Chapter 7

Interpolation

7.1 Introduction

Interpolation literally refers to introducing something additional or extraneous between other things or parts. In numerical analysis, interpolation is a method of constructing new data points within a discrete set of known data points, using finite differences. The process of obtaining function values outside (in the vicinity) the given range is called extrapolation.

In this chapter we shall extend the applications of differencing techniques to interpolate and extrapolate data points within a given range, for equal as well as well us unequal interval lengths.

7.2 Interpolation within Equal Intervals

Let $y = f(x)$ be an explicitly unknown function, with given discrete set of points (x_i, y_i), $i = 0,1,2,3,\dots,n$, where x_i 's are equispaced. The process of obtaining the values $f(x_i + ph)$, $-1 < p < 1$, where height of the interval (h) is fixed, is known as interpolation within equal intervals. There are several methods of interpolating data points within a given range, depending upon the location where the value is to be interpolated as given below:

i. Newton's forward interpolation formula
ii. Newton's backward interpolation formula
iii. Gauss's forward difference formula
iv. Gauss's backward difference formula
v. Stirling's central difference formula
vi. Bessels's interpolation formula

We shall discuss these methodologies one by one in the coming sections.

7.2.1 Newton's Forward Interpolation Formula

Newton's forward interpolation formula is used to interpolate the values of the function $y = f(x)$ near the beginning $(x > x_0)$ and to extrapolate the values when $(x < x_0)$, within the range of given data points (x_i, y_i), $i = 0,1,2,3,\dots,n$.

Let $f(x)$ take the values $y_0,\ y_1,\ y_2, \dots y_n$; for the independent variable x taking values $x_0,\ x_1,\ x_2, \dots x_n$, where height of the interval (h) is fixed, such that $x_1 = x_0 + h$, $x_2 = x_0 + 2h, \dots,\ x_n = x_0 + nh$.

Then to evaluate $f(x)$ for $x = x_0 + ph, -1 < p < 1$
We have $f(x) = f(x_0 + ph) = E^p f(x_0) \equiv (1 + \Delta)^p y_0$

$$\because E \equiv 1 + \Delta \text{ and } f(x_0) = y_0$$

$$\Rightarrow f(x) \equiv \left(1 + p\Delta + \frac{p(p-1)}{2!}\Delta^2 + \frac{p(p-1)(p-2)}{3!}\Delta^3 + \cdots\right) y_0\,, \quad x = x_0 + ph$$

$$\therefore f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots, p = \frac{x-x_0}{h}$$

This is Newton's forward interpolation formula and is used to interpolate or extrapolate values near the beginning of the table.

- In Newton's forward method, p is taken as 0 where $x = x_0$ and all the differences are evaluated taking $p = 0$ as reference point.
- Value which is to be interpolated i.e. $f(x)$ may be denoted by y_p, $|p < 1|$

Example1 Use Newton's forward interpolation formula to find the values of

$i.$ $f(1.4)$ $ii. f(0.9)$ for the given set of values

x	1	2	3	4
$f(x)$	6	11	18	27

Solution: Function has to be evaluated near the starting of the table, thereby constructing forward difference table for the function $y = f(x)$

x	P	$f(x)$	Δ	Δ^2	Δ^3
1	0	6			
			5		
2	1	11		2	
			7		0
3	2	18		2	
			9		
4	3	27			

Newton's forward interpolation formula given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots\,,\ p = \frac{x-x_0}{h} \quad \ldots ①$$

$i.$ To find $f(1.4)$

$x_0 = 1,\ y_0 = 6\,, x = 1.4, h = 1 \therefore p = \frac{1.4-1}{1} = 0.4$

Also from table $\Delta y_0 = 5, \Delta^2 y_0 = 2,\ \Delta^3 y_0 = 0$

Substituting these values in ①, we get

$$f(1.4) \equiv 6 + (0.4)(5) + \frac{0.4(0.4-1)}{2}(2) + 0 = 7.76$$

$ii.$ To find $f(0.9)$

$x_0 = 1,\ y_0 = 6\,, x = 0.9, h = 1 \therefore p = \frac{0.9-1}{1} = -0.1$

Also from table $\Delta y_0 = 5, \Delta^2 y_0 = 2,\ \Delta^3 y_0 = 0$

Substituting these values in ①, we get

$$f(0.9) \equiv 6 + (-0.1)(5) + \frac{-0.1(-0.1-1)}{2}(2) + 0 = 5.61$$

Example 2 From the following data, estimate the number of students who obtained marks between 40 and 45.

Marks	30–40	40–50	50–60	60–70	70–80
Number of Students	31	42	51	35	31

Solution: Preparing the cumulative frequency table,

Marks less than (x)	40	50	60	70	80
Number of Students (y_x)	31	73	124	159	190

Function has to be evaluated near the starting of the table, thereby constructing forward difference table for the function $y = f(x)$

Marks x	p	Number of Students y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$
less than 40	0	31				
			42			
less than 50	1	73		9		
			51		-25	
less than 60	2	124		-16		37
			35		12	
less than 70	3	159		-4		
			31			
less than 80	4	190				

Newton's forward interpolation formula given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$p = \frac{x - x_0}{h} \qquad \ldots ①$$

To find the number of students having marks less than 45 i.e. $f(45)$

$x_0 = 40,\ y_0 = 31,\ x = 45,\ h = 10 \therefore p = \frac{45-40}{10} = 0.5$

Also from table $\Delta y_0 = 42, \Delta^2 y_0 = 9,\ \Delta^3 y_0 = -25,\ \Delta^4 y_0 = 37$

Substituting these values in ①, we get

$$f(45) \equiv 31 + (0.5)(42) + \frac{0.5(0.5-1)}{2}(9) + \frac{0.5(0.5-1)(0.5-2)}{6}(-25) + \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(37)$$

$$\Rightarrow f(45) \equiv 31 + 21 - 1.125 - 1.5625 - 1.4453 = 47.87 \approx 48$$

$\therefore$ Number of students having marks less than 45 is 48.

Also number of students having marks less than 40 is 31. Hence number of students who obtained marks between 40 and 45 is $(48 - 31)$ i.e. 17

Example 3 Find the cubic polynomial with given set of points

x	0	1	2	3
$f(x)$	5	6	3	14

Hence or otherwise evaluate $f(0.5)$.

