CHAPTER 12

BESSEL'S AND LEGENDRE'S EQUATIONS

12.1 Introduction

Many linear differential equations having variable coefficients cannot be solved by usual methods and we need to employ series solution method to find their solutions in terms of infinite convergent series.

12.2 Bessel's Equation

The differential equation $x^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2}+x\frac{\mathrm{dy}}{\mathrm{d}x}+(x^2-n^2)y = 0$ ……①

is known as Bessel's equation of order n and it's particular solutions are called Bessel's functions. Series solution of ① in terms of Bessel's functions $J_n(x)$ and $J_{-n}(x)$ is given by $y = AJ_n(x) + BJ_{-n}(x)$

where $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(\mathrm{n+r+1})}\left(\frac{x}{2}\right)^{n+2r}$

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-\mathrm{n+r+1})}\left(\frac{x}{2}\right)^{-n+2r}$$

Proposition If n is any integer then $J_{-n}(x) = (-1)^nJ_n(x)$

Proof: Case I: n is a positive integer

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-\mathrm{n+r+1})}\left(\frac{x}{2}\right)^{-n+2r}$$

If n is a positive integer, values of r from 0 to $(n-1)$ will give gamma function of $-ve$ integers in the denominator, which being infinite all such terms will vanish.

$$\therefore J_{-n}(x) = \sum_{r=n}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-\mathrm{n+r+1})}\left(\frac{x}{2}\right)^{-n+2r}$$

Putting $\mathrm{r = n + k}$, we get

$$J_{-n}(x) = \sum_{\mathrm{k}=0}^{\infty}(-1)^{n+k}\frac{1}{(n+k)!\,\Gamma(\mathrm{k+1})}\left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n\sum_{\mathrm{k}=0}^{\infty}(-1)^k\frac{1}{k!\,\Gamma(n+\mathrm{k}+1}\left(\frac{x}{2}\right)^{n+2k}$$

$= \ (-1)^n J_n(x)$

Case II: $n = 0$

$J_{-0}(x) = (-1)^0 \, J_0(x)$

or $J_0(x) = \ J_0(x)$, which is true

Case III: n is a negative integer

Let $= -p$, where p is a positive integer

From case I $\quad J_p(x) = (-1)^{-p} J_{-p}(x)$

$\Rightarrow J_{-n}(x) = (-1)^n J_n(x)$

12.2.1 Expansions of $\boldsymbol{J_0(x)}$, $\boldsymbol{J_1(x)}$, $\boldsymbol{J_{\frac{1}{2}}(x)}$ and $\boldsymbol{J_{-\frac{1}{2}}(x)}$

We have $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(\mathrm{n+r+1})}$

1. $J_0(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{r!\,\Gamma(\mathrm{r+1})} = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{(r!)^2}$

$\because \Gamma(\mathrm{r} + 1 = \mathrm{r}!$ when r is a positive integer

$$\Rightarrow J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

2. $J_1(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{1+2r} \frac{1}{r!\,\Gamma(\mathrm{r+2})} = \frac{x}{2}\sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{\mathrm{r!\,(r+1)!}}$

$\because \Gamma(\mathrm{r} + 2 = (\mathrm{r}+1)!$ when r is a positive integer

$$\Rightarrow J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 + \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

3. $J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \frac{1}{r!\,\lceil\left(\mathrm{r}+\frac{3}{2}\right)}$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\lceil\frac{3}{2}} - \frac{1}{1!\,\lceil\frac{5}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\lceil\frac{7}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\lceil\frac{9}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\frac{1}{2}\lceil\frac{1}{2}} - \frac{1}{1!\,\frac{3}{2}\frac{1}{2}\lceil\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}\lceil\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\lceil\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(\mathrm{n} + 1 = \mathrm{n}\Gamma\mathrm{n}$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{1}{\frac{1}{2}} - \frac{1}{1!\,\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$\because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \sqrt{\frac{x}{2\pi}} \left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \frac{2x^6}{7!} + \cdots \right]$$

