Chapter 9

Numerical Solutions of Ordinary Differential Equations

9.1 Introduction

An ordinary differential equation is a mathematical equation that relates one or more functions of an independent variable with its derivatives. Differential equations are of extreme importance to scientists and engineers as they are inevitable tools for mathematical modeling of any problem involving rate of change. Sometimes, we encounter situations where these equations are not amenable to analytic solutions. They can either be solved using mathematical software or by using numerical techniques discussed in coming sections.

Many practical applications lead to second or higher order systems of ordinary differential equations, numerical methods for higher order initial value problems are entirely based on their reformulation as first order systems. Numerical solutions of ordinary differential equations require initial values as they are based on finite-dimensional approximations. In this chapter, we shall restrict our discussion to numerical methods for solving initial value problems of first-order ordinary differential equations.

The first-order differential equation and the given initial value constitute a first-order initial value problem given as: $\frac{dy}{dx} = f(x,y)$; $y(x_0) = y_0$, whose numerical solution may be given using any of the following methodologies:

(a) Taylor series method
(b) Picard's method
(c) Euler's method
(d) Modified Euler's method
(e) Runge-Kutta method
(f) Milne's Predictor corrector method
(g) Adams-Bashforth method

All these methods will be discussed in detail in coming sections.

9.2 Taylor Series Method

Taylor's series expansion of a function $y(x)$ about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0''' + \cdots \qquad \cdots ①$$

To approximate $y(x)$ numerically for the initial value problem given by

$\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$, we substitute the values of y_0 and its successive derivatives in Taylor's series given by ①. Working methodology is illustrated in the examples given below.

Example1 Solve the differential equation $\frac{dy}{dx} = x + y$; $y(0) = 1$, at $x = 0.2$, 0.4 correct to 3 decimal places, using Taylor's series method. Also compare the numerical solution obtained with the analytic solution.

Solution: Taylor's series expansion of $y(x)$ about $x = 0$ is given by:

$$y(x) = y_0 + (x-0)y_0' + \frac{1}{2!}(x-0)^2 y_0'' + \frac{1}{3!}(x-0)^3 y_0''' + \frac{1}{4!}(x-0)^4 y_0^{iv} + \cdots \quad \cdots ①$$

$$\begin{array}{llcl} \text{Given} & \frac{dy}{dx} = x + y & ; & y_0 = 1 \\ \text{or} & y' = x + y & ; & y_0' = 1 \\ \Rightarrow & y'' = 1 + y' & ; & y_0'' = 2 \\ & y''' = y'' & ; & y_0''' = 2 \\ & y^{iv} = y''' & ; & y_0^{iv} = 2 \\ & \vdots & & \end{array}$$

Substituting these values in ①, we get

$$y(x) = 1 + x(1) + \frac{1}{2!}x^2(2) + \frac{1}{3!}x^3(2) + \frac{1}{4!}x^4(2) + \cdots$$

$$\text{Or} \quad y(x) = 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{12} + \cdots$$

$$i. \quad y(0.2) = 1 + 0.2 + 0.04 + \frac{0.008}{3} + \frac{0.0016}{12} + \cdots$$

$$= 1 + 0.2 + 0.04 + 0.002667 + 0.00013 + \cdots$$

The fifth term in this series is $0.00013 < 0.0005$

Hence value of $y(0.2)$ correct to 3 decimal places may be obtained by adding first four terms.

$\therefore y(0.2) \approx 1.24280 \approx 1.243$

$$ii. \quad y(0.4) = 1 + 0.4 + 0.16 + \frac{0.064}{3} + \frac{0.0256}{12} + \frac{0.01024}{60} + \cdots$$

$$= 1 + 0.4 + 0.16 + 0.02133 + 0.00213 + 0.00017 + \cdots$$

The sixth term in this series is $0.00017 < 0.0005$

Hence value of $y(0.4)$ correct to 3 decimal places may be obtained by adding first five terms. $\therefore y(0.4) \approx 1.58346 \approx 1.583$ correct to three decimal places.

Again to find exact solution of $\frac{dy}{dx} - y = x$, which is a linear differential equation

Integrating Factor (I.F.) $= e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = \int xe^{-x}dx$

$$\Rightarrow \quad ye^{-x} = -xe^{-x} - e^{-x} + c$$

$$\Rightarrow \quad y = -x - 1 + ce^x$$

Given that $y(0) = 1 \quad \Rightarrow 1 = 0 - 1 + c \qquad \therefore c = 2$

$\Rightarrow y = -x - 1 + 2e^x$

$y(0.2) \approx 1.243$ and $y(0.4) \approx 1.584$ correct to three decimal places

Example2 Solve the differential equation $\frac{dy}{dx} = 4y$; $(0) = 1$, at $x = 0.1$ using Taylor's series method correct to three decimal places.

