, b = 1.11719$

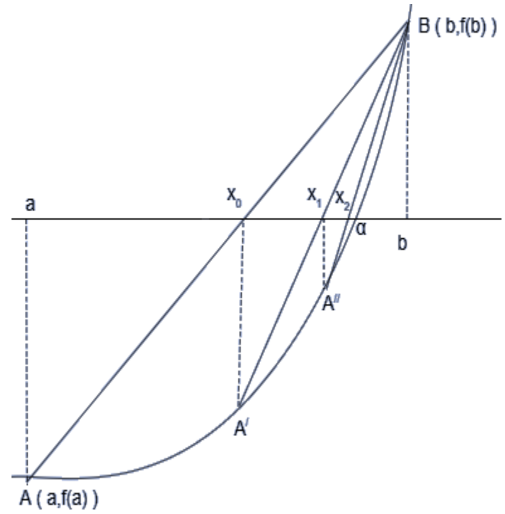
$x_7 = \frac{1.10937+1.11719}{2} = 1.11328, f(1.11328) = -.0012 \sim 0$

Hence 1.11328 is the real root correct to two decimal places.

5.2.2 Regula- Falsi Method (Geometrical Interpretation)

Regula-Falsi method is also known as method of false position as false position of curve is taken as initial approximation. Let $y = f(x)$ be represented by the curve AB.The real root of equation $f(x) = 0$ is α as shown in adjoining figure. The false position of curve AB is taken as chord AB and initial approximation x_0 is the point of intersection of chord

AB with x-axis. Successive approximations $x_1, x_2, ...$are given by point of intersection of chord $A'B, A''B, ...$with $x-$ axis, until the root is found to be of desired accuracy.



Now equation of chord AB in two-point form is given by:

$$y - f(a) = \frac{f(b)-f(a)}{b-a}(x-a)$$

To find x_0 (point of intersection of chord AB with x-axis), put $y = 0$

$$\Rightarrow -f(a) = \frac{f(b)-f(a)}{b-a}(x_0-a)$$

$$\Rightarrow (x_0 - a) = \frac{-(b-a)f(a)}{f(b)-f(a)}$$

$$\Rightarrow x_0 = a - \frac{(b-a)}{f(b)-f(a)}f(a)$$

Repeat the procedure until the root is found to the desired accuracy.

Remarks:

- Rate of convergence is much faster than that of bisection method.
- Unlike bisection method, one end point will converge to the actual root α , whereas the other end point always remains fixed. As a result Regula- Falsi method has linear convergence.

Example5 Apply Regula-Falsi method to find a root of the equation $x^3 + x - 1 = 0$ correct to two decimal places.

Solution:$f(x) = x^3 + x - 1$

Here $f(0) = -1$ and $f(1) = 1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$is continuous in [0,1], ∴ atleast one root exists in [0,1]

Initial approximation:$x_0 = a - \frac{(b-a)}{f(b)-f(a)}f(a)$; $a = 0, b = 1$

$$\Rightarrow x_0 = 0 - \frac{(1-0)}{f(1)-f(0)}f(0) = 0 - \frac{1}{1-(-1)}(-1) = 0.5$$

$f(0.5) = -0.375,\ f(0.5).f(1) < 0$

First approximation: $a = 0.5, b = 1$

$$x_1 = 0.5 - \frac{(1-0.5)}{f(1)-f(0.5)}f(0.5) = 0 - \frac{0.5}{1-(-0.375)}(-0.375) = 0.636$$

$f(0.636) = -0.107,\ f(0.636).f(1) < 0$

Second approximation: $a = 0.636, b = 1$

$$x_2 = 0.636 - \frac{(1-0.636)}{f(1)-f(0.636)}f(0.636) = 0.636 - \frac{0.364}{1-(-0.107)}(-0.107) = 0.6711$$

$f(0.6711) = -0.0267,\ f(0.6711).f(1) < 0$

Third approximation: $a = 0.6711, b = 1$

$x_3 = .6711 - \frac{(1-0.6711)}{f(1)-f(0.6711)} f(.6711) = .6711 - \frac{0.3289}{1-(-0.0267)}(-.0267) = 0.6796$

First 2 decimal places have been stabilized; hence 0.6796 is the real root correct to two decimal places.

Example6 Use Regula-Falsi method to find a root of the equation $x \log_{10} x - 1.2 = 0$ correct to two decimal places.

