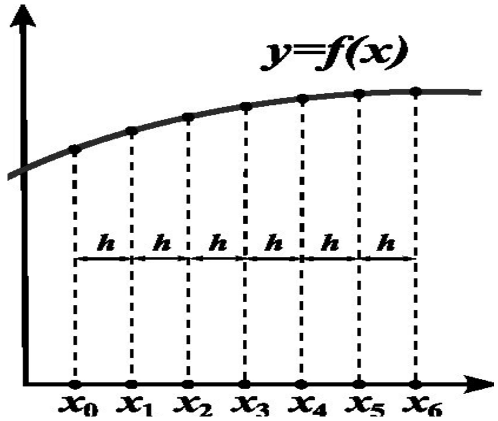
Chapter 6

Finite Differences

6.1 Introduction



For a function $y = f(x)$, finite differences refer to changes in values of y (dependent variable) for any finite (equal or unequal) variation in x (independent variable).

In this chapter, we shall study various differencing techniques for equal deviations in values of x and associated differencing operators; also their applications will be extended for finding missing values of a data and series summation.

6.2 Shift or Increment Operator (E)

Shift (Increment) operator denoted by 'E' operates on $f(x)$ as $Ef(x) = f(x+h)$

Or $Ey_x = y_{x+h}$, where 'h' is the step height for equi-spaced data points.

Clearly effect of the shift operator E is to shift the function value to the next higher value $f(x+h)$ or y_{x+h}

Also $E^2 f(x) = E(Ef(x)) = Ef(x+h) = f(x+2h)$

$\therefore E^n f(x) = f(x+nh)$

Moreover $E^{-1} f(x) = f(x-h)$, where E^{-1} is the inverse shift operator.

6.3 Differencing Operators

If $y_0, y_1, y_2, \dots y_n$ be the values of y for corresponding values of $x_0, x_1, x_2, \dots x_n$, then the differences of y are defined by $(y_1 - y_0), (y_2 - y_1), \dots, (y_n - y_{n-1})$, and are denoted by different operators discussed in this section.

6.3.1 Forward Difference Operator (Δ)

Forward difference operator 'Δ' operates on y_x as $\Delta y_x = y_{x+1} - y_x$

Or $\Delta f(x) = f(x+h) - f(x)$, where h is the height of differencing.

$$\therefore \Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

Also $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\vdots$$

$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^{n-1}\, {}^nC_{n-1} y_1 + (-1)^n y_0$$

Generalizing $\Delta^n y_r = y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

Here Δ^n is the n^{th} order forward difference; Table 6.1 shows the forward differences of various orders.

Table 6.1 Forward Differences

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_o	y_o					
		Δy_o				
x_1	y_1		$\Delta^2 y_o$			
		Δy_1		$\Delta^3 y_o$		
x_2	y_2		$\Delta^2 y_1$		$\Delta^4 y_0$	
		Δy_2		$\Delta^3 y_1$		$\Delta^5 y_0$
x_3	y_3		$\Delta^2 y_2$		$\Delta^4 y_1$	
		Δy_3		$\Delta^3 y_2$		
x_4	y_4		$\Delta^2 y_3$			
		Δy_4				
x_5	y_5					

The arrow indicates the direction of differences from top to bottom. Differences in each column notate difference of two adjoining consecutive entries of the previous column.

Relation between Δ and E

Δ and E are connected by the relation $\Delta \equiv E - 1$

Proof: we know that $\Delta y_n = y_{n+1} - y_n$

$$= E y_n - y_n$$

$$\Rightarrow \Delta y_n = (E-1)\, y_n$$

$$\Rightarrow \Delta \equiv E - 1 \text{ or } E \equiv 1 + \Delta$$

Properties of operator 'Δ'

- $\Delta C = 0$, C being a constant
- $\Delta C\, f(x) = C f(x)$
- $\Delta[af(x) \pm bg(x)] = a\,\Delta f(x) \pm b\,\Delta g(x)$
- $\Delta[f(x)g(x)] = f(x+h)\Delta\, g(x) + g(x)\Delta f(x)$, f & g may be interchanged
- $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$

Result 1: The n^{th} differences of a polynomial of degree 'n' are constant and all higher order differences are zero.

Proof: Consider the polynomial $f(x)$ of n^{th} degree

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \cdots + a_{n-1} x + a_n$$

First differences of the polynomial $f(x)$ are calculated as:

$$\Delta\, f(x) = f(x+h) - f(x)$$

$$= a_0[(x+h)^n - x^n] + a_1[(x+h)^{n-1} - x^{n-1}] + \cdots + a_{n-1}[(x+h) - x]$$

$$= a_0 nh\, x^{n-1} + a_1'\, x^{n-1} + a_2'\, x^{n-2} + \cdots + a_{n-1}' h + a_n'$$

where a_1' , a_2' ,... ,a_{n-1}', a_n' are new constants

$\Rightarrow$ First difference of a polynomial of degree n is a polynomial of degree $(n-1)$

Similarly $\Delta^2 f(x) = \Delta\, f(x+h) - \Delta\, f(x)$

$$= a_0 n(n-1)h^2\, x^{n-2} + a_1'' \, x^{n-3} + \dots + a_n''$$

$\therefore$ Second difference of a polynomial of degree n is a polynomial of degree $(n-2)$

Repeating the above process $\Delta^n f(x) = a_0 n(n-1) \dots 2.1 h^n\, x^{n-n}$

$$\Rightarrow \Delta^n f(x) = a_0 n!\, h^n \text{ which is a constant}$$

$\therefore n^{th}$ Difference of a polynomial of degree n is a polynomial of degree zero.

Thus $(n+1)^{th}$ and higher order differences of a polynomial of n^{th} degree are all zero.

- The converse of above result is also true , i.e. if the n^{th} difference of a polynomial given at equally spaced points are constant then the function is a polynomial of degree 'n'.