$$= \sqrt{\frac{x}{2\pi}} \frac{2}{x} \left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \right]$$

$$= \sqrt{\frac{2}{\pi x}} \sin x$$

4. $J_{-\frac{1}{2}}(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{-\frac{1}{2}+2r} \frac{1}{r!\,\Gamma\left(r+\frac{1}{2}\right)}$

$$= \sqrt{\frac{2}{x}} \left[\frac{1}{\Gamma\frac{1}{2}} - \frac{1}{1!\,\Gamma\frac{3}{2}} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\frac{5}{2}} \left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma\frac{7}{2}} \left(\frac{x}{2}\right)^6 + \cdots \right]$$

$$= \sqrt{\frac{2}{x}} \left[\frac{1}{\Gamma\frac{1}{2}} - \frac{1}{1!\ \frac{1}{2}\Gamma\frac{1}{2}} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\ \frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}} \left(\frac{x}{2}\right)^4 - \frac{1}{3!\ \frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}} \left(\frac{x}{2}\right)^6 + \cdots \right]$$

$$\because \Gamma(n+1 = n\Gamma n$$

$$= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{1}{1!\ \frac{1}{2}} \left(\frac{x}{2}\right)^2 + \frac{1}{2!\ \frac{3}{2}\frac{1}{2}} \left(\frac{x}{2}\right)^4 - \frac{1}{3!\ \frac{5}{2}\frac{3}{2}\frac{1}{2}} \left(\frac{x}{2}\right)^6 + \cdots \right]$$

$$\because \Gamma\tfrac{1}{2} = \sqrt{\pi}$$

$$= \sqrt{\frac{2}{\pi x}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \right] = \sqrt{\frac{2}{\pi x}} \cos x$$

12.2.2 Recurrence Relations of Bessel's Function

(1) $\boldsymbol{\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)}$ **or** $\boldsymbol{\int x^n J_{n-1}(x)dx = x^n J_n(x)}$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} (-1)^r \frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}} \frac{1}{r!\,(n+r)\Gamma(n+r)}$$

$$\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)$$

$$= x^n \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{r!\,\Gamma((n-1)+r+1)}$$

$$= x^n J_{n-1}(x)$$

(2) $\boldsymbol{\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)}$ **or** $\boldsymbol{\int x^{-n}J_{n+1}(x)\,dx = -\frac{d}{dx}[x^{-n}J_n(x)]}$

Proof: $J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow x^{-n}J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty} (-1)^r \frac{2r\,x^{2r-1}}{2^{n+2r}} \frac{1}{(r-1)!\; r\,\Gamma(n+r+1)}$$

$$= x^{-n} \sum_{r=1}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{(r-1)!\;\Gamma(n+r+1)}$$

$$= -x^{-n} \sum_{k=0}^{\infty} (-1)^k \left(\frac{x}{2}\right)^{(n+1)+2k} \frac{1}{k!\;\Gamma((n+1)+k+1)}$$

Putting $r = k + 1$

$$= -x^{-n}J_{n+1}(x)$$

(3) $\boldsymbol{J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)}$

Proof: From recurrence relation (1)

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n{}'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we get

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$$

$$\Rightarrow J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

(4) $\boldsymbol{J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)}$

Proof: From recurrence relation (2)

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow x^{-n}J_n{}'(x) - n x^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

Dividing by x^{-n}, we get

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$$

(5) $\boldsymbol{J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]}$

Proof: Adding recurrence relations (3) and (4), we get

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

(6) $\boldsymbol{2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$

Proof: Subtracting recurrence relations (3) from (4), we get

$$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow\ 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

Example 1 Evaluate $J_{\frac{3}{2}}(x), J_{-\frac{3}{2}}(x)$, $J_{\frac{5}{2}}(x)$ and $J_{-\frac{5}{2}}(x)$

Solution: From recurrence relation (6)

$$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \ \ldots\ldots\ldots\text{①}$$