Solution: Taylor's series of $y(x)$ about $x = 0$, is given by

$$y(x) = y_0 + (x-0)y_0' + \frac{1}{2!}(x-0)^2 y_0'' + \frac{1}{3!}(x-0)^3 y_0''' + \frac{1}{4!}(x-0)^4 y_0^{iv} + \cdots \quad \cdots ①$$

$$\begin{aligned} &\text{Given } \frac{dy}{dx} = 4y \quad &;& \quad y_0 = 1 \\ &\text{or } y' = 4y \quad &;& \quad y_0' = 4 \\ &\Rightarrow y'' = 4y' \quad &;& \quad y_0'' = 16 \\ &y''' = 4y'' \quad &;& \quad y_0''' = 64 \\ &y^{iv} = 4y''' \quad &;& \quad y_0^{iv} = 256 \end{aligned}$$

Substituting these values in ①, we get

$$y(x) = 1 + x(4) + \frac{1}{2!}x^2(16) + \frac{1}{3!}x^3(64) + \frac{1}{4!}x^4(256) + \cdots$$

or $y(x) = 1 + 4x + \frac{16x^2}{2!} + \frac{64x^3}{3!} + \frac{256x^4}{4!} + \frac{256x^4}{5!} \ldots$

$\Rightarrow \; y(x) = 1 + 4x + 8x^2 + \frac{32}{3}x^3 + \frac{32}{3}x^4 + \cdots$

$$y(0.1) = 1 + 4(0.1) + 8(0.1)^2 + \frac{32}{3}(0.1)^3 + \frac{32}{3}(0.1)^4 + \frac{128}{15}(0.1)^5 \ldots$$

$\Rightarrow \; y(0.1) = 1 + 0.4 + 0.08 + 0.01067 + 0.00107 + 0.00009$

$y(0.1) \approx 1.49183 \approx 1.492$ correct to three decimal places

Again to find analytical solution of $\frac{dy}{dx} = 4y \Rightarrow \frac{dy}{y} = 4dx$

This is a variable separable equation, whose solution is given by:

$\log y = 4x + \log c$

$\Rightarrow \; y = ce^{4x}$

Given that $y(0) = 1 \qquad \therefore c = 1$

$\Rightarrow y = e^{4x}$

$y(0.1) \approx 1.491824 \approx 1.492$ correct to three decimal places

Example3 Using Taylor's series method, solve the differential equation $\frac{dy}{dx} = y + 3e^x$; $(0) = 1$, at $x = 0.2$

Also compare the result with the exact solution.

Solution: Taylor's series expansion of $y(x)$ about $x = 0$ is given by:

$$y(x) = y_0 + (x-0)y_0' + \frac{1}{2!}(x-0)^2 y_0'' + \frac{1}{3!}(x-0)^3 y_0''' + \frac{1}{4!}(x-0)^4 y_0^{iv} + \cdots \quad \cdots ①$$

$$\begin{aligned} &\text{Given } \frac{dy}{dx} = y + 3e^x \quad &;& \quad y_0 = 1 \\ &\text{or } y' = y + 3e^x \quad &;& \quad y_0' = 4 \\ &\Rightarrow y'' = y' + 3e^x \quad &;& \quad y_0'' = 7 \\ &y''' = y'' + 3e^x \quad &;& \quad y_0''' = 10 \\ &y^{iv} = y''' + 3e^x \quad &;& \quad y_0^{iv} = 13 \end{aligned}$$

$$y^v = y^{iv} + 3e^x \quad ; \qquad y_0^v = 16$$

$$\vdots$$

Substituting these values in ①, we get

$$y(x) = 1 + x(4) + \frac{1}{2!}x^2(7) + \frac{1}{3!}x^3(10) + \frac{1}{4!}x^4(13) + \frac{1}{5!}x^5(16) + \cdots$$

or $\quad y(x) = 1 + 4x + \frac{7}{2}x^2 + \frac{5}{3}x^3 + \frac{13}{24}x^4 + \frac{2}{15}x^5 + \cdots$

$i.\ \ y(0.2) = 1 + 4(0.2) + \frac{7}{2}(0.2)^2 + \frac{5}{3}(0.2)^3 + \frac{13}{24}(0.2)^4 + \frac{2}{15}(0.2)^5 + \cdots$

$$= 1 + 0.8 + 0.14 + 0.01333 + 0.00087 + 0.00004 + \cdots$$

The sixth term in this series is $0.00004 < 0.0005$

Hence value of $y(0.2)$ correct to 3 decimal places may be obtained by adding first five terms.