6.3.2 Backward Difference Operator (∇)

Backward difference operator ' ∇ ' operates on y_n as $\nabla y_n = y_n - y_{n-1}$

$\therefore$ The differences $(y_1 - y_0)$, $(y_2 - y_1)$, $\dots, (y_n - y_{n-1})$ when denoted by $\nabla y_1, \nabla y_2, \dots, \nabla y_n$ are called first backward differences.

Also $\nabla^2 y_n = \nabla y_n - \nabla y_{n-1}$, $\nabla^3 y_n = \nabla^2 y_n - \nabla^2 y_{n-1}$ denote second and third backward differences respectively.

Table 6.2 shows the backward differences of various orders.

Table 6.2 Backward Differences

x	y	∇	∇^2	∇^3	∇^4	∇^5
x_o	y_o					
		∇y_1				
x_1	y_1		$\nabla^2 y_2$			
		∇y_2		$\nabla^3 y_3$		
x_2	y_2		$\nabla^2 y_3$		$\nabla^4 y_4$	
		∇y_3		$\nabla^3 y_4$		$\nabla^5 y_5$
x_3	y_3		$\nabla^2 y_4$		$\nabla^4 y_5$	
		∇y_4		$\nabla^3 y_5$		
x_4	y_4		$\nabla^2 y_5$			
		∇y_5				
x_5	y_5					

The arrow indicates the direction of differences from bottom to top. Differences in each column notate difference of two adjoining consecutive entries of the previous column, i.e. $\nabla y_1 = y_1 - y_o$, $\nabla^2 y_2 = \nabla y_2 - \nabla y_1, \dots, \nabla^5 y_5 = \nabla^4 y_5 - \nabla^4 y_4$.

<u>Relation between ∇ and E</u>

∇ and E are connected by the relation $\nabla \equiv 1 - E^{-1}$

Proof: we know that $\nabla y_n = y_n - y_{n-1}$

$$= y_n - E^{-1} y_n$$

$$\Rightarrow \nabla y_n = (1 - E^{-1})\, y_n$$

$$\Rightarrow \nabla \equiv 1 - E^{-1}$$

6.3.3 Central Difference Operator (δ)

Central difference operator ' δ ' operates on y_n as $\delta\, y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$

$\therefore$ The differences $(y_1 - y_0)$, $(y_2 - y_1)$, …, $(y_n - y_{n-1})$ when denoted by $\delta y_{\frac{1}{2}}, \delta y_{\frac{3}{2}}, \ldots, \delta y_{n-\frac{1}{2}}$ are called first central differences.

Also $\delta^2 y_n = \delta y_{n+\frac{1}{2}} - \delta y_{n-\frac{1}{2}}$, $\delta^3 y_n = \delta^2 y_{n+\frac{1}{2}} - \delta^2 y_{n-\frac{1}{2}}$ denote second and third central differences respectively as shown in Table 6.3.

Table 6.3 Central Differences

x	y	δ	δ^2	δ^3	δ^4	δ^5
x_o	y_o					
		$\delta y_{\frac{1}{2}}$				
x_1	y_1		$\delta^2 y_1$			
		$\delta y_{\frac{3}{2}}$		$\delta^3 y_{\frac{3}{2}}$		
x_2	y_2		$\delta^2 y_2$		$\delta^4 y_2$	
		$\delta y_{\frac{5}{2}}$		$\delta^3 y_{\frac{5}{2}}$		$\delta^5 y_{\frac{5}{2}}$
x_3	y_3		$\delta^2 y_3$		$\delta^4 y_3$	
		$\delta y_{\frac{7}{2}}$		$\delta^3 y_{\frac{7}{2}}$		
x_4	y_4		$\delta^2 y_4$			
		$\delta y_{\frac{9}{2}}$				
x_5	y_5					

Central differences in each column notate difference of two adjoining consecutive entries of the previous column, i.e. $\delta y_{\frac{1}{2}} = y_1 - y_o$, …, $\delta^5 y_{\frac{5}{2}} = \delta^4 y_3 - \delta^4 y_2$.

<u>Relation between δ and *E*</u>

δ and E are connected by the relation $\delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

Proof: we know that $\delta\, y_n = y_{n+\frac{1}{2}} - y_{n-\frac{1}{2}}$

$$= E^{\frac{1}{2}} y_n - E^{-\frac{1}{2}} y_n$$

$$\Rightarrow \delta\, y_n = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right) y_n$$

$$\therefore \delta \equiv \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

Observation: It is only the notation which changes and not the difference.

$$\therefore y_1 - y_o = \Delta y_0 = \nabla y_1 = \delta y_{\frac{1}{2}}$$

6.3.4 Averaging Operator (μ)

Averaging operator ' μ ' operates on y_x as $\mu\, y_x = \frac{1}{2}\left(y_{x+\frac{h}{2}} + y_{x-\frac{h}{2}}\right)$

Or $\mu\, f(x) = \frac{1}{2}\left(f\left(x+\frac{h}{2}\right) + f\left(x-\frac{h}{2}\right)\right)$, ‘ h ’ is the height of the interval.