Putting $n = \frac{1}{2}$ in ①

$$\Rightarrow \frac{1}{x}J_{\frac{1}{2}}(x) = J_{\frac{-1}{2}}(x) + J_{\frac{3}{2}}(x)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{\frac{-1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \quad \because J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,, J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Putting $n = -\frac{1}{2}$ in ①

$$\Rightarrow -\frac{1}{x}J_{\frac{-1}{2}}(x) = J_{\frac{-3}{2}}(x) + J_{\frac{1}{2}}(x)$$

$$\Rightarrow J_{\frac{-3}{2}}(x) = -\frac{1}{x}J_{\frac{-1}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= -\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) \quad \because J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,, J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Putting $n = \frac{3}{2}$ in ①

$$\Rightarrow \frac{3}{x} J_{\frac{3}{2}}(x) = J_{\frac{1}{2}}(x) + J_{\frac{5}{2}}(x)$$

$$\Rightarrow J_{\frac{5}{2}}(x) = \frac{3}{x} J_{\frac{3}{2}}(x) - J_{\frac{1}{2}}(x)$$

$$= \frac{3}{x} \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) - \sqrt{\frac{2}{\pi x}} \sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2} \sin x - \frac{3}{x} \cos x\right]$$

Putting $n = -\frac{3}{2}$ in ①

$$\Rightarrow -\frac{3}{x} J_{\frac{-3}{2}}(x) = J_{\frac{-5}{2}}(x) + J_{\frac{-1}{2}}(x)$$

$$\Rightarrow J_{\frac{-5}{2}}(x) = -\frac{3}{x} J_{\frac{-3}{2}}(x) - J_{\frac{-1}{2}}(x)$$

$$= \frac{3}{x} \sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x} + \sin x\right) - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3-x^2}{x^2} \cos x + \frac{3}{x} \sin x\right]$$

Example 2 Show that:

(i) $J_0'(x) = -J_1(x)$

(ii) $J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4)\, J_{n+4}(x)$

(iii) $J_n''(x) = \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$

(iv) $\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{-n}J_n(x)] = (-1)^m \frac{1}{x^{n+m}} J_{n+m}(x)$

(v) $\frac{d}{dx}[x\, J_n(x)\, J_{n+1}(x)] = x\left[{J_n}^2(x) - {J_{n+1}}^2(x)\right]$

(vi) $J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$

Solution: (i) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$\Rightarrow \frac{d}{dx}[J_0(x)] = -J_1(x)$, Putting $n = 0$

or $J_0'(x) = -J_1(x)$

(ii) From recurrence relation (6)

$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$ ....①

Replacing n by $(n+4)$ in ①

$$2(n+4)J_n(x) = x[J_{n+3}(x) + J_{n+5}(x)]$$

$$\Rightarrow J_{n+3}(x) + J_{n+5}(x) = \frac{2}{x}(n+4)\, J_{n+4}(x)$$

(iii) From recurrence relation (5)

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \quad \ldots\ldots\ldots\text{①}$$

$$\Rightarrow J_n''(x) = \frac{1}{2}[\, J_{n-1}'(x) - J_{n+1}'(x)] \quad \ldots\ldots\ldots\text{②}$$

Replacing $(n-1)$ in place of n in ①

$$\Rightarrow J_{n-1}'(x) = \frac{1}{2}[J_{n-2}(x) - J_n(x)] \quad \ldots\ldots\text{③}$$

Replacing $(n+1)$ in place of n in ①

$$\Rightarrow J_{n+1}'(x) = \frac{1}{2}[J_n(x) - J_{n+2}(x)] \quad \ldots\ldots\text{④}$$

Using ③ and ④ in ②

$$J_n''(x) = \frac{1}{2}\left[\frac{1}{2}[J_{n-2}(x) - J_n(x)] - \frac{1}{2}[J_n(x) - J_{n+2}(x)]\right]$$

$$= \frac{1}{4}[J_{n-2}(x) - 2J_n(x) + J_{n+2}(x)]$$

(iv) From recurrence relation (2)