$\therefore y(0.2) \approx 1.9542 \approx 1.954$

Again to find exact solution of $\frac{dy}{dx} - y = 3e^x$, which is a linear equation

Integrating Factor (I.F.) = $e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = 3\int e^x e^{-x} dx$

$$\Rightarrow \quad ye^{-x} = 3x + c$$

$$\Rightarrow \quad y = (3x + c)e^x$$

Given that $y(0) = 1 \quad \Rightarrow c = 1$

$\Rightarrow y = (3x + 1)e^x$

$y(0.2) \approx 1.954244 \approx 1.954$ correct to three decimal places

9.3 Picard's Method of Successive Approximations

Consider the initial value problem given by $\frac{dy}{dx} = f(x, y)$; $\ y(x_0) = y_0$

$\Rightarrow dy = f(x, y)dx$ Integrating, we get

$\int_{y_0}^{y} dy = \int_{x_0}^{x} f(x, y)dx$

$\Rightarrow y - y_0 = \int_{x_0}^{x} f(x, y)dx$

$\Rightarrow y = y_0 + \int_{x_0}^{x} f(x, y)dx$

To obtain the first approximation, replacing y by y_0 on R.H.S.

$$\Rightarrow \quad y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

Similarly $y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$

$$\vdots$$

$$y_n = y_0 + \int_{x_0}^{x} f(x, y_{n-1})dx$$, where $y(x_0) = y_0$

Remark: Picard's method can be applied only to limited types of problems, which can be integrated successively.

Example4 Using Picard's method, solve the initial value problem $\frac{dy}{dx} = x + y$; $y(0) = 1$, upto 3 approximations.

Solution: Given $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$\Rightarrow y_1 = 1 + \int_{0}^{x} (x + 1)dx$$

$$= 1 + \left[\frac{x^2}{2} + x\right]_0^x = 1 + x + \frac{x^2}{2}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$\Rightarrow y_2 = 1 + \int_{0}^{x} (x + y_1)dx$$

$$= 1 + \int_{0}^{x} \left(x + \left(1 + x + \frac{x^2}{2}\right)\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{6}$$

3rd approximation:

$$y_3 = y_0 + \int_{x_0}^{x} f(x, y_2)dx$$

$$\Rightarrow y_3 = 1 + \int_{0}^{x} (x + y_2)dx$$

$$= 1 + \int_{0}^{x} \left(x + \left(1 + x + x^2 + \frac{x^3}{6}\right)\right) dx$$

$$= 1 + x + x^2 + \frac{x^3}{3} + \frac{x^4}{24}$$

Example5 Using Picard's method, obtain the solution of $\frac{dy}{dx} = x(1 + x^3 y)$; $y(0) = 3$, at $x = 0.1$.

Solution: Given $f(x, y) = x(1 + x^3 y)$, $x_0 = 0$, $y_0 = 3$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^{x} f(x, y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^{x} f(x, y_0)dx$$

$$\Rightarrow y_1 = 3 + \int_{0}^{x} x(1 + x^3 y)\, dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5}$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^{x} f(x, y_1)dx$$

$$\Rightarrow\ y_2 = 3 + \int_0^x x\left[1 + x^3\left(3 + \frac{x^2}{2} + \frac{3x^5}{5}\right)\right]\,dx$$

$$= 3 + \frac{x^2}{2} + \frac{3x^5}{5} + \frac{x^7}{14} + \frac{3x^{10}}{50}$$

Clearly y_1 and y_2 are coincident upto 3 terms.

$\therefore$ Let $y = 3 + \frac{x^2}{2} + \frac{3x^5}{5}$

Also $y(0.1) = 3 + \frac{(0.1)^2}{2} + \frac{3(0.1)^5}{5} = 3.00501$

Example6 Using Picard's method, solve the initial value problem $\frac{dy}{dx} = xy$; $y(1) = 2$, upto 3 approximations.