Relation between μ and *E*

We know that $\mu\, y_n = \frac{1}{2}\left(y_{n+\frac{h}{2}} + y_{n-\frac{h}{2}}\right)$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} y_n + E^{-\frac{1}{2}} y_n\right)$$

$$\Rightarrow \mu\, y_n = \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) y_n$$

$$\therefore \mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

Result 2: Relation between *E* and *D*, where $D \equiv \frac{d}{dx}$

We know $y(x+h) = y(x) + h\, y'(x) + \frac{h^2}{2!}\, y''(x) + \ldots$ By Taylor’s theorem

$$= y(x) + h\, Dy(x) + \frac{h^2}{2!}\, D^2 y(x) + \ldots$$

$$= \left(1 + hD + \frac{h^2}{2!}\, D^2 + \cdots\right) y(x)$$

$$\Rightarrow E\, y(x) = e^{hD} y(x)$$

$$\therefore E = e^{hD},\ D \equiv \frac{d}{dx}$$

Result 3: Relation between Δ and *D*, where $D \equiv \frac{d}{dx}$

We know that $\Delta \equiv E - 1$

$$\Rightarrow \Delta \equiv e^{hD} - 1 \qquad \because E = e^{hD}$$

Result 4: Relation between ∇ and *D*, where $D \equiv \frac{d}{dx}$

We know that $\nabla \equiv 1 - E^{-1} = 1 - e^{-hD}$ $\quad \because E = e^{hD}$

Result 5: Relation between Δ and ∇

We know that $E \equiv 1 + \Delta$ ⋯ ①

Also $E^{-1} \equiv 1 - \nabla$

$$\Rightarrow \quad E \equiv \frac{1}{1-\nabla} \qquad \cdots ②$$

$$\Rightarrow 1 + \Delta \equiv \frac{1}{1-\nabla}$$ From ① and ②

$$\Rightarrow \quad \Delta \equiv \frac{1}{1-\nabla} - 1$$

$$\Rightarrow \quad \Delta \equiv \frac{\nabla}{1-\nabla}$$

Result 6: Relation between μ , *δ* and *E*

We have $\mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$

Also $\quad \delta \equiv \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$

$\Rightarrow \quad \mu\delta \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$

$\Rightarrow \quad \mu\delta \equiv \frac{1}{2}(E - E^{-1})$

Result 7: Relation between μ, δ, Δ and ∇

We have $\mu\delta \equiv \frac{1}{2}(E - E^{-1}) = \frac{1}{2}[(1+\Delta) - (1-\nabla)]$

$\Rightarrow \quad \mu\delta \equiv \frac{1}{2}(\Delta + \nabla)$

Result 8: $\quad \Delta^n y_r = \nabla^n y_{n+r}$

We have $\Delta^n y_r = (E-1)^n y_r \qquad \because \Delta = E - 1$

$= y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

$= (E^n - {}^nC_1 E^{n-1} + {}^nC_2 E^{n-2} - \cdots + (-1)^n) y_r$

$= E^n y_r - {}^nC_1 E^{n-1} y_r + {}^nC_2 E^{n-2} y_r - \cdots + (-1)^n y_r$

$= y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^n y_r$

Also $\nabla^n y_{n+r} = (1 - \mathrm{E}^{-1})^n y_{n+r} \qquad \because \nabla \equiv 1 - E^{-1}$

$= (1 - {}^nC_1 E^{-1} + {}^nC_2 E^{-2} - \cdots + (-1)^n E^{-n}) y_{n+r}$

$= y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^n y_r$

$\therefore \Delta^n y_r = \nabla^n y_{n+r}$

Example 1 Evaluate the following:

i. Δe^x ii. $\Delta^2 e^x$ iii. $\Delta\, tan^{-1} x$ iv. $\Delta\left(\frac{x+1}{x^2-3x+2}\right)$ v. $\Delta f_k^2 = (f_k + f_{k+1})\Delta f_k$

Solution: i. $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

$\Delta e^x = e^x(e-1)$, if $h = 1$

ii. $\Delta^2 e^x = \Delta(\Delta e^x)$

$= \Delta[e^x(e^h - 1)]$

$= (e^h - 1)\,\Delta e^x$

$= (e^h - 1)\,[e^{x+h} - e^x]$

$= (e^h - 1)\, e^x(e^h - 1)$

$= e^x\,(e^h - 1)^2$

iii. $\Delta tan^{-1} x = tan^{-1}(x+h) - tan^{-1} x$

$= tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$

$= tan^{-1} \frac{h}{1+(x+h)x}$

iv. $\Delta\left(\frac{x+1}{x^2-3x+2}\right) = \Delta\left(\frac{x+1}{(x-1)(x-2)}\right)$

$$= \Delta\left(\frac{-2}{x-1} + \frac{3}{x-2}\right) = \Delta\left(\frac{-2}{x-1}\right) + \Delta\left(\frac{3}{x-2}\right)$$

$$= -2\left(\frac{1}{x+1-1} - \frac{1}{x-1}\right) + 3\left(\frac{1}{x+1-2} - \frac{1}{x-2}\right)$$

$$= -2\left(\frac{1}{x} - \frac{1}{x-1}\right) + 3\left(\frac{1}{x-1} - \frac{1}{x-2}\right)$$

$$= -\frac{(x+4)}{x(x-1)(x-2)}$$

v. $\Delta f_k^2 = f_{k+1}^2 - f_k^2 = (f_{k+1} + f_k)(f_{k+1} - f_k) = (f_k + f_{k+1})\Delta f_k$

Example 2 Evaluate the following:

i. $\Delta e^x \log 2x$ ii. $\Delta\left(\frac{x^2}{\cos 2x}\right)$

Solution: i. Let $f(x) = e^x$ and $g(x) = \log 2x$

We have $\Delta[f(x)g(x)] = f(x+h)\Delta g(x) + g(x)\Delta f(x)$

$$\therefore \Delta e^x \log 2x = e^{x+h}\Delta \log 2x + \log 2x\, \Delta e^x$$

$$= e^{x+h}[\log 2(x+h) - \log 2x] + \log 2x\,[e^{x+h} - e^x]$$

$$= e^x e^h \log\left(1 + \frac{h}{x}\right) + e^x \log 2x\,[e^h - 1]$$

$$= e^x\left[e^h \log\left(1 + \frac{h}{x}\right) + \log 2x\,[e^h - 1]\right]$$

ii. Let $f(x) = x^2$ and $g(x) = \cos 2x$

We have $\Delta\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\Delta f(x) - f(x)\Delta g(x)}{g(x+h)g(x)}$

$$= \frac{\cos 2x\,[(x+h)^2 - x^2] - x^2[\cos 2(x+h) - \cos 2x]}{\cos 2(x+h)\cos 2x}$$

$$= \frac{(h^2 + 2hx)\cos 2x + 2x^2 \sin(2x+h)\sin h}{\cos 2(x+h)\cos 2x}$$

Example 3 Evaluate $\Delta^4[(1-2x)(1-3x)(1-4x)(1-x)]$,where interval of differencing is one.