Solution: Constructing forward difference table for the function $y = f(x)$

x	p	y	Δ	Δ^2	Δ^3
0	0	5			
			1		
1	1	6		-4	
			-3		18
2	2	3		14	
			11		
3	3	14			

Newton's forward interpolation formula given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots,\ p = \frac{x - x_0}{h} \quad \ldots ①$$

$x_0 = 0,\ y_0 = 5,\ h = 1 \therefore p = \frac{x-0}{1} = x$

Also from table $\Delta y_0 = 1, \Delta^2 y_0 = -4,\ \Delta^3 y_0 = 18$

Substituting these values in ①, we get

$$f(x) \equiv 5 + x(1) + \frac{x(x-1)}{2!}(-4) + \frac{x(x-1)(x-2)}{3!}(18)$$

$$\Rightarrow f(x) \equiv 5 + x + (-2)x(x-1) + (3)x(x-1)(x-2)$$

$$\Rightarrow f(x) \equiv 3x^3 - 11x^2 + 9x + 5$$

Also $f(0.5) \equiv 3(0.5)^3 - 11(0.5)^2 + 9(0.5) + 5 = 7.125$

7.2.2 Newton's Backward Interpolation Formula

Newton's backward interpolation formula is used to interpolate the values of $y = f(x)$ near the end $(x < x_n)$ and to extrapolate the values when $(x > x_n)$, within the range of given data points $(x_i, y_i),\ i = 0,1,2,3, \ldots, n$.

Let $f(x)$ take the values $y_0,\ y_1,\ y_2, \ldots y_n$; for the independent variable x taking values $x_0,\ x_1,\ x_2, \ldots x_n$, where height of the interval (h) is fixed, such that $x_1 = x_0 + h,\ x_2 = x_0 + 2h, \ldots,\ x_n = x_0 + nh$.

Then to evaluate $f(x)$ for $x = x_n + ph, -1 < p < 1$

We have $f(x) = f(x_n + ph) = E^p f(x_n) \equiv (1 - \nabla)^{-p} y_n$

$\because E \equiv (1 - \nabla)^{-1}$ and $f(x_n) = y_n$

$$\Rightarrow f(x) \equiv \left(1 + p\nabla + \frac{p(p+1)}{2!}\nabla^2 + \frac{p(p+1)(p+2)}{3!}\nabla^3 + \cdots\right) y_n\,, \quad x = x_n + ph$$

$$\therefore f(x) \equiv y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots, p = \frac{x-x_n}{h}$$

This is Newton's backward interpolation formula and is used to interpolate or extrapolate values near the end of the table.

- p is the index which is 0, where $x = x_n$

Example4 Following table gives the census population of a state for the years 1971 to 2011. Estimate the population for the years 1974 and 2005 by using appropriate interpolation technique.

Year	1971	1981	1991	2001	2011
Population (Million)	46	66	81	93	101

Solution: Function has to be evaluated near the beginning and also near the end of the table, thereby constructing difference table for the function $y = f(x)$

Year x	Population $f(x)$	1^{st}diff	2^{nd}diff	3^{rd}diff	4^{th}diff
1971	46				
		20			
1981	66		-5		
		15		2	
1991	81		-3		-3
		12		-1	
2001	93		-4		
		8			
2011	101				

To calculate the population for the year 1974, using Newton's forward interpolation formula given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 \ldots$$

$$p = \frac{x - x_0}{h} \qquad \ldots ①$$

$x_0 = 1971,\ y_0 = 46\,, x = 1974, h = 10 \therefore p = \frac{1974-1971}{10} = 0.3$

Also from table $\Delta y_0 = 20, \Delta^2 y_0 = -5,\ \Delta^3 y_0 = 2,\ \Delta^4 y_0 = -3$

Substituting these values in ①, we get

$$f(1974) \equiv 46 + (0.3)(20) + \frac{0.3(0.3-1)}{2}(-5) + \frac{0.3(0.3-1)(0.3-2)}{6}(2) + \frac{0.3(0.3-1)(0.3-2)(0.3-3)}{24}(-3)$$

$$\equiv 46 + 6 + 0.525 + 0.119 + 0.1205 = 52.7645 \text{ Million}$$

To calculate the population for the year 2005, using Newton's backward interpolation formula given by:

$$f(x) \equiv y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n \cdots$$

$$p = \frac{x-x_n}{h} \qquad \ldots ②$$

$$x_n = 2011,\ y_n = 101,\ x = 2005,\ h = 10 \therefore p = \frac{2005-2011}{10} = -0.6$$

Also from table $\Delta y_n = 8, \Delta^2 y_n = -4,\ \Delta^3 y_n = -1,\ \Delta^4 y_n = -3$

Substituting these values in ②, we get

$$f(2005) \equiv 101 + (-0.6)(8) + \frac{-0.6(-0.6+1)}{2}(-4) + \frac{-0.6(-0.6+1)(-0.6+2)}{6}(-1)$$

$$\frac{-0.6(-0.6+1)(-0.6+2)(-0.6+3)}{24}(-3)$$

$$\equiv 101 - 4.8 + 0.48 + 0.056 + 0.1008 = 96.837 \text{ Million}$$

Example5 Given a set of points for the function $f(x)$, evaluate $f(2.8)$ and $f(3.5)$.

x	0	1	2	3
$f(x)$	1	2	1	10

Solution: Function has to be evaluated near the end of the table, thereby constructing backward difference table for the function $y = f(x)$

x	$f(x)$	∇	∇^2	∇^3
0	1			
		1		
1	2		-2	
		-1		12
2	1		10	
		9		
3	10			

Newton's backward interpolation formula given by:

$$f(x) \equiv y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \cdots,\ p = \frac{x-x_n}{h} \ldots ①$$

i. To find $f(2.8)$

$$x_n = 3,\ y_n = 10,\ x = 2.8,\ h = 1 \therefore p = \frac{2.8-3}{1} = -0.2$$

Also from table $\Delta y_n = 9, \Delta^2 y_n = 10,\ \Delta^3 y_n = 12$

Substituting these values in ①, we get

$$f(2.8) \equiv 10 + (-.2)(9) + \frac{-.2(-.2+1)}{2}(10) + \frac{-.2(-.2+1)(-.2+2)}{6}(12) = 6.824$$

ii. To find $f(3.5)$

$$x_n = 3,\ y_n = 10,\ x = 3.5,\ h = 1 \therefore p = \frac{3.5-3}{1} = 0.5$$

Also from table $\Delta y_n = 9, \Delta^2 y_n = 10,\ \Delta^3 y_n = 12$

Substituting these values in ①, we get

$$f(3.5) \equiv 10 + (0.5)(9) + \frac{.5(.5+1)}{2}(10) + \frac{.5(.5+1)(.5+2)}{6}(12) = 22$$

Example6 For the given set of values, evaluate $\cos 22^\circ$ and $\cos 73^\circ$, using suitable interpolation techniques.

x	10°	20°	30°	40°	50°	60°	70°	80°
$\cos x$	0.9848	0.9397	0.8660	0.7660	0.6428	0.5000	0.3420	0.1737