Solution: $f(x) = x \log_{10} x - 1.2$

Here $f(2) = -0.5979$ and $f(3) = 0.2314 \Rightarrow f(2).f(3) < 0$

Also $f(x)$ is continuous in [2,3], ∴ atleast one root exists in [2,3]

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 2, b = 3$

$\Rightarrow x_0 = 2 - \frac{(3-2)}{f(3)-f(2)} f(2) = 2 - \frac{1}{0.2314-(-0.5979)}(-0.5979) = 2.721$

$f(2.721) = -0.0171,\ f(2.721).f(3) < 0$

First approximation: $a = 2.721, b = 3$

$x_1 = 2.721 - \frac{(3-2.721)}{f(3)-f(2.721)} f(2.721) = 2.721 - \frac{0.279}{.2314-(-0.0171)}(-0.0171) = 2.7402$

$f(2.7402) = -0.0004,\ f(2.7402).f(3) < 0$

Second approximation: $a = 2.7402, b = 3$

$x_2 = 2.7402 - \frac{(3-2.7402)}{f(3)-f(2.7402)} f(2.7402) = 2.7402 - \frac{0.2598}{.2314-(-.0004)}(-.0004) = 2.7407$

First two decimal places have been stabilized; hence 2.7407 is the real root correct to two decimal places.

Example7 Use Regula-Falsi method to find a root of the equation $\tan x + \tanh x = 0$ upto three iterations only.

Solution: $f(x) = \tan x + \tanh x$

Here $f(2) = -1.2210$ and $f(3) = 0.8525 \Rightarrow f(2).f(3) < 0$

Also $f(x)$ is continuous in [2,3], ∴ atleast one root exists in [2,3]

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 2, b = 3$

$\Rightarrow x_0 = 2 - \frac{(3-2)}{f(3)-f(2)} f(2) = 2 - \frac{1}{0.8525-(-1.221)}(-1.221) = 2.5889$

$f(2.5889) = 0.3720,\ f(2).f(2.5889) < 0$

First approximation: $a = 2$, $b = 2.5889$

$x_1 = 2 - \frac{(2.5889-2)}{f(2.5889)-f(2)} f(2) = 2 - \frac{0.5889}{0.3720-(-1.2210)}(-1.2210) = 2.4514$

$f(2.4514) = 0.1596$, $f(2).f(2.4514) < 0$

Second approximation: $a = 2$, $b = 2.4514$

$x_2 = 2 - \frac{(2.4514-2)}{f(2.4514)-f(2)} f(2) = 2 - \frac{0.4514}{0.1596-(-1.2210)}(-1.2210) = 2.3992$

$f(2.3992) = 0.0662$, $f(2).f(2.3992) < 0$

Third approximation: $a = 2$, $b = 2.3992$

$x_2 = 2 - \frac{(2.3992-2)}{f(2.3992)-f(2)} f(2) = 2 - \frac{0.3992}{0.0662-(-1.2210)}(-1.2210) = 2.3787$

∴Real root of the equationtan $x + \tanh x = 0$ after three iterations is 2.3787

Example8 Use Regula-Falsi method to find a root of the equation $xe^x - 2 = 0$ correct to three decimal places.

Solution: $f(x) = xe^x - 2$

Here $f(0) = -2$ and $f(1) = 0.7183 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in [0,1], ∴ atleast one root exists in [0,1]

Initial approximation: $x_0 = a - \frac{(b-a)}{f(b)-f(a)} f(a)$; $a = 0$, $b = 1$

$\Rightarrow x_0 = 0 - \frac{(1-0)}{f(1)-f(0)} f(0) = 0 - \frac{1}{0.7183-(-2)}(-2) = 0.7358$

$f(0.7358) = -0.4643$, $f(0.7358).f(1) < 0$

First approximation: $a = 0.7358$, $b = 1$

$x_1 = 0.7358 - \frac{(1-0.7358)}{f(1)-f(0.7358)} f(0.7358) = 0.7358 - \frac{0.2642}{0.7183-(-0.4643)}(-0.4643) = 0.8395$

$f(0.8395) = -0.0564$, $f(0.8395).f(1) < 0$

Second approximation: $a = 0.8395$, $b = 1$

$x_2 = 0.8395 - \frac{(1-0.8395)}{f(1)-f(0.8395)} f(0.8395) = 0.8395 - \frac{0.1605}{0.7183-(-0.0564)}(-0.0564) = 0.8512$

$f(0.8512) = -0.006$, $f(0.8512).f(1) < 0$

Third approximation: $a = 0.8512$, $b = 1$

$x_2 = 0.8512 - \frac{(1-0.8512)}{f(1)-f(0.8512)} f(0.8512) = 0.8512 - \frac{0.1488}{0.7183-(-0.006)}(-0.006) = 0.8524$

$f(0.8524) = -0.009$, $f(0.8524).f(1) < 0$

Fourth approximation: $a = 0.8474 \ b = 1$

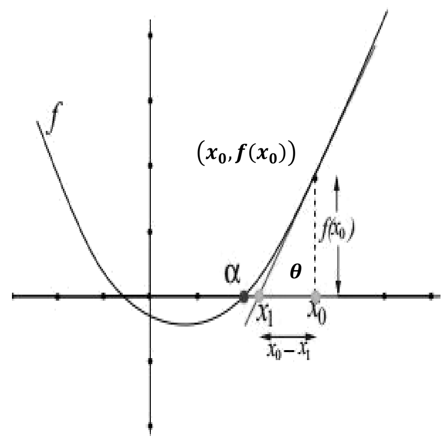
$x_4 = 0.8524 - \frac{(1-0.8524)}{f(1)-f(0.8524)} f(0.8524) = 0.8524 - \frac{0.1476}{0.7183-(-0.0009)}(-0.0009) = 0.8526$