Solution: $\Delta^4[(1-2x)(1-3x)(1-4x)(1-x)]$

$$= \Delta^4[24x^4 + \cdots + 1] = 24.4!.1^4 = 576$$

$\because \Delta^n f(x) = a_0 n!\, h^n$ and $\Delta^4 x^n = 0$ when $n < 4$

Example 4 Prove that $\Delta^3 y_3 = \nabla^3 y_6$

Solution: $\Delta^3 y_3 = (E-1)^3 y_3$ $\because \Delta = E - 1$

$$= (E^3 - 1 - 3E^2 + 3E)y_3$$

$$= E^3 y_3 - y_3 - 3E^2 y_3 + 3E y_3$$

$$= y_6 - y_3 - 3y_5 + 3y_4$$

Also $\nabla^3 y_6 = (1 - \mathrm{E}^{-1})^3 y_6 \qquad \because \nabla \equiv 1 - E^{-1}$

$$= (1 - \mathrm{E}^{-3} - 3E^{-1} + 3E^{-2}) y_6$$

$$= y_6 - y_3 - 3y_5 + 3y_4$$

Example 5 Prove that $\Delta + \nabla = \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

Solution: L.H.S. $= \Delta + \nabla = (E - 1) + (1 - E^{-1})$

$$= E - E^{-1}$$

R.H.S. $= \frac{\Delta}{\nabla} - \frac{\nabla}{\Delta}$

$$= \frac{E-1}{1-E^{-1}} - \frac{1-E^{-1}}{E-1}$$

$$= \frac{(E-1)^2 - \left(1 - E^{-1}\right)^2}{\left(1-E^{-1}\right)(E-1)}$$

$$= \frac{(E^2+1-2E) - \left(1+E^{-2}-2E^{-1}\right)}{E + E^{-1} - 2}$$

$$= \frac{E^2 - E^{-2} - 2E + 2E^{-1}}{E + E^{-1} - 2}$$

$$= \frac{\left(E+E^{-1}\right)\left(E-E^{-1}\right) - 2\left(E - E^{-1}\right)}{E + E^{-1} - 2}$$

$$= \frac{\left(E - E^{-1}\right)\left(E + E^{-1} - 2\right)}{E + E^{-1} - 2}$$

$$= E - E^{-1} = \text{R.H.S.}$$

Example 6 Prove that $E = 1 + \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

Solution: R.H.S. $= 1 + \frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

$$= 1 + \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{1 + \frac{1}{4}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2}$$

$$\because \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$= 1 + \frac{1}{2}(E + E^{-1} - 2) + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{1 + \frac{1}{4}(E + E^{-1} - 2)}$$

$$= 1 + \frac{1}{2}(E + E^{-1} - 2) + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{1}{4}(E + E^{-1} + 2)}$$

$$= 1 + \frac{1}{2}(E + E^{-1} - 2) + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{1}{4}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2}$$

$$= 1 + \frac{1}{2}(E + E^{-1} - 2) + \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

$$= \frac{1}{2}(E + E^{-1}) + \frac{1}{2}(E - E^{-1}) = E = \text{L.H.S.}$$

Example 7 Prove that $\nabla = -\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

Solution: R.H.S. $= -\frac{1}{2}\delta^2 + \delta\sqrt{1 + \frac{1}{4}\delta^2}$

$$= -\frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{1 + \frac{1}{4}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2}$$

$$\because \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$= -\frac{1}{2}(E + E^{-1} - 2) + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{1 + \frac{1}{4}(E + E^{-1} - 2)}$$

$$= -\frac{1}{2}(E + E^{-1} - 2) + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{1}{4}(E + E^{-1} + 2)}$$

$$= -\frac{1}{2}(E + E^{-1} - 2) + \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\sqrt{\frac{1}{4}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2}$$

$$= -\frac{1}{2}(E + E^{-1} - 2) + \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

$$= -\frac{1}{2}(E + E^{-1} - 2) + \frac{1}{2}(E - E^{-1}) = 1 - E^{-1} = \nabla = \text{L.H.S.}$$

Example 8 Prove that (i) $\Delta - \nabla = \delta^2$ (ii) $\mu = \sqrt{1 + \frac{1}{4}\delta^2} = \left(1 + \frac{\Delta}{2}\right)(1 + \Delta)^{-\frac{1}{2}}$

Solution: (i) $\delta^2 = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2 = E + E^{-1} - 2 \quad \because \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$

$$= (E - 1) - (1 - E^{-1}) = \Delta - \nabla$$

$$\because E - 1 \equiv \Delta \text{ and } 1 - E^{-1} = \Delta$$

(ii) $\sqrt{1 + \frac{1}{4}\delta^2} = \sqrt{1 + \frac{1}{4}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)^2} \quad \because \delta \equiv \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$

$$= \sqrt{1 + \frac{1}{4}(E + E^{-1} - 2)}$$

$$= \sqrt{\frac{1}{4}(E + E^{-1} + 2)}$$

$$= \sqrt{\frac{1}{4}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)^2}$$

$$= \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) = \mu \qquad \because \mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right)$$