Solution: Function has to be evaluated near the beginning and also near the end of the table, thereby constructing difference table for the function $y = f(x)$

x	$y = \cos x$	1^{st}diff	2^{nd}diff	3^{rd}diff	4^{th}diff	5^{th}diff
10°	0.9848					
		−0.0451				
20°	0.9397		−0.0286			
		−0.0737		0.0023		
30°	0.8660		−0.0263		0.0008	
		−0.1000		0.0031		−0.0003
40°	0.7660		−0.0232		0.0005	
		−0.1232		0.0036		0.0003
50°	0.6428		−0.0196		0.0008	
		−0.1428		0.0044		−0.0003
60°	0.5000		−0.0152		0.0005	
		−0.1580		0.0049		
70°	0.3420		−0.0103			
		−0.1683				
80°	0.1737					

To calculate the value of $\cos 22^\circ$, taking $x_0 = 20^\circ$ as 22° is nearest to this point and applying Newton's forward interpolation formula given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$p = \frac{x - x_0}{h} \qquad \text{...①}$$

$$x_0 = 20^\circ,\ y_0 = 0.9397,\ x = 22^\circ,\ h = 10^\circ \therefore p = \frac{22^\circ - 20^\circ}{10^\circ} = 0.2$$

Also from table $\Delta y_0 = -0.0737, \Delta^2 y_0 = -0.0263,\ \Delta^3 y_0 = 0.0031,$

$\Delta^4 y_0 = 0.0005,\ \Delta^5 y_0 = 0.0003$

Substituting these values in ①, we get

$$f(22^\circ) \equiv .9397 + (.2)(-.0737) + \frac{.2(.2-1)}{2}(-.0263) + \frac{.2(.2-1)(.2-2)}{6}(.0031) + \frac{.2(.2-1)(.2-2)(.2-3)}{24}(.0005) + \frac{.2(.2-1)(.2-2)(.2-3)(.2-4)}{120}(.00003)$$

$$\equiv 0.9397 - 0.0147 + 0.0021 + 0.0002 - 0.00002 + 0.000001 = 0.9272$$

To calculate the value of cos 73°, taking $x_n = 70^{\circ}$as 73° is nearest to this point and applying Newton's backward interpolation formula given by:

$$f(x) \equiv y_n + p\nabla y_n + \frac{p(p+1)}{2!}\nabla^2 y_n + \frac{p(p+1)(p+2)}{3!}\nabla^3 y_n + \frac{p(p+1)(p+2)(p+3)}{4!}\nabla^4 y_n \ldots$$

$$p = \frac{x - x_n}{h} \qquad \ldots②$$

$$x_n = 70^{\circ},\ y_n = 0.3420\ , x = 73^{\circ}, h = 10^{\circ} \therefore p = \frac{73^{\circ} - 70^{\circ}}{10^{\circ}} = 0.3$$

Also from table $\Delta y_n = -0.158, \Delta^2 y_n = -.0152, \Delta^3 y_n = 0.0044,$
$\Delta^4 y_n = 0.0008,\ \Delta^5 y_n = 0.0003$

Substituting these values in ②, we get

$$f(73^{\circ}) \equiv 0.3420 + (0.3)(-.158) + \frac{.3(.3+1)}{2}(-.0152) + \frac{.3(.3+1)(.3+2)}{6}(.0044) +$$
$$\frac{.3(.3+1)(.3+2)(.3+3)}{24}(.0008) + \frac{.3(.3+1)(.3+2)(.3+3)(.3+4)}{120}(.0003)$$
$$\equiv 0.3420 - 0.0474 - 0.003 + .0007 + .0001 + .00003 = 0.2924$$

7.2.3 Gauss's Forward and Backward Difference Formulae

Gauss central difference formula is used to interpolate the values of y near the middle of the table.

Newton's forward difference formula is given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^4 y_0 + \cdots$$

$$p = \frac{x - x_0}{h} \quad \ldots①$$

Now $\Delta^3 y_{-1} = \Delta^2 y_0 - \Delta^2 y_{-1}$

$$\Rightarrow \Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1} \qquad \ldots②$$

Similarly $\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1}$ ...③

$$\Delta^4 y_0 = \Delta^4 y_{-1} + \Delta^5 y_{-1} \qquad \ldots④$$

Substituting $\Delta^2 y_0, \Delta^3 y_0, \Delta^4 y_0$ from ② ③ ④ in ①, we get

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) +$$
$$\frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

Rewriting by collecting the coefficients of $\Delta^2 y_{-1}, \Delta^3 y_{-1}, \Delta^4 y_{-1}$ ..., we get

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} +$$
$$\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \cdots \qquad \ldots⑤$$

Again $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ ...⑥

Using ⑥in ⑤, we get

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} +$$
$$\frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-2} + \cdots \qquad \ldots⑦$$

Expression given by ⑦is known as **Gauss forward interpolation formula**

Again $\Delta^2 y_{-1} = \Delta y_0 - \Delta y_{-1}$

$$\Rightarrow \Delta y_0 = \Delta y_{-1} + \Delta^2 y_{-1} \qquad \ldots ②'$$

Similarly $\Delta^2 y_0 = \Delta^2 y_{-1} + \Delta^3 y_{-1}$ ... ③′

$$\Delta^3 y_0 = \Delta^3 y_{-1} + \Delta^4 y_{-1} \qquad \ldots ④'$$

Substituting $\Delta y_0, \Delta^2 y_0, \Delta^3 y_0$ from ②′, ③′ , ④′ in ①, we get

$$f(x) \equiv y_0 + p(\Delta y_{-1} + \Delta^2 y_{-1}) + \frac{p(p-1)}{2!}(\Delta^2 y_{-1} + \Delta^3 y_{-1}) + \frac{p(p-1)(p-2)}{3!}(\Delta^3 y_{-1} + \Delta^4 y_{-1}) + \frac{p(p-1)(p-2)(p-3)}{4!}(\Delta^4 y_{-1} + \Delta^5 y_{-1}) + \cdots$$

Rewriting by collecting the coefficients of $\Delta y_{-1}, \Delta^2 y_{-1}, \Delta^3 y_{-1}, \Delta^4 y_{-1}$, we get

$$f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \frac{(p+1)p(p-1)(p-2)}{4!}\Delta^4 y_{-1} + \frac{p(p-1)(p-2)(p-3)}{4!}\Delta^5 y_{-1} + \cdots \qquad \ldots ⑤'$$

Again $\Delta^3 y_{-1} = \Delta^3 y_{-2} + \Delta^4 y_{-2}$ and $\Delta^4 y_{-1} = \Delta^4 y_{-2} + \Delta^5 y_{-2}$ ... ⑥′

Using ⑥′ in ⑤′, we get

$$f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}(\Delta^3 y_{-2} + \Delta^4 y_{-2}) + \frac{(p+1)p(p-1)(p-2)}{4!}(\Delta^4 y_{-2} + \Delta^5 y_{-2}) + \cdots$$

$$\Rightarrow f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} + \frac{(p+2)(p+1)p(p-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad \ldots ⑦'$$