$f(0.8526) = -0.00002 \sim 0,$

$\therefore$ Real root of the equation $xe^x - 2 = 0$ correct to three decimal places is 0.8526

5.2.3 Newton-Raphson Method (Geometrical Interpretation)

Newton-Raphson method named after Isaac Newton and Joseph Raphson is a powerful technique for solving equations numerically. The Newton-Raphson method in one variable is implemented as follows:



Let α be an exact root and x_0 be the initial approximate root of the equation $f(x) = 0$. First approximation x_1 is taken by drawing a tangent to curve $y = f(x)$ at the point $(x_0, f(x_0))$. If θ is the angle which tangent through the point $(x_0, f(x_0))$ makes with x- axis, then slope of the tangent is given by:

$$\tan\theta = \frac{f(x_0)}{x_0 - x_1} = f'(x_0)$$

$$\Rightarrow x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

Similarly $x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

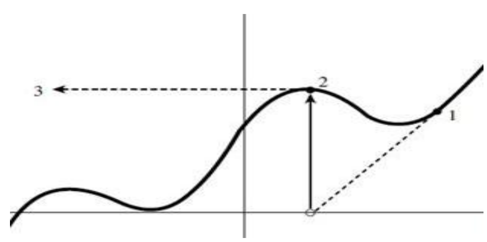
$$\vdots$$

The required root to desired accuracy is obtained by drawing tangents to the curve at points $(x_n, f(x_n))$ successively.

$$\therefore \quad x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

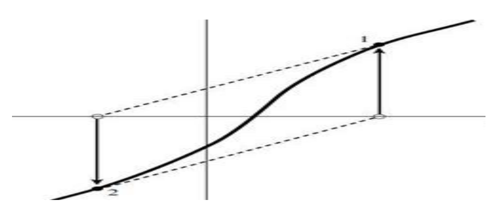
Newton-Raphson method works very fast but sometimes it fails to converge as shown below:

Case I:



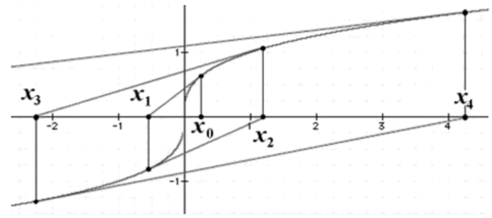
If any of the approximations encounters a zero derivative (extreme point), then the tangent at that point goes parallel to x-axis, resulting in no further approximations as shown in given figure where third approximation tends to infinity.

Case II:



Sometimes Newton-Raphson method may run into an infinite cycle or loop as shown in adjoining figure. Change in initial approximation may untangle the problem.

Case III:
In case of a point of discontinuity, as shown in given figure, subsequent roots may diverge instead of converging.



Remarks:

- Newton-Raphson method can be used for solving both algebraic and transcendental equations and it can also be used when roots are complex.
- Initial approximation x_0 can be taken randomly in the interval $[a, b]$, such that $f(a).f(b) < 0$
- Newton-Raphson method has quadratic convergence, but in case of bad choice of x_0 (the initial guess), Newton- Raphson method may fail to converge
- This method is useful in case of large value of $f'(x_n)$ i.e. when graph of $f(x)$ while crossing x -axis is nearly vertical

Example 9 Use Newton-Raphson method to find a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places.

Solution: $f(x) = x^3 - 5x + 3$

$\Rightarrow f'(x) = 3x^2 - 5$

Here $f(0) = 3$ and $f(1) = -1 \Rightarrow f(0).f(1) < 0$

Also $f(x)$ is continuous in $[0,1]$, $\therefore$ atleast one root exists in $[0,1]$

Initial approximation: Let initial approximation (x_0) in the interval $[0,1]$ be 0.8

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = 0.8,\ f(0.8) = -0.488,\ f'(0.8) = -3.08$

$\Rightarrow x_1 = 0.8 - \frac{-0.488}{-3.08} = 0.6416$

Second approximation:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where $x_1 = 0.6415, f(0.6416) = 0.0561,\ f'(0.6416) = -3.7650$

$\Rightarrow x_2 = 0.6416 - \frac{0.05611}{-3.7650} = 0.6565$

Third approximation:

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, where $x_2 = 0.6565,\ f(0.6565) = 0.0004,\ f'(0.6565) = -3.7070$

$\Rightarrow x_3 = 0.6565 - \frac{0.0004}{-3.7070} = 0.6566$

Hence a root of the equation $x^3 - 5x + 3 = 0$ correct to three decimal places is 0.6566

Example 10 Find the approximate value of $\sqrt{28}$ correct to 3 decimal places using Newton Raphson method.