Also $\left(1 + \frac{\Delta}{2}\right)(1 + \Delta)^{-\frac{1}{2}} = \left(1 + \frac{E-1}{2}\right)(1 + E - 1)^{-\frac{1}{2}}$

$$\because \Delta \equiv E - 1$$

$$= \left(\frac{E+1}{2}\right)E^{-\frac{1}{2}}$$

$$= \frac{1}{2}\left(E^{-\frac{1}{2}} + E^{\frac{1}{2}}\right) = \mu$$

Example 9 Prove that (i) $\Delta \equiv E\nabla \equiv \nabla E = \delta E^{\frac{1}{2}}$ (ii) $E^r = \left(\mu + \frac{\delta}{2}\right)^{2r}$

Solution: (i) $E\nabla = E(1 - E^{-1}) = E - 1 = \Delta \qquad \because \nabla \equiv 1 - E^{-1}$

$$\nabla E = (1 - E^{-1})E = E - 1 = \Delta$$

$$\delta E^{\frac{1}{2}} = \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)E^{\frac{1}{2}} = E - 1 = \Delta \qquad \because \delta \equiv \left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)$$

(ii) R.H.S. $= \left(\mu + \frac{\delta}{2}\right)^{2r} = \left(\frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) + \frac{1}{2}\left(E^{\frac{1}{2}} - E^{-\frac{1}{2}}\right)\right)^{2r}$

$$\because \mu \equiv \frac{1}{2}\left(E^{\frac{1}{2}} + E^{-\frac{1}{2}}\right) \text{ and } \delta \equiv E^{\frac{1}{2}} - E^{-\frac{1}{2}}$$

$$= \left(\frac{1}{2}\left(2E^{\frac{1}{2}}\right)\right)^{2r} = \left(E^{\frac{1}{2}}\right)^{2r} = E^r = \text{L.H.S}$$

Example 10 Prove that (i) $D \equiv \frac{1}{h}\log E$ (iii) $hD \equiv \log(1 + \Delta) \equiv -\log(1 - \nabla)$

(iii) $\nabla^2 \equiv h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 + \cdots$

Solution: (i) We know that $E \equiv e^{hD}$

$$\Rightarrow \log E \equiv \log e^{hD}$$

$$\Rightarrow \log E \equiv hd \log e$$

$$\Rightarrow \quad D \equiv \frac{1}{h}\log E \qquad \because \log e = 1$$

(ii) $h\,D \equiv \log E$ From relation (i)

$$\equiv \log(1 + \Delta) \qquad \because E \equiv 1 + \Delta$$

Also $h\,D \equiv \log E \equiv -\log E^{-1}$

$$\equiv -\log(1 - \nabla) \qquad \because \nabla \equiv 1 - E^{-1}$$

(iii) We know that $\nabla \equiv 1 - E^{-1}$

$$\Rightarrow \nabla \equiv 1 - \frac{1}{E}$$

$$\equiv 1 - e^{-hD} \qquad \because\ E = e^{hD}$$

$$\Rightarrow\ \nabla \equiv 1 - \left(1 - hd + \frac{h^2D^2}{2!} - \frac{h^3D^3}{3!} + \cdots\right)$$

$$\Rightarrow\ \nabla \equiv\ hd - \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots$$

$$\therefore \nabla^2 \equiv \left(hd - \frac{h^2D^2}{2!} + \frac{h^3D^3}{3!} + \cdots\right)^2$$

$$\Rightarrow \nabla^2 \equiv h^2D^2 + \left(\frac{h^2D^2}{2!}\right)^2 + \cdots - 2(hd)\left(\frac{h^2D^2}{2!}\right) + 2(hd)\left(\frac{h^3D^3}{3!}\right) - \cdots$$

$$\Rightarrow \nabla^2 \equiv h^2D^2 - h^3D^3 + \left(\frac{h^4D^4}{4} + \frac{h^4D^4}{3}\right) - \cdots$$

$$\Rightarrow \nabla^2 \equiv h^2D^2 - h^3D^3 + \frac{7}{12}h^4D^4 - \cdots$$

Remark: In order to prove any relation, we can express the operators (Δ ,∇,δ) in terms of fundamental operator E.

Example 11 Form the forward difference table for the function

$f(x) = x^3 - 2x^2 - 3x - 1$ for $x = 0, 1, 2, 3, 4.$

Hence or otherwise find $\Delta^3 f(x)$, also show that $\Delta^4 f(x) = 0$

Solution: $f(0) = -1, f(1) = -5, f(2) = -7, f(3) = -1, f(4) = 19$

Constructing the forward difference table:

x	$f(x)$	Δ	Δ^2	Δ^3	Δ^4
0	-1				
		-4			
1	-5		2		
		-2		6	
2	-7		8		0
		6		6	
3	-1		14		
		20			
4	19				

From the table, we see that $\Delta^3 f(x) = 6$ and $\Delta^4 f(x) = 0$

Note: Using the formula $\Delta^n f(x) = a_0 n!\, h^n$, $\Delta^3 f(x) = 1.3!.\, 1^n = 6$

Also $\Delta^{n+1} f(x) = 0$ for a polynomial of degree n, $\therefore$ $\Delta^4 f(x) = 0$

Example 12 If for a polynomial, five observations are recorded as: $y_0 = -8$, $y_1 = -6,\ y_2 = 22,\ y_3 = 148,\ y_4 = 492$, find y_5.