Expression given by ⑦′ is known as **Gauss backward interpolation formula**

Example7 Given a set of points for the function $y = f(x)$, evaluate $f(33)$ using
$i.$ Gauss's forward $ii.$ Gauss's backward interpolation formulae

x	25	30	35	40
$f(x)$	0.25	0.3	0.33	0.37

Solution: Function has to be evaluated near centre of the table, thereby constructing difference table for the function $y = f(x)$, taking $x_0 = 30$

x	p	$f(x)$	Δ	Δ^2	Δ^3
25	-1	0.25			
			0.05		
<u>30</u>	0	<u>0.3</u>		-0.02	
			0.03		0.03
35	1	0.33		0.01	
			0.04		
40	2	0.37			

$i.$ Gauss forward interpolation formula is given by

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-1} + \cdots \ , \ p = \frac{x - x_0}{h} \quad \ldots ①$$

To find $f(33)$, $x_0 = 30,\ y_0 = 0.3$, $x = 33,\ h = 5 \therefore p = \frac{33-30}{5} = 0.6$

Also $\Delta^n y_0$, $n = 1, 2$ lie along the dotted line as shown

$\therefore$ From table $\Delta y_0 = 0.03, \Delta^2 y_{-1} = -0.02,\ \Delta^3 y_{-1} = 0.03$

Substituting these values in ①, we get

$$f(33) \equiv 0.3 + (0.6)(0.03) + \frac{0.6(0.6-1)}{2}(-0.02) + \frac{(0.6+1)0.6(0.6-1)}{6}(0.03)$$

$$\Rightarrow f(33) \equiv 0.3185$$

$ii.$ Gauss backward interpolation formula is given by

$$f(x) \equiv y_0 + p\Delta y_{-1} + \frac{(p+1)p}{2!}\Delta^2 y_{-1} + \frac{(p+1)p(p-1)}{3!}\Delta^3 y_{-2} \ , \ p = \frac{x - x_0}{h} \quad \ldots ②$$

To find $f(33)$, $x_0 = 30,\ y_0 = 0.3$, $x = 33,\ h = 5 \therefore p = \frac{33-30}{5} = 0.6$

Also $\Delta^n y_0$, $n = 1, 2$ lie along the dotted line as shown

$\therefore$ From table $\Delta y_{-1} = 0.05, \Delta^2 y_{-1} = -0.02,\ \Delta^3 y_{-2} = 0$

Substituting these values in ②, we get

$$f(33) \equiv 0.3 + (0.6)(0.05) + \frac{0.6(0.6-1)}{2}(-0.02) + \frac{(0.6+1)0.6(0.6-1)}{6}(0)$$

$$\Rightarrow f(33) \equiv 0.3324$$

7.2.4 Stirling's Central Difference Formula

Stirling gave the most general formula for interpolating values near the centre of the table by taking mean of Gauss forward and Gauss backward interpolation formulae.

Taking mean of expressions given by ⑦ and ⑦$'$ respectively, we get

$$f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \left(\frac{p(p-1)}{2!} + \frac{(p+1)p}{2!}\right)\frac{\Delta^2 y_{-1}}{2} + \frac{(p+1)p(p-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right) +$$
$$\left(\frac{(p+1)p(p-1)(p-2)}{4!} + \frac{(p+2)(p+1)p(p-1)}{4!}\right)\frac{\Delta^2 y_{-2}}{2} + \cdots$$

$$\Rightarrow f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+ \frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \quad \ldots ⑧$$

Expression given in ⑧ is known as **Stirling's central difference formula**

Putting $\frac{1}{2}(\Delta y_0 + \ \Delta y_{-1}) = \frac{1}{2}\left(\delta y_{\frac{1}{2}} + \ \delta y_{-\frac{1}{2}}\right) = \mu\delta y_0$

$$\frac{1}{2}(\Delta^3 y_{-1} + \Delta^3 y_{-2}) = \frac{1}{2}\left(\delta^3 y_{\frac{1}{2}} + \delta^3 y_{-\frac{1}{2}}\right) = \mu\delta^3 y_0$$

$\vdots$

In terms of central differences, ⑧ takes the form

$$f(x) \equiv y_0 + p\mu\delta y_0 + \frac{p^2}{2!}\delta^2 y_0 + \frac{p(p^2-1^2)}{3!}\mu\delta^3 y_0 + \frac{p^2(p^2-1^2)}{4!}\delta^4 y_0 + \cdots$$

This is another form of **Stirling's central difference formula.**

The difference table used to evaluate $f(x)$ as per Stirling's formula, is shown below. Column wise averages of differences (shown in boxes) are taken while evaluation phase.

x	p	$f(x)$	1^{st}diff	2^{nd}diff	3^{rd}diff	4^{th}diff
x_{-2}	-2	y_{-2}				
			Δy_{-2}			
x_{-1}	-1	y_{-1}				
			$\boxed{\Delta y_{-1}}$		$\boxed{\Delta^3 y_{-2}}$	
$\underline{x_0}$	0	$\underline{y_0}$		$\boxed{\Delta^2 y_{-1}}$		$\boxed{\Delta^4 y_{-2}}$
			$\boxed{\Delta y_0}$		$\boxed{\Delta^3 y_{-1}}$	
x_1	1	y_1		$\Delta^2 y_0$		
			Δy_1			
x_2	2	y_2				

Example8 Given a set of points for the function $y = f(x)$, evaluate $f(33)$ using Stirling's central difference formula.

x	25	30	35	40	45
$f(x)$	0.25	0.3	0.33	0.37	0.43

Solution: Function has to be evaluated near centre of the table, thereby constructing difference table for the function $y = f(x)$, taking $x_0 = 35$. Also $\Delta^n y_0$, $n = 1, 2$ lie along the dotted line as shown.

x	p	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
25	-2	0.25				
			0.05			
30	-1	0.3		-0.02		
			$\boxed{0.03}$		$\boxed{0.03}$	
$\underline{35}$	0	$\underline{0.33}$		$\boxed{0.01}$		$\boxed{-0.02}$
			$\boxed{0.04}$		$\boxed{0.01}$	
40	1	0.37		0.02		
			0.06			
45	2	0.43				

Stirling's central differences formula is given by

$$f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad , \ p = \frac{x - x_0}{h} \quad \ldots ①$$

To find $f(33)$, $x_0 = 35,\ y_0 = 0.33$, $x = 33,\ h = 5 \therefore p = \frac{33-35}{5} = -0.4$,
Also from the table $\Delta y_0 = 0.04$, $\Delta y_{-1} = 0.03$ $\Delta^2 y_{-1} = 0.01$, $\Delta^3 y_{-1} = 0.01$, $\Delta^3 y_{-2} = 0.03$, $\Delta^4 y_{-2} = -0.02$. All the positions have been shown, enclosed in boxes.