Solution: $x = \sqrt{28} \quad \Rightarrow x^2 - 28 = 0$

$\therefore \; f(x) = x^2 - 28$

$\Rightarrow \; f'(x) = 2x$

Here $f(5) = -3$ and $f(6) = 8 \Rightarrow f(5).f(6) < 0$

Also $f(x)$ is continuous in [5,6], ∴ atleast one root exists in [5,6]

Initial approximation: Let initial approximation (x_0) in the interval [5,6] be 5.5

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$, where $x_0 = 5.5,\; f(5.5) = 2.25, f'(5.5) = 11$

$\Rightarrow x_1 = 5.5 - \frac{\mathbf{2.25}}{11} = 5.2955$

Second approximation:

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$, where $x_1 = 5.2955, f(5.2955) = 0.0423, f'(5.2955) = 10.591$

$\Rightarrow x_2 = 5.2955 - \frac{0.0423}{10.591} = 5.2915$

Third approximation:

$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$, where $x_2 = 5.2915, f(5.2915) = -0.00003, f'(5.2915) = 10.583$

$\Rightarrow x_3 = 5.2915 - \frac{-0.00003}{10.583} = 5.2915$

Hence value of $\sqrt{28}$ correct to three decimal places is 5.2915

Example 11 Use Newton-Raphson method to find a root of the equation $x \sin x + \cos x = 0$ correct to three decimal places.

Solution: $f(x) = x \sin x + \cos x$

$\Rightarrow \; f'(x) = x \cos x + \sin x - \sin x = x \cos x$

Here $f\left(\frac{\pi}{2}\right) = 1.5708$ and $f(\pi) = -1 \Rightarrow f\left(\frac{\pi}{2}\right).f(\pi) < 0$

Also $f(x)$ is continuous in $\left[\frac{\pi}{2}, \pi\right]$ ∴ atleast one root exists in $\left[\frac{\pi}{2}, \pi\right]$

Initial approximation: Let initial approximation (x_0) in the interval $\left[\frac{\pi}{2}, \pi\right]$ be π

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

First approximation:

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}\text{, where } x_0 = \pi,\ f(\pi) = -1, f'(\pi) = -3.1416$$

$$\Rightarrow x_1 = 3.1416 - \frac{-1}{-3.1416} = 2.8233$$

Second approximation:

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}\text{, where } x_1 = 2.8233,\ f(2.8233) = -0.0662, f'(2.8233) = -2.6815$$

$$\Rightarrow x_2 = 2.8233 - \frac{-0.0662}{-2.6815} = 2.7986$$

Third approximation:

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}\text{, where } x_2 = 2.798, f(2.7986\ ) = -0.0006, f'(2.7986\ ) = -2.6356$$

$$\Rightarrow x_3 = 2.7986\ - \frac{-0.0006}{-2.6356} = 2.7984$$

Hence a root of the equation $x \sin x + \cos x = 0$ correct to three decimal places is 2.7984

Example 12 Use Newton Raphson method to derive a formula to find $\sqrt[5]{N}$, $N \epsilon R$.

Hence evaluate $\sqrt[5]{43}$ correct to 3 decimal places.

Solution: $x = \sqrt[5]{N} \quad \Rightarrow x^5 - N = 0$

$$f(x) = x^5 - N$$

$$\Rightarrow f'(x) = 5x^4$$

By Newton-Raphson method $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

$$\Rightarrow x_{n+1} = x_n - \frac{x_n^5 - N}{5x_n^4} = \frac{4}{5}x_n + \frac{N}{5x_n^4}$$

To evaluate $\sqrt[5]{43}$, putting $N = 43$, ∴Newton-Raphson formula is given by

$$x_{n+1} = \frac{4}{5}x_n + \frac{43}{5x_n^4}$$

Let initial approximation x_0 be 2

First approximation:

$$x_1 = \frac{4}{5}x_0 + \frac{43}{5x_0^4}\text{, where } x_0 = 2$$

$$\Rightarrow x_1 = \frac{8}{5} + \frac{43}{80} = 2.1375$$

Second approximation:

$$x_2 = \frac{4}{5}x_1 + \frac{43}{5x_1^4}\text{, where } x_1 = 2.1375$$

$$\Rightarrow x_2 = \frac{4(2.1375)}{5} + \frac{43}{5(2.1375)^4} = 2.1220$$

Third approximation:

$$x_3 = \frac{4}{5}x_2 + \frac{43}{5x_2^4}\text{, where } x_2 = 2.1220$$

$$\Rightarrow x_3 = \frac{4(2.1220)}{5} + \frac{43}{5(2.1220)^4} = 2.1217$$

Fourth approximation:

$$x_4 = \frac{4}{5}x_3 + \frac{43}{5x_3^4}\text{, where } x_3 = 2.1217$$

$$\Rightarrow x_4 = \frac{4(2.1217)}{5} + \frac{43}{5(2.1217)^4} = 2.1217$$

Hence value of $\sqrt[5]{43}$ correct to four decimal places is 2.1217

5.2.3.1 Generalized Newton's Method for Multiple Roots

Result: If α is a root of equation $f(x) = 0$ with multiplicity m, then it is also a root of equation $f'(x) = 0$ with multiplicity $(m-1)$ and also of the equation $f''(x) = 0$ with multiplicity $(m-1)$ and so on.