Solution: $y_5 = E^5 y_0 = (1+\Delta)^5 y_0 \qquad \because E \equiv 1 + \Delta$

$$= y_0 + {}^5C_1 \Delta y_0 + {}^5C_2 \Delta^2 y_0 + {}^5C_3 \Delta^3 y_0 + {}^5C_4 \Delta^4 y_0 + \Delta^5 y_0 \quad \ldots①$$

Constructing the forward difference table:

x	y	Δ	Δ^2	Δ^3	Δ^4
x_0	-8				
		2			
x_1	-6		26		
		28		72	
x_2	22		98		48
		126		120	
x_3	148		218		
		344			
x_4	492				

From table $\Delta y_0 = 2,\ \Delta^2 y_0 = 26$, $\Delta^3 y_0 = 72$, $\Delta^3 y_0 = 48$ ... ②

$\Rightarrow y_5 = -8 + 5(2) + 10(26) + 10(72) + 5(48) = 1222$ using ② in ①

6.4 Missing values of Data

Missing data or missing values occur when an observation is missing for a particular variable in a data sample. Concept of finite differences can help to locate the requisite value using known concepts of curve fitting.

To determine the equation of a line (equation of degree one), we need at least two given points. Similarly to trace a parabola (equation of degree two), at least three points are imperative. Thus we essentially require $(n + 1)$ known observations to determine a polynomial of n^{th} degree.

To find missing values of data using finite differences, we presume the degree of the polynomial by the number of known observations and use the result $\Delta^{n+1} f(x) = 0$ for a polynomial of degree n.

Example 13 Use the concept of missing data to find y_5 if $y_0 = -8$, $y_1 = -6$, $y_2 = 22$, $y_3 = 148$, $y_4 = 492$

Solution: Constructing the forward difference table taking y_5 as missing value

x	y	Δ	Δ^2	Δ^3	Δ^4	Δ^5
x_o	-8					
		2				
x_1	-6		26			
		28		72		
x_2	22		98		48	
		126		120		$y_5 - 1222$
x_3	148		218		$y_5 - 1174$	
		344		$y_5 - 1054$		
x_4	492		$y_5 - 836$			
		$y_5 - 492$				
x_5	y_5					

Since 5 observations are known, let us assume that the polynomial represented by given data is of 4^{th} degree. $\therefore\ \Delta^5 y = 0 \Rightarrow y_5 - 1222 = 0$ or $y_5 = 1222$

Example 14 Find the missing values in the following table

x	**0**	**5**	**10**	**15**	**20**	**25**
$f(x)$	6	?	13	17	22	?

Solution: Since there are 4 known values of $f(x)$ in the given data, let us assume the polynomial represented by the given data to be of 3^{rd} degree.
Constructing the forward difference table taking missing values as a and b.

x	y	Δ	Δ^2	Δ^3	Δ^4
0	6				
		$a-6$			
5	a		$19-2a$		
		$13-a$		$3a-28$	
10	13		$a-9$		$38-4a$
		4		$10-a$	
15	17		1		$a+b-38$
		5		$b-28$	
20	22		$b-27$		
		$b-22$			
25	b				

Since the polynomial represented by the given data is considered to be of 3^{rd} degree, 4^{th} and higher order differences are zero i.e. $\Delta^4 y = 0$

$\therefore\ 38 - 4a = 0$ and $a + b - 38 = 0$

Solving these two equations, we get $a = 9.5$ $b = 28.5$

6.5 Finding Differences Using Factorial Notation

We can conveniently find the forward differences of a polynomial using factorial notation.

6.5.1 Factorial Notation of a Polynomial

A product of the form $x(x-1)(x-2)\ldots(x-r+1)$ is called a factorial polynomial and is denoted by $[x]^r$

$$\therefore\quad [x] = x$$
$$[x]^2 = x(x-1)$$
$$[x]^3 = x(x-1)(x-2)$$
$$\vdots$$
$$[x]^n = x(x-1)(x-2)\ldots(x-n+1)$$

In case, the interval of differencing is h, then

$$[x]^n = x(x-h)(x-h)\ldots\left(x-\overline{n-1}\ h\right)$$

The results of differencing $[x]^r$ are analogous to that differentiating x^r

$$\therefore \quad \Delta[x]^n = n[x]^{n-1}$$
$$\Delta^2[x]^n = n(n-1)[x]^{n-2}$$
$$\Delta^3[x]^n = n(n-1)(n-2)[x]^{n-3}$$
$$\vdots$$
$$\Delta^n[x]^n = n(n-1)(n-2)\ldots 3.2.1 = n!$$
$$\Delta^{n+1}[x]^n = 0$$

Also $\frac{1}{\Delta}[x] = \frac{[x]^2}{2}, \frac{1}{\Delta}[x]^2 = \frac{[x]^3}{3}$ and so on

$$\frac{1}{\Delta^2}[x] = \frac{1}{\Delta}\left[\frac{[x]^2}{2}\right] = \frac{[x]^3}{6}$$
$$\vdots$$

Remark:

i. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice-versa.

ii. The coefficient of highest power of x and also the constant term remains unchanged while transforming a polynomial to factorial notation.

Example15 Express the polynomial $2x^2 + 3x + 1$ in factorial notation.

Solution: $2x^2 - 3x + 1 = 2x^2 - 2x + 5x + 1$

$$= 2x(x-1) + 5x + 1$$
$$= 2[x]^2 + 5[x] + 1$$

Example16 Express the polynomial $2x^3 - x^2 + 3x - 4$ in factorial notation.