Solution: Given $f(x, y) = xy,\quad x_0 = 1,\ y_0 = 2$

Using Picard's approximation

$$y = y_0 + \int_{x_0}^x f(x, y)dx$$

1st approximation:

$$y_1 = y_0 + \int_{x_0}^x f(x, y_0)dx$$

$$\Rightarrow\ y_1 = 2 + \int_1^x x(2)dx$$

$$= 2 + [x^2]_1^x = 1 + x^2$$

2nd approximation:

$$y_2 = y_0 + \int_{x_0}^x f(x, y_1)dx$$

$$\Rightarrow\ y_2 = 2 + \int_1^x (x.y_1)dx$$

$$= 2 + \int_1^x (x(1 + x^2))dx$$

$$= \frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}$$

3rd approximation:

$$y_3 = y_0 + \int_{x_0}^x f(x, y_2)dx$$

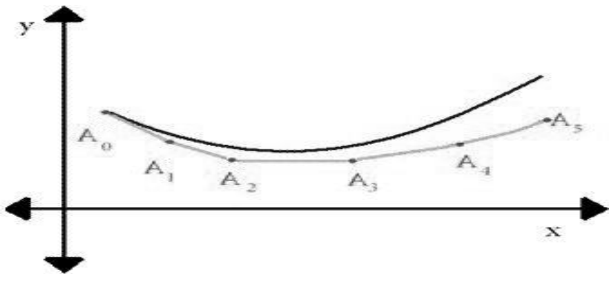
$$\Rightarrow\ y_3 = 2 + \int_1^x (x.y_2)dx$$

$$= 2 + \int_1^x x\left(\frac{5}{4} + \frac{x^2}{2} + \frac{x^4}{4}\right)dx$$

$$= \frac{29}{24} + \frac{5x^2}{8} + \frac{x^4}{8} + \frac{x^6}{24}$$

9.4 Euler's Method

Euler's Method provides us with a numerical solution of the initial value problem $\frac{dy}{dx} = f(x, y)\ ;\ y(x_0) = y_0 \cdots ①$, by joining multiple small line segments A_0A_1 , A_1A_2, $A_2A_3, \cdots$, making an approximation of the actual curve, as shown in the adjoining figure.



Thus if $[x_0, x_1]$ is the small interval, where $x_1 = x_0 + h$, we approximate the curve by the tangent drawn to curve at point A_0 , having coordinates (x_0, y_0), whose equation is given by

$y - y_0 = m(x - x_0)$, where m is slope of tangent at the point (x_0, y_0)

Also $m = \left.\frac{dy}{dx}\right]_{(x_0,y_0)} = f(x_0, y_0)$ from ①

$\Rightarrow y = y_0 + f(x_0, y_0)\,(x - x_0)$

$\Rightarrow\ y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0)$ $\qquad \because y(x_1) = y_1$

$\Rightarrow\ y_1 = y_0 + hf(x_0, y_0)$ $\qquad \because\ x_1 - x_0 = h$

Similarly for range $[x_1, x_2]$

$y_2 = y_1 + hf(x_1, y_1)$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

It is evident from the given figure that h has to be kept small to avoid the approximations diverging away from the curve. As a result, this method is very slow and needs to be improved.

Example7 Using Euler's method, Compute $y(0.12)$ for the initial value problem:

$$\frac{dy}{dx} = x^3 + y\ ;\ y(0) = 1 \text{ , taking } h = 0.02 .$$

Solution: Given $f(x, y) = x^3 + y$, $x_0 = 0,\ y_0 = 1,\ x_n = x_{n-1} + h$, $h = 0.02$

$\therefore x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$, $x_4 = 0.08$, $x_5 = 0.1$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$\Rightarrow\ y_n = y_{n-1} + h({x_{n-1}}^3 + y_{n-1})$ $\qquad \cdots$①