For example $(x-1)^3 = 0$ has '1' as a root with multiplicity 3

$3(x-1)^2 = 0$ has '1' as the root with multiplicity 2

$6(x-1) = 0$ has '1' as the root with multiplicity 1

$\therefore$ The expressions $x_n - m\frac{f(x_n)}{f'(x_n)}$, $x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$, $x_n - (m-2)\frac{f''(x_n)}{f'''(x_n)}$ are equivalent

Generalized Newton's method is used to find repeated roots of an equation as is given as:

If α be a root of equation $f(x) = 0$ which is repeated m times,

Then $x_{n+1} = x_n - m\frac{f(x_n)}{f'(x_n)} \sim x_n - (m-1)\frac{f'(x_n)}{f''(x_n)}$

Example 13 Use Newton-Raphson method to find a double root of the equation

$x^3 - x^2 - x + 1 = 0$ upto three iterations.

Solution: $f(x) = x^3 - x^2 - x + 1$

$f'(x) = 3x^2 - 2x - 1$

$f''(x) = 6x - 2$

Let the initial approximation $x_0 = 0.7$

First approximation:

$x_1 = x_0 - \frac{2f(x_0)}{f'(x_0)}$ Also $x_1 = x_0 - \frac{f'(x_0)}{f''(x_0)}$

$\Rightarrow x_1 = 0.7 - \frac{0.306}{-0.93} = 1.0290$ And $x_1 = 0.7 - \frac{-0.93}{2.2} = 1.1227$

$\therefore x_1 = \frac{1.029+1.1227}{2} = 1.0759, f(x_1) = .012$

Second approximation:

$x_2 = x_1 - \frac{2f(x_1)}{f'(x_1)}$ Also $x_2 = x_1 - \frac{f'(x_1)}{f''(x_1)}$

$\Rightarrow x_2 = 1.0759 - \frac{0.0239}{0.3209} = 1.001$ And $x_2 = 1.0759 - \frac{0.3209}{4.4554} = 1.004$

$\therefore x_2 = \frac{1.001+1.004}{2} = 1.0025$, $f(x_2) = .00001$

Third approximation:

$x_3 = x_2 - \frac{2f(x_2)}{f'(x_2)}$ Also $x_3 = x_2 - \frac{f'(x_2)}{f''(x_2)}$

$\Rightarrow x_3 = 1.0025 - \frac{0.00003}{0.0100} = 0.995$ And $x_3 = 1.0025 - \frac{0.0100}{4.015} = 1.0000$

$\therefore x_3 = \frac{0.995+1.000}{2} = 0.9975$, $f(x_3) = .00001$

The double root of the equation $x^3 - x^2 - x + 1 = 0$ after three iterations is 0.9975.

5.2.3.1 Convergence of Newton Raphson Method

Let α be an exact root of the equation $f(x) = 0$

$\Rightarrow f(\alpha) = 0$

Also let x_n and x_{n+1} be two successive approximations to the root α.

If ϵ_n and ϵ_{n+1} are the corresponding errors in the approximations x_n and x_{n+1}

Then $x_n = \alpha + \epsilon_n$ ... ①

and $x_{n+1} = \alpha + \epsilon_{n+1}$ ... ②

Now by Newton Raphson method

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ ... ③

Using ① and ② in ③

$\Rightarrow \alpha + \epsilon_{n+1} = \alpha + \epsilon_n - \frac{f(\alpha + \epsilon_n)}{f'(\alpha + \epsilon_n)}$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n f'(\alpha+\epsilon_n) - f(\alpha+\epsilon_n)}{f'(\alpha+\epsilon_n)}$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n\left[f'(\alpha)+\epsilon_n f''(\alpha)+\ldots\right] - \left[f(\alpha)+\epsilon_n f'(\alpha)+\frac{{\epsilon_n}^2}{2!}f''(\alpha)+\cdots\right]}{f'(\alpha)+\epsilon_n f''(\alpha)+\ldots}$$ By Taylor's expansion

$$\Rightarrow \epsilon_{n+1} = \frac{{\epsilon_n}^2 f''(\alpha) - \frac{\epsilon_n^2}{2!}f''(\alpha)+\cdots}{f'(\alpha)\left[1+\frac{\epsilon_n f''(\alpha)}{f'(\alpha)}+\cdots\right]} \qquad \because f(\alpha) = 0$$

$$\Rightarrow \epsilon_{n+1} = \left[\frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 + \frac{\epsilon_n f''(\alpha)}{f'(\alpha)} + \cdots\right]^{-1}$$

$$\Rightarrow \epsilon_{n+1} = \left[\frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)} + \cdots\right]\left[1 - \frac{\epsilon_n f''(\alpha)}{f'(\alpha)} + \cdots\right]$$

$$\Rightarrow \epsilon_{n+1} = \frac{\epsilon_n^2}{2}\frac{f''(\alpha)}{f'(\alpha)}$$ Neglecting higher order terms

$$\Rightarrow \epsilon_{n+1} = K\,\epsilon_n^2$$ Where $k = \frac{1}{2}\frac{f''(\alpha)}{f'(\alpha)}$

∴ Newton Raphson method has convergence of order 2 or quadratic convergence.