Solution: $2x^3 - x^2 + 3x - 4 = 2[x]^3 + A[x]^2 + B[x] - 4$

Using remarks i and ii

$$= 2x(x-1)(x-2) + Ax(x-1) + Bx - 4$$
$$= 2x^3 + (A-6)x^2 + (-A+B+4)x - 4$$

Comparing the coefficients on both sides

$$\mathrm{A} - 6 = -1, \quad -A + \mathrm{B} + 4 = 3$$
$$\Rightarrow \mathrm{A} = 5, \quad \mathrm{B} = 4$$
$$\therefore 2x^3 - x^2 + 3x - 4 = 2[x]^3 + 5[x]^2 + 4[x] - 4$$

- We can also find factorial polynomial using synthetic division as shown: Coefficients A and B can be found as remainders under x^2 and x columns

	x^3	x^2	x	
1	2	–1	3	–4
	–	2	1	
2	2	1	$4 = B$	
	–	4		
	2	5 = A		

Example 17 Find $\Delta^3 f(x)$ for the polynomial $f(x) = x^3 - 2x^2 - 3x - 1$

Also show that $\Delta^4 f(x) = 0$

Solution: Finding factorial polynomial of $f(x)$ as shown:

Let $x^3 - 2x^2 - 3x - 1 \ = [x]^3 + A[x]^2 + B[x] - 1$

Coefficients A and B can be found as remainders under x^2 and x columns

	x^3	x^2	x	
1	1	–2	–3	–1
	–	1	–1	
2	1	–1	$-4 = B$	
	–	2		
	1	1 = A		

$$\therefore \quad f(x) = x^3 - 2x^2 - 3x - 1 \ = [x]^3 + [x]^2 - 4[x] - 1$$

$$\Delta^3 f(x) = \Delta^3[[x]^3 + [x]^2 - 4[x] - 1 \ ]$$

$$= 3! + 0 = 6 \qquad \because \ \Delta^n [x]^n = n! \text{ and } \Delta^{n+1}[x]^n = 0$$

Also $\Delta^4 f(x) = \Delta^4[[x]^3 + [x]^2 - 4[x] - 1 \ ] = 0$

Note: Results obtained are same as in Example 11, where we have used forward difference table to compute the differences.

Example 18: Obtain the function whose first difference is $8x^3 - 3x^2 + 3x - 1$

Solution: Let $f(x)$ be the function whose first difference is $8x^3 - 3x^2 + 3x - 1$

$$\Rightarrow \Delta f(x) = 8x^3 - 3x^2 + 3x - 1$$

Let $8x^3 - 3x^2 + 3x - 1 \ = 8[x]^3 + A[x]^2 + B[x] - 1$

Coefficients A and B can be found as remainders under x^2 and x columns

	x^3	x^2	x	
1	8	–3	3	–1
	–	8	5	
2	8	5	$8 = B$	
	–	16		
	8	21 = A		

$$\therefore \Delta f(x) = 8x^3 - 3x^2 + 3x - 1 \ = 8[x]^3 + 21[x]^2 + 8[x] - 1$$

$$f(x) = \frac{1}{\Delta}[8[x]^3 + 21[x]^2 + 8[x] - 1 \ ]$$

$$= \frac{8[x]^4}{4} + \frac{21[x]^3}{3} + \frac{8[x]^2}{2} - [x] \quad \because \frac{1}{\Delta}[x] = \frac{[x]^2}{2}, \frac{1}{\Delta}[x]^2 = \frac{[x]^3}{3}, \ldots$$

$$= 2[x]^4 + 7[x]^3 + 4[x]^2 - [x]$$

$$= 2x(x-1)(x-2)(x-3) + 7x(x-1)(x-2) + 4x(x-1) - x$$

$$= x[2(x-1)(x-2)(x-3) + 7(x-1)(x-2) + 4(x-1) - 1]$$

$$= x[2x^3 - 5x^2 + 5x - 3] = 2x^4 - 5x^3 + 5x^2 - 3x$$

$\Rightarrow f(x) = 2x^4 - 5x^3 + 5x^2 - 3x$

6.6 Series Summation Using Finite Differences

The method of finite differences may be used to find sum of a given series by applying the following algorithm:

1. Let the series be represented by $u_0, u_1, u_2, u_3, \ldots$
2. Use the relation $u_r = E^r u_0$ to introduce the operator E in the series.
3. Replace E by Δ by substituting $E \equiv 1 + \Delta$ and find the sum the series by any of the applicable methods like sum of a G.P., exponential or logarithmic series or by binomial expansion and operate term by term on u_0 to find the required sum.

Example 19 Prove the following using finite differences:

i. $u_0 + u_1 \frac{x}{1!} + u_2 \frac{x^2}{2!} + \cdots = e^x \left[u_0 + x \frac{\Delta u_0}{1!} + x^2 \frac{\Delta^2 u_0}{2!} + \cdots \right]$

ii. $u_0 - u_1 + u_2 - u_3 + \cdots = \frac{1}{2} u_0 - \frac{1}{4} \Delta u_0 + \frac{1}{8} \Delta^2 u_0 - \cdots$

Solution: i. $u_0 + u_1 \frac{x}{1!} + u_2 \frac{x^2}{2!} + \cdots = u_0 + \frac{x}{1!} E u_0 + \frac{x^2}{2!} E^2 u_0 + \cdots$

$$= \left[1 + \frac{xE}{1!} + \frac{x^2E^2}{2!} u_0 + \cdots \right] u_0$$

$$= e^{xE} u_0 = e^{x(1+\Delta)} u_0$$

$$= e^x e^{x\Delta} u_0$$

$$= e^x \left[1 + \frac{x\Delta}{1!} + \frac{x^2\Delta^2}{2!} + \cdots \right] u_0$$

$$= e^x \left[u_0 + x \frac{\Delta u_0}{1!} + x^2 \frac{\Delta^2 u_0}{2!} + \cdots \right]$$

ii. $u_0 - u_1 + u_2 - u_3 + \cdots = u_0 - Eu_0 + E^2 u_0 - E^3 u_0 + \cdots$

$$= [1 - E + E^2 - E^3 + \cdots] u_0$$

$$= [1 + E]^{-1} u_0$$

$$= [2 + \Delta]^{-1} u_0$$

$$= 2^{-1} \left[1 + \frac{\Delta}{2}\right]^{-1} u_0$$

$$= \frac{1}{2} \left[1 - \frac{\Delta}{2} + \frac{\Delta^2}{4} - \cdots \right] u_0$$

$$= \frac{1}{2} u_0 - \frac{1}{4} \Delta u_0 + \frac{1}{8} \Delta^2 u_0 - \cdots$$

Example 20 Sum the series $1^2, 2^2, 3^2, \ldots, n^2$ using finite differences.