Putting $n = 1$ in ①, $y_1 = y(0.02) = y_0 + h\,({x_0}^3 + y_0)$

$\therefore\ y_1 = 1 + 0.02(\,0 + 1) = 1.02$

Putting $n = 2$ in ①, $y_2 = y(0.04) = y_1 + h\,({x_1}^3 + y_1)$

$\therefore\ y_2 = 1.02 + 0.02(\,(0.02)^3 + 1.02) = 1.04040016$

Putting $n = 3$ in ①, $y_3 = y(0.06) = y_2 + h\,({x_2}^3 + y_2)$

$\therefore\ y_3 = 1.04040016 + 0.02(\,(0.04)^3 + 1.04040016) = 1.061209443$

Putting $n = 4$ in ①, $y_4 = y(0.08) = y_3 + h\,({x_3}^3 + y_3)$

$\therefore\ y_4 = 1.061209443 + 0.02(\,(0.06)^3 + 1.061209443) = 1.082437952$

Putting $n = 5$ in ①, $y_5 = y(0.1) = y_4 + h\,({x_4}^3 + y_4)$

$\therefore\ y_5 = 1.082437952 + 0.02(\,(0.08)^3 + 1.082437952) = 1.104096951$

Putting $n = 6$ in ①, $y_6 = y(0.12) = y_5 + h\,({x_5}^3 + y_5)$

$\therefore\ y_6 = 1.104096951 + 0.02(\,(0.1)^3 + 1.104096951) = 1.126198890$

Thus at $x = 0.12,\ y = 1.126198890\ \Rightarrow y(0.12) = 1.126198890$

Example8 Using Euler's method, solve $\frac{dy}{dx} = \frac{x-y}{2}$; $y(0) = 1$, over the interval [0,2], taking the step size $\frac{1}{2}$

Solution: Given $f(x, y) = \frac{x-y}{2}$, $x_0 = 0,\ y_0 = 1,\ x_n = x_{n-1} + h$, $h = \frac{1}{2}$

$\therefore x_1 = \frac{1}{2} = 0.5\ ,\ x_2 = 1\ ,\ x_3 = \frac{3}{2} = 1.5\ ,\ x_4 = 2$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$\Rightarrow\ y_n = y_{n-1} + \frac{h}{2}(x_{n-1} - y_{n-1})$

or $y_n = y_{n-1} + 0.25(x_{n-1} - y_{n-1})$ $\cdots$ ①

Putting $n = 1$ in ①, $y_1 = y\left(\frac{1}{2}\right) = y_0 + 0.25(x_0 - y_0)$

$\therefore\ y_1 = 1 + 0.25(0 - 1) = 0.75$

Putting $n = 2$ in ①, $y_2 = y(1) = y_1 + 0.25(x_1 - y_1)$

$\therefore\ y_2 = 0.75 + 0.25(0.5 - 0.75) = 0.6875$

Putting $n = 3$ in ①, $y_3 = y\left(\frac{3}{2}\right) = y_2 + 0.25(x_2 - y_2)$

$\therefore\ y_3 = 0.6875 + 0.25(1 - 0.6875) = 0.765625$

Putting $n = 4$ in ①, $y_4 = y(2) = y_3 + 0.25(x_3 - y_3)$

$\therefore\ y_4 = 0.765625 + 0.25(1.5 - 0.765625) = 0.94921875$

9.5 Modified Euler's Method

Though Euler's method is quite easy to implement, but unless the step size h is very small, the truncation error will be large and the results will be inaccurate.

As per Modified Euler's method, a better approximation of y_1 is given by improving $f(x_0, y_0)$ obtained by Euler's method as shown:

$y_1^{(1)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]$

$\vdots$

Continue approximating y_1 until two consecutive values are coincident to a specific degree of accuracy.

$\therefore y_1^{(k)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(k-1)}\right)]$

Repeat the procedure for $y_2\ ,\ y_3,\ y_4$ ... to find y_n

Example9 Use Modified Euler's method to obtain $y(0.2)$, $y(0.4)$ correct to 3 decimal places, given that $\frac{dy}{dx} = y - x^2$; $y(0) = 1$

Solution: Given $f(x, y) = y - x^2$, $x_0 = 0,\ y_0 = 1$

By Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

i. To evaluate $y(0.2)$, $h = 0.2, x_1 = 0 + 0.2 = 0.2$

$y_1 = y(0.2) = y_0 + hf(x_0, y_0), f(x_0, y_0) = y_0 - x_0^2 = 1 - 0 = 1$

$\therefore\ y_1 = 1 + 0.2(1) = 1.2$

$f(x_1, y_1) = y_1 - x_1^2 = 1.2 - (0.2)^2 = 1.16$

Now improving y_1 using Modified Euler's method

$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$

$$\therefore y_1^{(1)} = 1 + \frac{0.2}{2}(1 + 1.16) = 1.216$$

$$f\left(x_1, y_1^{(1)}\right) = y_1^{(1)} - x_1^2 = 1.216 - (0.2)^2 = 1.176$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(1)}\right)]$$