Substituting these values in ①, we get

$$f(33) \equiv .33 + (-.4)\left(\frac{.04 + .03}{2}\right) + \frac{(-.4)^2}{2}(.01) + \frac{-.4((-.4)^2-1)}{6}\left(\frac{.01 + .03}{2}\right)$$

$$+\frac{(-.4)^2((-.4)^2-1)}{24}(-0.02)$$

$$\Rightarrow f(33) \equiv 0.33 - 0.014 + 0.0008 + 0.0011 + 0.0001 = 0.318$$

Example9 Use Stirling's formula to evaluate $f(1.22)$ given that

x	1.0	1.1	1.2	1.3	1.4
$f(x)$	0.841	0.891	0.932	0.963	0.985

Solution: Function has to be evaluated near centre of the table, thereby constructing difference table for the function $y = f(x)$, taking $x_0 = 1.2$. Also $\Delta^n y_0$, $n = 1, 2$ lie along the dotted line as shown.

x	p	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
1.0	-2	0.841				
			0.050			
1.1	-1	0.891		-0.009		
			$\boxed{0.041}$		$\boxed{-0.001}$	
$\underline{1.2}$	0	$\underline{0.932}$		$\boxed{-0.01}$		$\boxed{0.002}$
			$\boxed{0.031}$		$\boxed{0.001}$	
1.3	1	0.963		-0.009		
			0.022			
1.4	2	0.985				

Stirling's central differences formula is given by

$$f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$

$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad , \ p = \frac{x - x_0}{h} \quad \ldots ①$$

To find $f(1.22)$, $x_0 = 1.2,\ y_0 = 0.932$, $x = 1.22$, $h = 0.1 \therefore p = \frac{1.22-1.2}{0.1} = 0.2$

Also from the table $\Delta y_0 = 0.031$, $\Delta y_{-1} = 0.041$, $\Delta^2 y_{-1} = -0.01$,

$\Delta^3 y_{-1} = 0.001,\ \Delta^3 y_{-2} = -0.001,\ \Delta^4 y_{-2} = 0.002$. All the positions have been shown, enclosed in boxes.

Substituting these values in ①, we get

$$f(1.22) \equiv .932 + (0.2)\left(\frac{.031+.041}{2}\right) + \frac{(.2)^2}{2}(-.01) + \frac{.2((.2)^2-1)}{6}\left(\frac{.001-.001}{2}\right)$$
$$+\frac{(0.2)^2((0.2)^2-1)}{24}(0.002)$$

$$\Rightarrow f(1.22) \equiv 0.932 + 0.0072 - 0.0002 + 0 - 0.0000032 = 0.9390$$

Example10 Given following data $f(x) = 10^5 u_x$, for values of x in degrees, where $u_x = 1 + \log\sin x$. Use Stirling's formula to compute $u_{11.8^\circ}$.

x°	10	11	12	13	14
$f(x) = \mathbf{10^5 u_x}$	23,967	28,060	31,788	35,209	38,368

Solution: Function has to be evaluated near centre of the table, thereby constructing difference table for the function $y = f(x)$, taking $x_0 = 12$. Also $\Delta^n y_0$, $n = 1$, 2 lie along the dotted line as shown.

x°	p	$f(x) = \mathbf{10^5 u_x}$	Δ	Δ^2	Δ^3	Δ^4
10	-2	23967				
			4093			
11	-1	28060		-365		
			$\boxed{3728}$		$\boxed{58}$	
12	0	31788		$\boxed{-307}$		$\boxed{-13}$
			$\boxed{3421}$		$\boxed{45}$	
13	1	35209		-262		
			3159			
14	2	38368				

Stirling's central differences formula is given by

$$f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad ,\ p = \frac{x - x_0}{h} \quad \ldots ①$$

To find $f(11.8^\circ)$

$x_0 = 12,\ y_0 = 31788$, $x = 11.8,\ h = 1,\ \therefore p = \frac{11.8-12}{1} = -0.2$

Also from the table $\Delta y_0 = 3421$, $\Delta y_{-1} = 3728,\ \Delta^2 y_{-1} = -307$,

$\Delta^3 y_{-1} = 45,\ \Delta^3 y_{-2} = 58,\ \Delta^4 y_{-2} = -13$. All the positions have been shown, enclosed in boxes.

Substituting these values in ①, we get

$$f(11.8^\circ) \equiv 31788 + (-0.2)\left(\frac{3421+3728}{2}\right) + \frac{(-0.2)^2}{2}(-307) +$$

$$\frac{-0.2((-0.2)^2-1)}{6}\left(\frac{45+58}{2}\right)+\frac{(-0.2)^2((-0.2)^2-1)}{24}(-13)$$

$$\Rightarrow f(11.8^\circ) \equiv 31788 + -714.9 - 6.14 + 1.648 + 0.0208 = 31068.6288$$

$$\therefore f(11.8^\circ) = 10^5 u_{11.8^\circ} \equiv 31068.6288$$

$$\Rightarrow u_{11.8^\circ} \equiv 31068.6288(10^{-5}) = 0.31069$$

Example11 Use Stirling's formula to evaluate $\sin 57^\circ$, given that $\sin 45^\circ = .7071,\ \sin 50^\circ = .7660,\ \sin 55^\circ = .8192.,\ \sin 60^\circ = .8660,\ \sin 65^\circ = .9063$

Also compare the results by evaluating $\sin 57^\circ$using Newton's forward interpolation formula.

Solution: Function has to be evaluated near centre of the table, thereby constructing difference table for the function $y = \sin x$, taking $x_0 = 55$. Also $\Delta^n y_0$, $n = 1, 2$ lie along the dotted line as shown.

x	p	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
45°	-2	0.7071				
			0.0589			
50°	-1	0.7660		-0.0057		
			$\boxed{0.0532}$		$\boxed{-0.0007}$	
$\underline{55^\circ}$	0	$\underline{0.8192}$		$\boxed{-0.0064}$		$\boxed{0.0006}$
			$\boxed{0.0468}$		$\boxed{-0.0001}$	
60°	1	0.8660		-0.0065		
			0.0403			
65°	2	0.9063				

Stirling's central differences formula is given by

$$f(x) \equiv y_0 + p\left(\frac{\Delta y_0 + \Delta y_{-1}}{2}\right) + \frac{p^2}{2!}\Delta^2 y_{-1} + \frac{p(p^2-1)}{3!}\left(\frac{\Delta^3 y_{-1} + \Delta^3 y_{-2}}{2}\right)$$
$$+\frac{p^2(p^2-1)}{4!}\Delta^4 y_{-2} + \cdots \qquad , \ p = \frac{x - x_0}{h} \quad \ldots ①$$

To find $f(57^\circ)$, $x_0 = 55^\circ$, $y_0 = 0.8192$, $x = 57^\circ$, $h = 5^\circ$ $\therefore p = \frac{57^\circ - 55^\circ}{5^\circ} = 0.4$

Also from the table $\Delta y_0 = 0.0468$, $\Delta y_{-1} = 0.0532$, $\Delta^2 y_{-1} = -0.0064$,

$\Delta^3 y_{-1} = -0.0001$, $\Delta^3 y_{-2} = -0.0007$, $\Delta^4 y_{-2} = 0.0006$. All the positions have been shown, enclosed in boxes.