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

Multiplying both sides by $\frac{1}{x}$

$$\frac{1}{x}\frac{d}{dx}\left[\frac{1}{x^n}\, J_n(x)\right] = -\frac{1}{x^{n+1}}\, J_{n+1}(x) \quad \ldots\ldots\ldots\text{①}$$

Multiplying both sides of ① by $\left(\frac{1}{x}\frac{d}{dx}\right)$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^2 [x^{-n}J_n(x)] = -\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x^{n+1}}\, J_{n+1}(x)\right]$$

$$= (-1)^2 \frac{1}{x^{n+2}} J_{n+2}(x) \text{ using ①} \ldots\ldots\ldots\text{②}$$

Again multiplying both sides of ② by $\left(\frac{1}{x}\frac{d}{dx}\right)$

$$\left(\frac{1}{x}\frac{d}{dx}\right)^3 [x^{-n}J_n(x)] = (-1)^2\left(\frac{1}{x}\frac{d}{dx}\right)\left[\frac{1}{x^{n+2}}J_{n+2}(x)\right]$$

$$= (-1)^3\frac{1}{x^{n+3}}J_{n+3}(x) \text{ again using ①}$$

Continuing in this manner m times, we get

$$\left(\frac{1}{x}\frac{d}{dx}\right)^m [x^{-n}J_n(x)] = (-1)^m\frac{1}{x^{n+m}}J_{n+m}(x)$$

(v) $\frac{d}{dx}[x\,J_n(x)\,J_{n+1}(x)] = J_n(x)\,J_{n+1}(x) + x\,J_n{}'(x)\,J_{n+1}(x) + x\,J_n(x)\,J_{n+1}{}'(x)$…①

From recurrence relation (4)

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x) \qquad \text{………②}$$

Also from recurrence relation (3)

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

$$\Rightarrow J_{n+1}'(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x) \qquad \text{……..③}$$

Using ② and ③ in ① we get

$$\frac{d}{dx}[x\,J_n(x)\,J_{n+1}(x)] = J_n(x)\,J_{n+1}(x) + x\left[-J_{n+1}(x) + \frac{n}{x}J_n(x)\right]J_{n+1}(x) +$$

$$x\,J_n(x)\left[J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right]$$

$$= J_n(x)\,J_{n+1}(x) - xJ_{n+1}{}^2(x) + nJ_n(x)J_{n+1}(x) +$$

$$xJ_n{}^2(x) - (n+1)J_n(x)J_{n+1}(x)$$

$$= x\left[J_n{}^2(x) - J_{n+1}{}^2(x)\right]$$

(vi) $J_0'(x) = -J_1(x)$ from (i)

$$\Rightarrow J_0''(x) = -J_1'(x)$$

$$= -\frac{1}{2}[J_0(x) - J_2(x)] \qquad \text{Using } J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

Differentiating again we get

$$J_0'''(x) = -\frac{1}{2}[J_0'(x) - J_2'(x)]$$

$$= -\frac{1}{2}\left[J_0'(x) - \frac{1}{2}\big(J_1(x) - J_3(x)\big)\right] \text{ Using } J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

$$= -\frac{1}{2}\left[J_0'(x) + \frac{1}{2}J_0'(x) + \frac{1}{2}J_3(x)\right] \quad \text{Using} \quad -J_1(x) = J_0'(x)$$

$$= -\frac{1}{4}[3J_0'(x) + J_3(x)]$$

$$\Rightarrow J_3(x) + 3J_0'(x) + 4J_0'''(x) = 0$$

Example 3 Show that:

(i) $\int x\,{J_0}^2(x)dx = \frac{1}{2}x^2\left[{J_0}^2(x) + {J_1}^2(x)\right]$

(ii) $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\; J_{\frac{1}{2}}(2x)dx = 1$