$$\therefore y_1^{(2)} = 1 + \frac{0.2}{2}(1 + 1.176) = 1.2176$$

$$f\left(x_1, y_1^{(2)}\right) = y_1^{(2)} - x_1^2 = 1.2176 - (0.2)^2 = 1.1776$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f\left(x_1, y_1^{(2)}\right)]$$

$$\therefore y_1^{(3)} = 1 + \frac{0.2}{2}(1 + 1.1776) = 1.21776 = y(0.2)$$

Thus by Modified Euler's method, we have improved $y(0.2)$ from 1.2 to 1.21776

$ii.$ To evaluate $y(0.4)$, $h = 0.2, x_2 = 0.2 + 0.2 = 0.4$

$$y_2 = y(0.4) = y_1 + hf(x_1, y_1),$$

$$f(x_1, y_1) = y_1 - x_1^2 = 1.21776 - (0.2)^2 = 1.17776$$

$$\therefore y_2 = 1.21776 + 0.2(1.17776) = 1.453312$$

$$f(x_2, y_2) = y_2 - x_2^2 = 1.453312 - (0.4)^2 = 1.293312$$

Now improving y_1 using Modified Euler's method

$$y_2^{(1)} = y_1 + \frac{h}{2}(f(x_1, y_1) + f(x_2, y_2))$$

$$\therefore y_2^{(1)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.293312) = 1.4648672$$

$$f\left(x_2, y_2^{(1)}\right) = y_2^{(1)} - x_2^2 = 1.4648672 - (0.4)^2 = 1.3048672$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f\left(x_2, y_2^{(1)}\right)\right)$$

$$\therefore y_2^{(2)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.3048672) = 1.46602272$$

$$f\left(x_2, y_2^{(2)}\right) = y_2^{(2)} - x_2^2 = 1.46602272 - (0.4)^2 = 1.30602272$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left(f(x_1, y_1) + f\left(x_2, y_2^{(2)}\right)\right)$$

$$\therefore y_2^{(3)} = 1.21776 + \frac{0.2}{2}(1.17776 + 1.30602272) = 1.466138272$$

Thus by Modified Euler's method, we have improved $y(0.4)$ from 1.453312 to 1.466138272 correct to 3 decimal places.

Example10 Use Modified Euler's method to obtain $y(1.2)$ correct to 3 decimal places, given that $\frac{dy}{dx} = \ln(x + y)$; $y(1) = 2$

Solution: Given $f(x, y) = \ln(x + y)$, $x_0 = 1$, $y_0 = 2$

By Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

To evaluate $y(1.2)$, $h = 0.2, x_1 = 1 + 0.2 = 1.2$

$$y_1 = y(1.2) = y_0 + hf(x_0, y_0)$$

$$f(x_0, y_0) = ln\,(x_0 + y_0) = \ln(1 + 2) = 1.09861$$

$$\therefore y_1 = 2 + 0.2(1.09861) = 2.21972$$

$$f(x_1, y_1) = ln(x_1 + y_1) = \ln(1 + 2.21972) = 1.16929$$

Now improving y_1 using Modified Euler's method

$$y_1^{(1)} = y_0 + \frac{h}{2}(f(x_0, y_0) + f(x_1, y_1))$$

$$\therefore y_1^{(1)} = 2 + \frac{0.2}{2}(1.09861 + 1.16929) = 2.22679$$

$$f(x_1, y_1^{(1)}) = \ln(x_1 + y_1^{(1)}) = \ln(1 + 2.22679) = 1.17149$$

$$y_1^{(2)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(1)})]$$

$$\therefore y_1^{(2)} = 2 + \frac{0.2}{2}(1.09861 + 1.17149) = 2.22701$$

$$f(x_1, y_1^{(2)}) = \ln(x_1 + y_1^{(2)}) = \ln(1 + 2.22701) = 1.17156$$

$$y_1^{(3)} = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1^{(2)})]$$

$$\therefore y_1^{(3)} = 2 + \frac{0.2}{2}(1.09861 + 1.17156) = 2.227017 = y(1.2)$$

Thus by Modified Euler's method, we have improved $y(1.2)$ from 2.21972 to 2.227017 correct to 4 decimal places

9.6 Runge- Kutta's Method

Runge-Kutta method is preferment of the concepts used in Euler's and Modified Euler's methods.