5.3 Iterative Methods for Solving Simultaneous Linear Equations

Consider a system of linear equations:

$$\left.\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}\right\} \ldots ①$$

We have been using direct methods for solving a system of linear equations. Direct methods produce exact solution after a finite number of steps whereas iterative methods give a sequence of approximate solutions until solution is obtained up to desired accuracy. Common iterative methods for solving a system of linear equations are:

1. Gauss-Jacobi's iteration method
2. Gauss-Seidal's iteration method

These methods require partial pivoting before application.

Partial pivoting: It is about changing rows of a system of linear equations given by ① such that $a_1 \geq a_2, a_3$; $b_2 \geq b_3$.

Complete pivoting: It is the process of selecting the largest element in the magnitude as the pivot element by interchanging row as well as columns of the system. Order of variables is also changed in the procedure. In particular for the system given by ①, complete pivoting would require $a_1 \geq a_2, a_3$; $b_2 \geq b_1, b_3$, if a_1 and b_2 are to be taken as pivots.

5.3.1 Gauss-Jacobi's Iteration Method

The concept of the Gauss- Jacobi's iteration scheme is extremely simple with the assumptions that the system has unique solution and diagonal elements are non-zeros.

Algorithm: Gauss-Jacobi's iteration method

1. Take the system of linear equations given by ① after partial pivoting and solve each equation in the system for the diagonal value of variables such that

$$\left.\begin{aligned} x &= \frac{1}{a_1}(d_1 - b_1 y - c_1 z) \\ y &= \frac{1}{b_2}(d_2 - a_2 x - c_2 z) \\ z &= \frac{1}{c_3}(d_3 - a_3 x - b_3 y) \end{aligned}\right\} \ldots ②$$

2. Rewrite ② in generalized form given by:

$$\left.\begin{aligned} x_{n+1} &= \frac{1}{a_1}(d_1 - b_1 y_n - c_1 z_n) \\ y_{n+1} &= \frac{1}{b_2}(d_2 - a_2 x_n - c_2 z_n) \\ z_{n+1} &= \frac{1}{c_3}(d_3 - a_3 x_n - b_3 y_n) \end{aligned}\right\} \ldots ③$$

3. Take $x_0 = y_0 = z_0 = 0$ as initial approximation (in general if a better approximation can not be judged) and substitute in the system given by ③

$$\therefore\ x_1 = \frac{d_1}{a_1},\ y_1 = \frac{d_2}{b_2},\ z_1 = \frac{d_3}{c_3}$$

4. Putting $n = 1$, substitute the values of x_1, y_1 and z_1 in ③ to get next approximations of x_2, y_2 and z_2. Continue the procedure until the difference between two consecutive approximations is negligible.

Example 14 Solve the following system of equations using Gauss Jacobi's method

$$5x - 2y + 3z = -1$$

$$-3x + 9y + z = 2$$

$$2x - y - 7z = 3$$

Solution: The given system of equations is satisfying rules of partial pivoting.

Rewriting in general form as given in ③

$$x_{n+1} = \frac{1}{5}(-1 + 2y_n - 3z_n)$$

$$y_{n+1} = \frac{1}{9}(2 + 3x_n - z_n)$$

$$z_{n+1} = \frac{1}{7}(-3 + 2x_n - y_n)$$

Taking $x_0 = y_0 = z_0 = 0$ as initial approximation

First Approximation:

$$x_1 = -\frac{1}{5} = -0.2\ y_1 = \frac{2}{9} = 0.222\ ,\ z_1 = -\frac{3}{7} = -0.429$$

Second Approximation:

$x_2 = \frac{1}{5}(-1 + 2y_1 - 3z_1)$, $y_2 = \frac{1}{9}(2 + 3x_1 - z_1)$, $z_2 = \frac{1}{7}(-3 + 2x_1 - y_1)$

$\Rightarrow x_2 = \frac{1}{5}(-1 + 2(0.222) - 3(-0.429)) = 0.146$

$y_2 = \frac{1}{9}(2 + 3(-0.2) + 0.429) = 0.203$

$z_2 = \frac{1}{7}(-3 + 2(-0.2) - 0.222) = -0.517$

Third Approximation:

$x_3 = \frac{1}{5}(-1 + 2y_2 - 3z_2)$, $y_3 = \frac{1}{9}(2 + 3x_2 - z_2)$, $z_3 = \frac{1}{7}(-3 + 2x_2 - y_2)$

$\Rightarrow x_3 = \frac{1}{5}(-1 + 2(0.203) - 3(-0.517)) = 0.191$

$y_3 = \frac{1}{9}(2 + 3(0.146) + 0.517) = 0.328$

$z_3 = \frac{1}{7}(-3 + 2(0.146) - 0.203) = -0.416$

Fourth Approximation:

$x_4 = \frac{1}{5}(-1 + 2y_3 - 3z_3)$, $y_4 = \frac{1}{9}(2 + 3x_3 - z_3)$, $z_4 = \frac{1}{7}(-3 + 2x_3 - y_3)$

$\Rightarrow x_4 = \frac{1}{5}(-1 + 2(0.328) - 3(-0.416)) = 0.181$

$y_4 = \frac{1}{9}(2 + 3(0.191) + 0.416) = 0.332$

$z_4 = \frac{1}{7}(-3 + 2(0.191) - 0.328) = -0.421$

Fifth Approximation:

$x_5 = \frac{1}{5}(-1 + 2y_4 - 3z_4)$, $y_5 = \frac{1}{9}(2 + 3x_4 - z_4)$, $z_5 = \frac{1}{7}(-3 + 2x_4 - y_4)$

$\Rightarrow x_5 = \frac{1}{5}(-1 + 2(0.332) - 3(-0.421)) = 0.185$

$y_5 = \frac{1}{9}(2 + 3(0.181) + 0.421) = 0.329$

$z_5 = \frac{1}{7}(-3 + 2(0.181) - 0.332) = -0.424$

Sixth Approximation:

$x_6 = \frac{1}{5}(-1 + 2y_5 - 3z_5)$, $y_6 = \frac{1}{9}(2 + 3x_5 - z_5)$, $z_6 = \frac{1}{7}(-3 + 2x_5 - y_5)$

$\Rightarrow x_6 = \frac{1}{5}(-1 + 2(0.329) - 3(-0.424)) = 0.186$

$y_6 = \frac{1}{9}(2 + 3(0.185) + 0.424) = 0.331$

$z_6 = \frac{1}{7}(-3 + 2(0.185) - 0.329) = -0.423$

Values of variables have been stabilized, ∴ approximate solution is given by

$x = 0.186$, $y = 0.331$and $z = -0.423$

Example 15 Compute 4 iterations to find an approximate solution of the given system of equations using Gauss Jacobi's method.

$$x + y + 5z = -1$$
$$5x - y + z = 10$$
$$2x + 4y = 12$$

Solution: Rearranging the given equations by partial pivoting

$$5x - y + z = 10$$
$$2x + 4y = 12$$
$$x + y + 5z = -1$$

Rewriting in general form as given in ③

$$x_{n+1} = \frac{1}{5}(10 + y_n - z_n)$$
$$y_{n+1} = \frac{1}{4}(12 - 2x_n)$$
$$z_{n+1} = \frac{1}{5}(-1 - x_n - y_n)$$

Taking $x_0 = y_0 = z_0 = 0$ as initial approximation

First Approximation:

$$x_1 = \frac{10}{5} = 2 \text{ , } y_1 = \frac{12}{4} = 3 \text{ , } z_1 = -\frac{1}{5}$$

Second Approximation:

$$x_2 = \frac{1}{5}(10 + y_1 - z_1) \text{ , } y_2 = \frac{1}{4}(12 - 2x_1) \text{ , } z_2 = \frac{1}{5}(-1 - x_1 - y_1)$$
$$\Rightarrow x_2 = \frac{1}{5}\left(10 + 3 + \frac{1}{5}\right) \text{ , } y_2 = \frac{1}{4}(12 - 4) \text{ , } z_2 = \frac{1}{5}(-1 - 2 - 3)$$
$$\therefore x_2 = 2.64 \text{ , } y_2 = 2 \text{ , } z_2 = -1.2$$

Third Approximation:

$$x_3 = \frac{1}{5}(10 + y_2 - z_2) \text{ , } y_3 = \frac{1}{4}(12 - 2x_2) \text{ , } z_3 = \frac{1}{5}(-1 - x_2 - y_2)$$
$$\Rightarrow x_3 = \frac{1}{5}(10 + 2 + 1.2) \text{ , } y_3 = \frac{1}{4}(12 - 2(2.64)) \text{ , } z_3 = \frac{1}{5}(-1 - 2.64 - 2)$$
$$\therefore x_3 = 2.64 \text{ , } y_3 = 1.68 \text{ , } z_3 = -0.928$$

Fourth Approximation:

$$x_4 = \frac{1}{5}(10 + y_3 - z_3) \text{ , } y_3 = \frac{1}{4}(12 - 2x_3) \text{ , } z_3 = \frac{1}{5}(-1 - x_3 - y_3)$$
$$\Rightarrow x_4 = \frac{1}{5}(10 + 1.68 + 0.928) \text{ , } y_4 = \frac{1}{4}(12 - 2(2.64)) \text{ , } z_4 = \frac{1}{5}(-1 - 2.64 - 1.68)$$

$$\therefore x_4 = 2.52 \text{ , } y_4 = 1.68 \text{ , } z_4 = -1.064$$

Approximate solution after 4 iterations is given by $x = 2.52,\ y = 1.68, z = -1.064$

5.3.2Gauss-Seidal's Iteration Method

Gauss-Seidel method is an improvement of the basic Gauss-Jordan method. Here the improved values of variables are utilized as soon as they are obtained.