(iii) $\int J_3(x)\,dx = -\frac{2}{x}J_1(x) - J_2(x)$

(iv) $\int x^{-1}J_4(x)\,dx = -\frac{1}{x}J_3(x) - \frac{2}{x^2}J_2(x)$

(v) $\int J_5(x)\,dx = -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x)$

Solution: (i) $\int x{J_0}^2(x)\,dx = {J_0}^2(x).\frac{x^2}{2} - \int 2J_0(x)J_0'(x).\frac{x^2}{2}\,dx$

$$= {J_0}^2(x).\frac{x^2}{2} + \int x^2 J_0(x)\,J_1(x)dx \qquad \because J_0'(x) = -J_1(x)$$

$$= {J_0}^2(x).\frac{x^2}{2} + \int\; x\,J_1(x)\;\; xJ_0(x)dx$$

$$= {J_0}^2(x).\frac{x^2}{2} + \int x\,J_1(x)\,\frac{d}{dx}[x\,J_1(x)]dx$$

$\because xJ_0(x) = \frac{d}{dx}[x\,J_1(x)]$ from recurrence relation (1) by putting $n = 1$

$$\therefore \int x{J_0}^2(x)\,dx = {J_0}^2(x).\frac{x^2}{2} + \frac{(x\,J_1(x))^2}{2}$$

$$= \frac{1}{2}x^2\left[{J_0}^2(x) + {J_1}^2(x)\right]$$

(ii) $\int_0^{\frac{\pi}{2}} \sqrt{\pi x}\; J_{\frac{1}{2}}(2x)dx = \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\pi} \sqrt{t}\; J_{\frac{1}{2}}(t)dt$ Putting $2x = t$

$$= \frac{\sqrt{\pi}}{2\sqrt{2}}\int_0^{\pi}\left(\sqrt{t}\sqrt{\frac{2}{\pi t}}\sin t\right)dt \qquad \because J_{\frac{1}{2}}(t) = \sqrt{\frac{2}{\pi t}}\sin t$$

$$= \int_0^{\pi} \sin t\; dt = 1$$

(iii) $\int J_3(x)\,dx = \int x^2 x^{-2} J_3(x)dx$

$$= -\int x^2\,\frac{d}{dx}[x^{-2}J_2(x)]dx \qquad \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

$= -x^2[x^{-2}J_2(x)] + \int 2x[x^{-2}J_2(x)]dx$ Integrating by parts

$= -J_2(x) + 2\int[x^{-1}J_2(x)]\,dx$

$= -J_2(x) - 2\int \frac{d}{dx}[x^{-1}J_1(x)]\,dx \;\; \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -J_2(x) - \frac{2}{x}J_1(x)$

(iv) $\int x^{-1}J_4(x)\,dx = \int x^2(x^{-3}J_4(x))\,dx$

$= -\int x^2 \frac{d}{dx}(x^{-3}J_3(x))\,dx \;\; \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -[x^2x^{-3}J_3(x) - 2\int x\,x^{-3}J_3(x)dx]$ Integrating by parts

$= -\frac{1}{x}J_3(x) + 2\int x^{-2}J_3(x)dx$

$= -\frac{1}{x}J_3(x) - 2\int \frac{d}{dx}(x^{-2}J_2(x))dx$

$= -\frac{1}{x}J_3(x) - \frac{2}{x^2}J_2(x)$

(v) $\int J_5(x)\,dx = \int x^4x^{-4}J_5(x)dx$

$= -\int x^4 \frac{d}{dx}[x^{-4}J_4(x)]dx \;\; \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -x^4[x^{-4}J_4(x)] + \int 4x^3[x^{-4}J_4(x)]dx$ Integrating by parts

$= -J_4(x) + 4\int[x^2x^{-3}J_4(x)]\,dx$

$= -J_4(x) - 4\int x^2 \frac{d}{dx}[x^{-3}J_3(x)]dx \;\; \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -J_4(x) - 4x^2x^{-3}J_3(x) + 8\int x^{-2}J_3(x)dx$ Integrating by parts