Substituting these values in ①, we get

$$f(57^\circ) \equiv 0.8192 + (0.4)\left(\frac{0.0468+0.0532}{2}\right) + \frac{(.4)^2}{2}(-0.0064) +$$
$$\frac{0.4((0.4)^2-1)}{6}\left(\frac{-0.0001-0.0007}{2}\right) + \frac{(0.4)^2((0.4)^2-1)}{24}(0.0006)$$

$$\Rightarrow f(57^\circ) \equiv 0.8192 + 0.02 - 0.000512 + 0.0000224 - 0.00000336$$

$\therefore f(57^\circ) \equiv 0.83870704$ using Stirling's central differences formula.

To evaluate the value of $\sin 57^\circ$, taking $x_0 = 55^\circ$ as 57° is nearest to this point and applying Newton's forward interpolation formula given by:

$$f(x) \equiv y_0 + p\Delta y_0 + \frac{p(p-1)}{2!}\Delta^2 y_0 + \frac{p(p-1)(p-2)}{3!}\Delta^3 y_0 + \cdots,\ p = \frac{x - x_0}{h} \qquad \ldots②$$

$$x_0 = 55^\circ,\ y_0 = 0.8192\ ,\ x = 57^\circ,\ h = 10^\circ \ \therefore p = \frac{57^\circ - 55^\circ}{5^\circ} = 0.4$$

Also from table $\Delta y_0 = 0.0468\ ,\ \Delta^2 y_0 = -0.0065,\ \Delta^3 y_0 = 0, \ldots$

Substituting these values in ②, we get

$$f(57^\circ) \equiv 0.8192 + (0.4)(0.0468) + \frac{0.4(0.4-1)}{2}(-.0065) + 0$$

$$\therefore f(57^\circ) \equiv 0.8192 + 0.01872 + 0.00078 = 0.8387$$

7.3 Interpolation with Unequal Intervals

For the function $y = f(x)$, with given discrete set of points (x_i, y_i), $i = 0,1,2,3,\ldots,n$, where x_i's are not equispaced, common methods of interpolating data points are listed below:

i. Newton's divided difference formula
ii. Lagrange's interpolation formula

7.3.1 Newton's Divided Difference Method

Let the function $y = f(x)$ take the values $f(x_0) = y_0$, $f(x_1) = y_1$, $f(x_2) = y_2$,..., $f(x_n) = y_n$; for the argument x taking values $x_0,\ x_1,\ x_2$, ..., x_n, which are not equally spaced. Divided difference may be defined as the difference between two successive values of the ordinates divided by the difference between the corresponding values of the abscissa.

So the first divided difference denoted by $[x_0, x_1]$ or $f(x_0, x_1)$ is defined as:

$$f(x_0, x_1) \equiv [x_0, x_1] = \frac{y_1 - y_o}{x_1 - x_o} = \frac{f(x_1) - f(x_0)}{x_1 - x_o}$$

Similarly $$f(x_1, x_2) \equiv [x_1, x_2] = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$\vdots$$

Second divided differences, denoted by $[x_0, x_1, x_2]$ or $f(x_0, x_1, x_2)$ are defined as:

$$f(x_0, x_1, x_2) \equiv [x_0, x_1, x_2] = \frac{[x_1, x_2] - [x_0, x_1]}{x_2 - x_0}$$

$$f(x_1, x_2, x_3) \equiv [x_1, x_2, x_3] = \frac{[x_2, x_3] - [x_1, x_2]}{x_3 - x_1}$$

$$\vdots$$

Third divided differences, denoted by $[x_0, x_1, x_2]$ or $f(x_0, x_1, x_2)$ are defined as:

$$f(x_0, x_1, x_2, x_3) \equiv [x_0, x_1, x_2, x_3] = \frac{[x_1, x_2, x_3] - [x_0, x_1, x_2]}{x_3 - x_0}$$ and so on.

Remark: Divided differences are symmetrical in arguments,

i.e. $[x_r, x_s] = [x_s, x_r]$ or $f(x_r, x_s) = f(x_s, x_r)$

7.3.1.1 Newton's Divided Difference Formula

By definition of divided differences $f(x, x_0) = \frac{f(x_0)-f(x)}{x_o-x} = \frac{f(x)-f(x_0)}{x-x_o}$

$$\Rightarrow f(x) = f(x_0) + (x - x_o)f(x, x_0) \qquad \text{...①}$$

Also $f(x, x_0, x_1) = \frac{f(x_0,x_1)-f(x,x_0)}{x_1-x} = \frac{f(x,x_0)-f(x_0,x_1)}{x-x_1}$

$$\Rightarrow f(x,\ x_0) = f(x_0,\ x_1) + (x - x_1)f(x, x_0, x_1) \qquad \text{...②}$$

Similarly $f(x,\ x_0, x_1\ ) = f(x_0,\ x_1, x_2) + (x - x_2)f(x, x_0, x_1, x_2)$ ...③

$$\vdots$$

$f(x,\ x_0, x_1, \dots x_n\ ) = f(x_0,\ x_1, x_2, \dots x_n) + (x - x_n)f(x, x_0, x_1, x_2, \dots x_n)$ ... ④

Multiplying ② by $(x - x_0)$, ③ by$(x - x_0)(x - x_1)$, ...

④ by $(x - x_0)(x - x_1) \dots (x - x_{n-1})$ and adding to equation ①, we get

$$f(x) \equiv f(x_0) + (x - x_0)f(x_0,\ x_1) + \ (x - x_0)(x - x_1)f(x_0,\ x_1, x_2) + \cdots$$
$$+(x - x_0)(x - x_1) \dots (x - x_{n-1})(x - x_n)f(x_0, x_1, x_2, \dots x_n) + \ R^n$$

Where R^n denotes remainder terms which vanish being $(n + 1)^{th}$ order divided differences.