Solution: Let the series $1^2, 2^2, 3^2, \ldots, n^2$ be represented by $u_0, u_1, u_2, \ldots, u_{n-1}$

$\therefore\ S = u_0 + u_1 + u_2 + \cdots + u_{n-1}$

$\Rightarrow S = u_0 + Eu_0 + E^2 u_0 + \cdots + E^{n-1} u_0$

$= (1 + E + E^2 + \cdots + E^{n-1}) u_0$

$$= \frac{1-E^n}{1-E} u_0 = \frac{E^n - 1}{E-1} u_0 \qquad \because S_n = a\frac{1-r^n}{1-r}$$

$$\Rightarrow S = \frac{(1+\Delta)^n - 1}{(1+\Delta)-1} u_0$$

$$= \frac{1}{\Delta}\left\{\left[1 + n\Delta + \frac{n(n-1)}{2!}\Delta^2 + \frac{n(n-1)(n-2)}{3!}\Delta^3 + \cdots\right] - 1\right\} u_0$$

$$= nu_0 + \frac{n(n-1)}{2!}\Delta u_0 + \frac{n(n-1)(n-2)}{3!}\Delta^2 u_0 + \cdots$$

Now $u_0 = 1^2 = 1$

$\Delta u_0 = u_1 - u_0 = 2^2 - 1^2 = 3$

$\Delta^2 u_0 = \Delta u_1 - \Delta u_0 = u_2 - 2u_1 + u_0 = 3^2 - 2(2^2) + 1^2 = 2$

$\Delta^3 u_0,\ \Delta^4 u_0$ ... are all zero as given series is an expression of degree 2

$$\therefore\ S = n + \frac{n(n-1)}{2!}(3) + \frac{n(n-1)(n-2)}{3!}(2) + 0$$

$$= n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3}$$

$$= \frac{1}{6}\big(6n + 9n(n-1) + 2n(n-1)(n-2)\big)$$

$$= \frac{1}{6}n(6 + 9n - 9 + 2n^2 - 6n + 4)$$

$$= \frac{1}{6}n(2n^2 + 3n + 1) = \frac{1}{6}n(n+1)(2n+1)$$

Example 21 Prove that $u_0 + u_1 x + u_2 x^2 + \cdots = \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta_{u_0}^2}{(1-x)^3} + \cdots$

and hence evaluate $1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \cdots$

Solution: $u_0 + u_1 x + u_2 x^2 + \cdots = u_0 + xEu_0 + x^2E^2u_0 + \cdots$

$$= \left(1 + xE + x^2E^2 + \cdots\right)u_0$$

$$= \frac{1}{1-xE}u_0 \qquad \because S_\infty = \frac{a}{1-r}$$

$$= \frac{1}{1-x(1+\Delta)}u_0 = \frac{1}{(1-x)-x\Delta}u_0$$

$$= \frac{1}{1-x}\,\frac{1}{1-\frac{x\Delta}{1-x}}u_0$$

$$= \frac{1}{1-x}\left(1 - \frac{x\Delta}{1-x}\right)^{-1}u_0$$

$$= \frac{1}{1-x}\left(1 + \frac{x\Delta}{1-x} + \frac{x^2\Delta^2}{(1-x)^2} + \cdots\right)u_0$$

$$= \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta_{u_0}^2}{(1-x)^3} + \cdots = \text{R.H.S.}$$

Now to evaluate the series $1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \cdots$

Let $u_0 = 1.2 = 2,\ \ u_1 = 2.3 = 6,\ \ u_2 = 3.4 = 12,\ \ u_3 = 4.5 = 20, \ldots$

Forming forward difference table to calculate the differences

$\boldsymbol{u}$	Δ	Δ^2	Δ^3	Δ^4
$\boldsymbol{u_0 = 1.2 = 2}$				
	4			
$\boldsymbol{u_1 = 2.3 = 6}$		2		
	6		0	
$\boldsymbol{u_2 = 3.4 = 12}$		2		0
	8		0	
$\boldsymbol{u_3 = 4.5 = 20}$		2		
	10			
$\boldsymbol{u_4 = 5.6 = 30}$				

$$\therefore 1.2 + 2.3x + 3.4x^2 + 4.5x^3 + \cdots = \frac{u_0}{1-x} + \frac{x\Delta u_0}{(1-x)^2} + \frac{x^2\Delta^2_{u_0}}{(1-x)^3} + \cdots$$

$$= \frac{2}{1-x} + \frac{4x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} + 0$$

$$= \frac{2}{(1-x)^3}$$

Exercise 6A

1. Express y_4 in terms of successive forward differences.
2. Prove that $\Delta^n e^{3x+5} = (e^3 - 1)^n e^{3x+5}$
3. Evaluate $\Delta^2\left(\frac{5x+12}{x^2+5x+6}\right)$
4. If $u_0 = 3, u_1 = 12, u_2 = 81, u_3 = 2000, u_4 = 100$, calculate $\Delta^4 u_0$.
5. Prove that $\mu = \frac{2+\Delta}{2\sqrt{1+\Delta}} = \sqrt{1 + \frac{1}{4}\delta^2}$
6. Find the missing value in the following table

x	0	5	10	15	20	25
y	6	10	-	17	-	31

7. Sum the series $1^3,\ 2^3,\ 3^3, \ldots,\ n^3$ using finite differences.

Answers

1. $y_4 = y_0 + 4\Delta y_0 + 6\Delta^2 y_0 + 4\Delta^3 y_0 + \Delta^4 y_0$
3. $\frac{-3(x^2+9x+15)}{x(x+1)(x+4)(x+5)(x+8)(x+9)}$
4. -7459
6. $13.25, 22.5$