Consider the initial value problem

$$\frac{dy}{dx} = f(x, y)\ ; \quad y(x_0) = y_0 \qquad \cdots ①$$

Taylor's series expansion of a function $y(x)$ about $x = x_0$ is given by

$$y(x) = y_0 + (x - x_0)y_0' + \frac{1}{2!}(x - x_0)^2 y_0'' + \frac{1}{3!}(x - x_0)^3 y_0''' + \cdots$$

Now $y_1 = y(x_0 + h)$, $\therefore$ Putting $x = x_0 + h$ in Taylor's series, we get

$$y_1 = y(x_0 + h) = y_0 + hy_0' + \frac{h^2}{2!}y_0'' + \cdots \qquad \cdots ②$$

Also by Euler's method $y_1 = y_0 + hf(x_0, y_0) = y_0 + hy_0'$ $\qquad \cdots ③$

From ② and ③, Euler's method is in consonant to Taylor's series expansion upto first 2 terms i.e. till the term containing h of order one.

Euler's method itself is **first order Runge-Kutta method**.

Similarly it can be shown that Modified Euler's method coincides with Taylor's series expansion upto first 3 terms.

Modified Euler's method is given by $y_1 = y_0 + \frac{h}{2}[f(x_0, y_0) + f(x_1, y_1)]$

$$\Rightarrow \quad y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + h\, f(x_1, y_1)]$$

Now $x_1 = x_0 + h$ and $y_1 = y_0 + hf(x_0, y_0)$ by Euler's method

$$\Rightarrow \quad y_1 = y_0 + \frac{1}{2}[hf(x_0, y_0) + hf(x_0 + h, y_0 + hf(x_0, y_0))]$$

$$\Rightarrow y_1 = y_0 + \frac{1}{2}[K_1 + K_2]$$

Where $K_1 = hf(x_0, y_0)$, $K_2 = hf(x_0 + h, y_0 + K_1)$

$\therefore$ Modified Euler's method itself is **second order Runge-Kutta method.**

It is in consonant to Taylor's series expansion upto first 3 terms i.e. till the term containing h of order two.

Similarly **third order Runge-Kutta method** tallies with Taylor's series expansion upto first 4 terms i.e. till the term containing h of order three and is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 4K_2 + K_3]$$

where $K_1 = hf(x_0, y_0)$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right),$$

$$K_3 = hf(x_0 + h, y_0 + hf(x_0 + h, y_0 + K_1)$$

On the similar lines, **Runge- Kutta's method of order four** is collateral with Taylor's series expansion upto first 5 terms i.e. till the term containing h of order four.

Numerical solution of initial value problem given by ①, using fourth order Runge-Kutta method is: $y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4]$

where $K_1 = hf(x_0, y_0)$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right)$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right)$$

$$K_4 = hf(x_0 + h, y_0 + K_3)$$

Fourth order Runge- Kutta's method (commonly known as Runge- Kutta method)**,** provides most accurate result and is widely used to approximate initial value problems.

Example11 Solve the differential equation $\frac{dy}{dx} = y - x$; $y(0) = 1$, at $x = 0.1$, using Runge-Kutta method. Also compare the numerical solution obtained with the exact solution.

Solution: Given $f(x, y) = x + y$, $x_0 = 0$, $y_0 = 1$, $h = 0.1$

Runge-Kutta method of 4^{th} order is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \quad \cdots ①$$

$$K_1 = hf(x_0, y_0) = h(y_0 - x_0) = 0.1(1 - 0) = 0.1$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left(\left(1 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right) = 0.1$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left(\left(1 + \frac{0.1}{2}\right) - \left(0 + \frac{0.1}{2}\right)\right) = 0.1$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1\big((1 + 0.1) - (0 + 0.1)\big) = 0.1$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y_1 = 1 + \frac{1}{6}[0.1 + 2(0.1) + 2(0.1) + 0.1] = 1.1$$

Again to find exact solution of the initial value problem

$\frac{dy}{dx} - y = -x$, which is a linear differential equation

Integrating Factor (I.F.) $= e^{\int -dx} = e^{-x}$

Solution is given by $ye^{-x} = -\int xe^{-x}dx$

$$\Rightarrow \quad ye^{-x} = xe^{-x} + e^{-x} + c$$

$$\Rightarrow \quad y = x + 1 + ce^{x}$$

Given that $y(0) = 1 \quad \Rightarrow 1 = 0 + 1 + c \qquad \therefore c = 0$

$$\Rightarrow y = x + 1$$

$$y(0.1) = 0.1 + 1 = 1.1$$

Example12 Solve the differential equation $\frac{dy}{dx} = \ln(x + y);\ y(0) = 2$, at $x = 0.3$, using Runge-Kutta method of 4th order by dividing into two steps of $h = 0.15$ each. Compare the results with one step solution.