∴ Newton's divided difference formula is given by:

$$f(x) \equiv f(x_0) + (x - x_0)f(x_0,\ x_1) + \ (x - x_o)(x - x_1)f(x_0,\ x_1, x_2) +$$
$$(x - x_0)(x - x_1)(x - x_2)f(x_0,\ x_1, x_2,\ x_3) + \cdots$$

Example12 Estimate $f(2)$ from the following data, using Newton's divided differences method.

x	0	1	3	6
y	1	3	55	343

Solution: The divided difference table is given as follows:

x	y	1^{st}diff	2^{nd}diff	3^{rd}diff
0	1			
		$\frac{3-1}{1-0} = 2$		
1	3		$\frac{26-2}{3-0} = 8$	
		$\frac{55-3}{3-1} = 26$		$\frac{14-8}{6-0} = 1$
3	55		$\frac{96-26}{6-1} = 14$	
		$\frac{343-55}{6-3} = 96$		
6	343			

Newton's divided difference formula is given by:

$$f(x) \equiv f(x_0) + (x - x_0)f(x_0,\ x_1) + \ (x - x_o)(x - x_1)f(x_0,\ x_1, x_2) +$$
$$(x - x_0)(x - x_1)(x - x_2)f(x_0,\ x_1, x_2,\ x_3) + \cdots \qquad \text{...①}$$

Here $x = 2,\ x_0 = 0\ ,\ x_1 = 1, x_2 = 3\ , f(x_0) = 1, f(x_0,\ x_1) = 2\ ,$ $f(x_0,\ x_1, x_2) = 8,\ f(x_0,\ x_1, x_2,\ x_3) = 1$

Substituting these values in ①, we get

$$f(2) \equiv 1 + (2-0)(2) + \ (2-0)(2-1)(8) + (2-0)(2-1)(2-3)(1)$$

$$\therefore f(2) \equiv 1 + 4 + 16 - 2 = 19$$

Example13 Find $\log_{10} 656$, given that $\log_{10} 654 = 2.8156$, $\log_{10} 658 = 2.8182$, $\log_{10} 659 = 2.8189$, $\log_{10} 661 = 2.8202$.

Solution: The divided difference table is given as follows:

x	$f(x) = \log_{10} x$	1^{st}diff	2^{nd}diff	3^{rd}diff
654	2.8156			
		$\frac{2.8182 - 2.8156}{658 - 654} = .00065$		
			$\frac{.0007 - .00065}{659 - 654} = .00001$	
658	2.8182			
		$\frac{2.8189 - 2.8182}{659 - 658} = .0007$		$\frac{-.00002 - .00001}{661 - 654} = -.000004$
			$\frac{.00065 - .0007}{661 - 658} = -.00002$	
659	2.8189			
		$\frac{2.8202 - 2.8189}{661 - 659} = .00065$		
661	2.8202			

Newton's divided difference formula is given by:

$$f(x) \equiv f(x_0) + (x - x_0)f(x_0,\ x_1) + \ (x - x_0)(x - x_1)f(x_0,\ x_1, x_2) +$$
$$(x - x_0)(x - x_1)(x - x_2)f(x_0,\ x_1, x_2,\ x_3) + \cdots \qquad \text{...①}$$

Here $x = 656$, $x_o = 654$, $x_1 = 658$, $x_2 = 659$, $f(x_0) = 2.8156$, $f(x_0,\ x_1) = .00065,\ f(x_0,\ x_1, x_2) = .00001, f(x_0,\ x_1, x_2,\ x_3) = -.000004$

Substituting these values in ①, we get

$$f(656) \equiv 2.8156 + (656 - 654)(.00065) + \ (656 - 654)(656 - 658).$$
$$(.00001) + (656 - 654)(656 - 658)(656 - 659)(-.00004)$$

$$\therefore f(656) = \log_{10} 656 = 2.8168$$

Example14 Employing Newton's divided difference interpolation formula, estimate $f(x)$ from the following data:

x	0	1	2	4	5	6
$f(x)$	1	14	15	5	6	9

Hence or otherwise find $f(2.5)$.

Solution: Constructing divided difference table for the function $y = f(x)$

x	$f(x)$	1^{st}diff	2^{nd}diff	3^{rd}diff	4^{th}diff
0	1				
		$\frac{14-1}{1-0} = 13$			
1	14		$\frac{1-13}{2-0} = -6$		
		$\frac{15-14}{2-1} = 1$		$\frac{-2+6}{4-0} = 1$	
2	15		$\frac{-5-1}{4-1} = -2$		$\frac{1-1}{5-0} = 0$
		$\frac{5-15}{4-2} = -5$		$\frac{2+2}{5-1} = 1$	
4	5		$\frac{1+5}{5-2} = 2$		$\frac{1-1}{6-1} = 0$
		$\frac{6-5}{5-4} = 1$		$\frac{6-2}{6-2} = 1$	
5	6		$\frac{13-1}{6-4} = 6$		
		$\frac{19-6}{6-5} = 13$			
6	19				

Newton's divided difference formula is given by

$$f(x) \equiv f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_o)(x-x_1)f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \cdots \quad \ldots ①$$

Here $x_0 = 0$, $x_1 = 1, x_2 = 2$, $f(x_0) = 1$, $f(x_0, x_1) = 13$, $f(x_0, x_1, x_2) = -6$, $f(x_0, x_1, x_2, x_3) = 1, f(x_0, x_1, x_2, x_3, x_4) = 0$

Substituting these values in ①, we get

$$f(x) \equiv 1 + (x-0)(13) + (x-0)(x-1)(-6) + (x-0)(x-1)(x-2)(1) + 0$$

$$\therefore f(x) \equiv 1 + 13x - 6(x^2 - x) + (x^3 - 3x^2 + 2x)$$

$$\Rightarrow f(x) \equiv x^3 - 9x^2 + 21x + 1$$

Also $f(2.5) = (5.5)^3 - 9(5.5)^2 + 21(5.5) + 1 = 12.875$

7.3.2 Lagrange's Interpolation Formula

Let $\boldsymbol{y = f(x)}$ take the values $\boldsymbol{y_0}$, $\boldsymbol{y_1}$, $\boldsymbol{y_2}$, ..., $\boldsymbol{y_n}$; for the argument $\boldsymbol{x}$ taking values $\boldsymbol{x_0}$, $\boldsymbol{x_1}$, $\boldsymbol{x_2}$, ..., $\boldsymbol{x_n}$, then the polynomial by Lagrange's interpolation formula is given by

$$\boldsymbol{f(x) = \sum_{i=0}^{n} L_i\, y_i = L_0 y_0 + L_1 y_1 + L_2 y_2 + \cdots + L_n y_n}$$

where $L_0 = \frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}$

$$L_1 = \frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}$$

$$\vdots$$

$$L_n = \frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\ldots(x_n-x_{n-1})}$$

$$\therefore f(x) = \left(\frac{(x-x_1)(x-x_2)\ldots(x-x_n)}{(x_0-x_1)(x_0-x_2)\ldots(x_0-x_n)}\right) y_0 + \left(\frac{(x-x_0)(x-x_2)\ldots(x-x_n)}{(x_1-x_0)(x_1-x_2)\ldots(x_1-x_n)}\right) y_1 +$$
$$\ldots + \left(\frac{(x-x_0)(x-x_1)\ldots(x-x_{n-1})}{(x_n-x_0)(x_n-x_2)\ldots(x_n-x_{n-1})}\right) y_n$$

Remarks:

- This formula can be used irrespective of whether the values x_0, x_1, x_2, ..., x_n are equispaced or not.
- It is easy to remember but cumbersome to apply.