∴ System of equations given in ③ is improved by taking latest values of the variables as

$$x_{n+1} = \frac{1}{a_1}(d_1 - b_1 y_n - c_1 z_n)$$
$$y_{n+1} = \frac{1}{b_2}(d_2 - a_2 x_{n+1} - c_2 z_n)$$
$$z_{n+1} = \frac{1}{c_3}(d_3 - a_3 x_{n+1} - b_3 y_{n+1})$$

Gauss-Seidel scheme usually converges faster than Jacobi's iteration method.

Example 16 Solve the system of equations given in Example 14 using Gauss Seidal's method

$$5x - 2y + 3z = -1$$
$$-3x + 9y + z = 2$$
$$2x - y - 7z = 3$$

Also compare the results obtained in two methods.

Solution: The given system of equations is satisfying rules of partial pivoting.

Using Gauss Seidal's approximations, system can be rewritten as

$$x_{n+1} = \frac{1}{5}(-1 + 2y_n - 3z_n)$$
$$y_{n+1} = \frac{1}{9}(2 + 3x_{n+1} - z_n)$$
$$z_{n+1} = \frac{1}{7}(-3 + 2x_{n+1} - y_{n+1})$$

Taking $x_0 = y_0 = z_0 = 0$ as initial approximation

First Approximation:

$$x_1 = -\frac{1}{5} = -0.2\ y_1 = \frac{2}{9} = 0.222 \text{ , } z_1 = -\frac{3}{7} = -0.429$$

Second Approximation:

$$x_2 = \frac{1}{5}(-1 + 2y_1 - 3z_1) \text{ , } y_2 = \frac{1}{9}(2 + 3x_2 - z_1) \text{ , } z_2 = \frac{1}{7}(-3 + 2x_2 - y_2)$$

$$\Rightarrow x_2 = \frac{1}{5}(-1 + 2(0.222) - 3(-0.429)) = 0.146$$

$$y_2 = \frac{1}{9}(2 + 3(0.146) + 0.429) = 0.319$$

$z_2 = \frac{1}{7}(-3 + 2(0.146) - 0.319) = -0.432$

Third Approximation:

$x_3 = \frac{1}{5}(-1 + 2y_2 - 3z_2)$, $y_3 = \frac{1}{9}(2 + 3x_3 - z_2)$, $z_3 = \frac{1}{7}(-3 + 2x_3 - y_3)$

$\Rightarrow x_3 = \frac{1}{5}(-1 + 2(0.319) - 3(-0.432)) = 0.187$

$y_3 = \frac{1}{9}(2 + 3(0.187) + 0.432) = 0.333$

$z_3 = \frac{1}{7}(-3 + 2(0.187) - 0.333) = -0.423$

Fourth Approximation:

$x_4 = \frac{1}{5}(-1 + 2y_3 - 3z_3)$, $y_4 = \frac{1}{9}(2 + 3x_4 - z_3)$, $z_4 = \frac{1}{7}(-3 + 2x_4 - y_4)$

$\Rightarrow x_4 = \frac{1}{5}(-1 + 2(0.333) - 3(-0.423)) = 0.187$

$y_4 = \frac{1}{9}(2 + 3(0.187) + 0.423) = 0.332$

$z_4 = \frac{1}{7}(-3 + 2(0.187) - 0.332) = -0.423$

Values of variables have been stabilized, ∴ approximate solution is given by

$x = 0.187$, $y = 0.332$ and $z = -0.423$

Clearly numbers of iterations for the solution to converge in Gauss Seidal's method are much less than Gauss Jacobi's method.

Exercise 5A

1. Find the real root of the equation $x^3 - 2x - 5 = 0$ correct to three decimal places using Bisection method.
2. Perform three iterations to find root of the equation $xe^x - \cos x = 0$ using Regula-Falsi method.
3. Find the real root of the equation $x^3 - 3x - 5 = 0$ correct to three decimal places using Newton- Raphson method.
4. Solve the system of equations:

   $10x_1 - 2x_2 - x_3 - x_4 = 3$

   $2x_1 - 10x_2 + x_3 + x_4 = -15$

   $x_1 + x_2 - 10x_3 + 2x_4 = -27$

   $x_1 + x_2 + 2x_3 - 10x_4 = 9$

   using Gauss- Jacobi method. Compute results for 2 iterations.
5. Solve the system of equations given in Q4 upto 2 iterations, using Gauss-Seidal method.

Answers

1. 2.0944
2. 0.494015
3. 2.279
4. $x_1 = 0.78$, $x_2 = 1.74$, $x_3 = 2.7$, $x_4 = -0.18$ taking initial approximations as zero.
5. $x_1 = 0.8869$, $x_2 = 1.9523$, $x_3 = 2.9566$, $x_4 = -0.0248$ taking initial approximations as zero.