Solution: $i.$ Given $f(x,y) = \ln(x + y), \quad x_0 = 0,\ y_0 = 2, h = 0.15$

Runge-Kutta method of 4th order is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \qquad \cdots ①$$

$$K_1 = hf(x_0, y_0) = 0.15\ln(x_0 + y_0) = 0.15\ln(0 + 2) = 0.10397$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.15\ln\left(0 + \frac{0.15}{2} + 2 + \frac{0.10397}{2}\right) = 0.11321$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.15\ln\left(0 + \frac{0.15}{2} + 2 + \frac{0.11321}{2}\right) = 0.11353$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.15\ln(0 + 0.15 + 2 + 0.11353) = 0.12254$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y_1 = y(0.15) = 2 + \frac{1}{6}[0.10397 + 2(0.11321) + 2(0.11353) + 0.12254]$$

$$= 2.11333$$

Now taking $x_0 = 0.15,\ y_0 = 2.11333, h = 0.15$

$$K_1 = hf(x_0, y_0) = 0.15\ln(x_0 + y_0) = .15\ln(.15 + 2.11333) = .12253$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = .15\ln\left(.15 + \frac{.15}{2} + 2.11333 + \frac{.12253}{2}\right) = .13129$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = .15\ln\left(.15 + \frac{.15}{2} + 2.11333 + \frac{.13129}{2}\right) = .13157$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = .15\ln(.15 + .15 + 2.11333 + .13157) = .14011$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y(0.3) = 2.11333 + \frac{1}{6}[.12253 + 2(.13129) + 2(.13157) + .14011]$$

$$= 2.24472$$

$ii.$ Solving in single step of $h = 0.3$

Given $f(x,y) = \ln(x + y), \quad x_0 = 0,\ y_0 = 2,\ h = 0.3$

Runge-Kutta method of 4th order is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \qquad \cdots ①$$

$$K_1 = hf(x_0, y_0) = 0.3\ln(x_0 + y_0) = 0.3\ln(0 + 2) = 0.20794$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.3\ln\left(0 + \frac{0.3}{2} + 2 + \frac{0.20794}{2}\right) = 0.24381$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.3\ln\left(0 + \frac{0.3}{2} + 2 + \frac{0.24381}{2}\right) = 0.24619$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.3\ln(0 + 0.3 + 2 + 0.24619) = 0.28038$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y_1 = 2 + \frac{1}{6}[0.20794 + 2(0.24381) + 2(0.24619) + 0.28038] = 2.24472$$

Example13 Solve the differential equation $\frac{dy}{dx} = x^2 + y^2$; $y(0) = 2$, at $x = 0.1$, using Runge-Kutta method.

Solution: Given $f(x,y) = x^2 + y^2$, $x_0 = 0$, $y_0 = 2$, $h = 0.1$

Runge-Kutta method of 4th order is given by

$$y_1 = y_0 + \frac{1}{6}[K_1 + 2K_2 + 2K_3 + K_4] \quad \cdots ①$$

$$K_1 = hf(x_0, y_0) = h({x_0}^2 + {y_0}^2) = 0.1(0 + 4) = 0.4$$

$$K_2 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_1}{2}\right) = 0.1\left(\left(0 + \frac{0.1}{2}\right)^2 + \left(2 + \frac{0.4}{2}\right)^2\right) = 0.48425$$

$$K_3 = hf\left(x_0 + \frac{h}{2}, y_0 + \frac{K_2}{2}\right) = 0.1\left(\left(0 + \frac{0.1}{2}\right)^2 + \left(2 + \frac{0.48425}{2}\right)^2\right) = 0.50296$$

$$K_4 = hf(x_0 + h, y_0 + K_3) = 0.1((0 + 0.1)^2 + (2 + 0.50296)^2) = 0.62748$$

Substituting values of K_1, K_2, K_3, K_4 in ①, we get the solution as:

$$y_1 = 2 + \frac{1}{6}[0.4 + 2(0.48425) + 2(0.50296) + 0.62748] = 2.50032$$