Example15 Estimate $f(10)$ from the following data, using Lagrange's interpolation formula.

x	5	6	9	11
y	12	13	14	16

Solution: By Lagrange's interpolation formula

$$f(x) = \sum_{i=0}^{3} L_i\, y_i = L_0 y_0 + L_1 y_1 + L_2 y_2 + L_3 y_3$$

$$\Rightarrow f(x) = \left(\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right) y_0 + \left(\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right) y_1 +$$
$$\left(\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right) y_2 + \left(\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right) y_3$$

Putting $x = 10$ and remaining values from given data

$$f(10) = \frac{(10-6)(10-9)(10-11)}{(5-6)(5-9)(5-11)}(12) + \frac{(10-5)(10-9)(10-11)}{(6-5)(6-9)(6-11)}(13)$$
$$+ \frac{(10-5)(10-6)(10-11)}{(9-5)(9-6)(9-11)}(14) + \frac{(10-5)(10-6)(10-9)}{(11-5)(11-6)(11-9)}(16)$$

$$\Rightarrow f(10) = 2 - 4.3333 + 11.6667 + 5.3333 = 14.6667$$

Example16 Find the polynomial of the lowest possible degree which assumes the values $-21, 15, 12, 3$ for x taking the values $-1, 1, 2, 3$ respectively, using Newton's divided difference formula and hence find $f(1.5)$.

Compare the results by finding $f(1.5)$ using Lagrange's interpolation formula.

Solution: i. The divided difference table is given as follows:

x	y	1^{st}diff	2^{nd}diff	3^{rd}diff
-1	-21			
		$\frac{15+21}{1+1}=18$		
1	15		$\frac{-3-18}{2+1}=-7$	
		$\frac{12-15}{2-1}=-3$		$\frac{-3+7}{3+1}=1$
2	12		$\frac{-9+3}{3-1}=-3$	
		$\frac{3-12}{3-2}=-9$		
3	3			

Newton's divided difference formula is given by:

$$f(x) \equiv f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) +$$
$$(x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3) + \cdots \qquad \ldots①$$

Here $x_0 = -1$, $x_1 = 1, x_2 = 2$, $f(x_0) = -21, f(x_0, x_1) = 18$, $f(x_0, x_1, x_2) = -7$, $f(x_0, x_1, x_2, x_3) = 1$

Substituting these values in ①, we get

$$f(x) \equiv -21 + (x+1)(18) + (x+1)(x-1)(-7)$$
$$+(x+1)(x-1)(x-2)(1) + 0$$
$$\therefore f(x) \equiv 1 + 13x - 6(x^2 - x) + (x^3 - 3x^2 + 2x)$$
$$\Rightarrow f(x) \equiv x^3 - 9x^2 + 17x + 6$$

Also $f(1.5) \equiv (1.5)^3 - 9(1.5)^2 + 17(1.5) + 6 = 14.625$

ii. To find $f(1.5)$ using Lagrange's interpolation formula:

$$f(x) = \sum_{i=0}^{3} L_i\, y_i = L_0y_0 + L_1y_1 + L_2y_2 + L_3y_3$$
$$\Rightarrow f(x) = \left(\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}\right)y_0 + \left(\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}\right)y_1 +$$
$$\left(\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}\right)y_2 + \left(\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}\right)y_3$$
$$\Rightarrow f(1.5) = \left(\frac{(1.5-1)(1.5-2)(1.5-3)}{(-1-1)(-1-2)(-1-3)}\right)(-21) + \left(\frac{(1.5+1)(1.5-2)(1.5-3)}{(1+1)(1-2)(1-3)}\right)(15) +$$
$$\left(\frac{(1.5+1)(1.5-1)(1.5-3)}{(2+1)(2-1)(2-3)}\right)(12) + \left(\frac{(1.5+1)(1.5-1)(1.5-2)}{(3+1)(3-1)(3-2)}\right)(3)$$
$$\therefore f(1.5) = 0.328125 + 7.03125 + 7.5 - 0.234375 = 14.625$$

Example16 Find the polynomial of the lowest degree which assumes the values 1, 27, 64 for x taking the values1, 3, 4 respectively, using Lagrange's interpolation formula and hence find $f(2)$.

Solution: To find $f(x)$ using Lagrange's interpolation formula:

$$f(x) = \sum_{i=0}^{3} L_i\, y_i = L_0 y_0 + L_1 y_1 + L_2 y_2$$

$$\Rightarrow f(x) = \left(\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\right) y_0 + \left(\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\right) y_1 + \left(\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\right) y_2 \qquad \ldots ①$$

Here $x_0 = 1,\ x_1 = 3,\ x_2 = 4,\ \ y_0 = 1, y_1 = 27, y_2 = 64$

Substituting these values in ①, we get

$$f(x) = \left(\frac{(x-3)(x-4)}{(1-3)(1-4)}\right)(1) + \left(\frac{(x-1)(x-4)}{(3-1)(3-4)}\right)(27) + \left(\frac{(x-1)(x-3)}{(4-1)(4-3)}\right)(64)$$

$$\Rightarrow f(x) = \left(\frac{x^2-7x+12}{(-2)(-3)}\right)(1) + \left(\frac{x^2-5x+4}{(2)(-1)}\right)(27) + \left(\frac{x^2-4x+3}{(3)(1)}\right)(64)$$

$$\Rightarrow f(x) = \frac{1}{6}(48x^2 - 114x + 72) = 8x^2 - 19x + 12$$

$$\therefore f(2) = 8(2)^2 - 19(2) + 12 = 6$$

Exercise 7

1. Find the mean number of men getting wages between Rs. 10 and Rs. 15 from the following data

Wages in Rs.	0-10	10-20	20-30	30-40
Frequency	9	30	35	42

2. Find approximate value of $\cos 23^0$ using interpolation from the given data

x	10^0	20^0	30^0	40^0	50^0	60^0	70^0	80^0
$\cos x$	.9848	.9397	.8660	.7660	.6428	.5	.3420	.1737

3. Find the 4th order divided differences from the given data

x	0.5	1.5	3.0	5.0	6.5	8.0
y	1.625	5.875	31	131	282.125	521

4. Use Stirling's formula to evaluate $f(32)$ given that

$\boldsymbol{x}$	22	25	30	35	40
$\boldsymbol{f(x)}$	14.035	13.674	13.257	12.089	11.309

5. Find the polynomial of the lowest possible degree which assumes the values $1245,\ 33,\ 5,\ 9,\ 1335$ for x taking the values $-4,\ -1,\ 0,\ 2,\ 5$ respectively, using Newton's divided difference formula and hence find $f(1)$.

6. Estimate $f(7)$ from the following data, using Lagrange's interpolation formula.

x	5	6	9	11
y	12	13	14	16

Answers

1. 15
2. 0.9205
3. 0
4. 13.0622
5. $3x^4 - 5x^3 + 6x^2 - 14x + 5, f(1) = -5$
6. 13.46