$= -J_4(x) - \frac{4}{x}J_3(x) - 8\int \frac{d}{dx}[x^{-2}J_2(x)]dx \;\; \because \frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$

$= -J_4(x) - \frac{4}{x}J_3(x) - \frac{8}{x^2}J_2(x)$

Example 4 Express $J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$

Solution: From recurrence relation (6)

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \;\Rightarrow J_{n+1}(x) = \frac{2n}{x}J_n(x) - J_{n-1}(x) \;\ldots.①$$

Putting $n = 4$ in ①, we get $J_5(x) = \frac{8}{x}J_4(x) - J_3(x)$ …….②

Putting $n = 3$ in ①, we get $J_4(x) = \frac{6}{x} J_3(x) - J_2(x)$ ....③

Putting $n = 2$ in ①, we get $J_3(x) = \frac{4}{x} J_2(x) - J_1(x)$ ....④

Putting $n = 1$ in ①, we get $J_2(x) = \frac{2}{x} J_1(x) - J_0(x)$ ....⑤

Using ⑤ in ④, we get $J_3(x) = \frac{4}{x}\left[\frac{2}{x} J_1(x) - J_0(x)\right] - J_1(x)$

$\Rightarrow J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$ ......⑥

$\therefore J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$ = $\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)$

Using ⑥and ⑤ in ③ , we get

$$J_4(x) = \frac{6}{x}\left[\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right] - \left[\frac{2}{x} J_1(x) - J_0(x)\right]$$

$$= \left[\left(\frac{48}{x^3} - \frac{6}{x} - \frac{2}{x}\right) J_1(x)\right] + \left[\left(-\frac{24}{x^2} + 1\right) J_0(x)\right]$$

$$= \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x) \text{ .... ⑦}$$

$\therefore J_4(x)$ in terms of $J_0(x)$ and $J_1(x)$ = $\left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$

Using ⑦ and ⑥ in ②, we get

$$J_5(x) = \frac{8}{x}\left\{\left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)\right\} - \left\{\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x} J_0(x)\right\}$$

$$= \left(\frac{384}{x^4} - \frac{64}{x^2} - \frac{8}{x^2} + 1\right) J_1(x) + \left(\frac{8}{x} - \frac{192}{x^3} + \frac{4}{x}\right) J_0(x)$$

$$= \left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0(x)$$

$\therefore J_5(x)$ in terms of $J_0(x)$ and $J_1(x)$ = $\left(\frac{384}{x^4} - \frac{72}{x^2} + 1\right) J_1(x) + \left(\frac{12}{x} - \frac{192}{x^3}\right) J_0(x)$

12.2.3 **Orthogonality of Bessel's Function**

If α and β be roots of the equation $J_n(x) = 0$, then

$$\int_0^1 x J_n(\alpha x) J_n(\beta x) dx = \begin{cases} 0 & if\ \beta \neq \alpha \\ \frac{1}{2} J_{n+1}^2(\alpha) & if\ \beta = \alpha \end{cases}$$

Proof: Given that α and β be roots of the equation $J_n(x) = 0$

$\therefore J_n(\alpha) = J_n(\beta) = 0$ ……①

Consider the Bessel's equations:

$x^2u'' + xu' + (\alpha^2x^2 - n^2)u = 0$ …..②

$x^2v'' + xv' + (\beta^2x^2 - n^2)v = 0$….. ③

Solutions of ② and ③are respectively $u = J_n(\alpha x)$ and $v = J_n(\beta x)$

Multiplying ② by $\frac{v}{x}$ and ③by $\frac{u}{x}$ and subtracting

$x(u''v - uv'') + (u'v - uv') + (\alpha^2 - \beta^2)uvx = 0$

$\frac{d}{dx}[x(u'v - uv') + (\alpha^2 - \beta^2)uvx = 0]$ ….④

Integrating ④ with respect to x from 0 to 1

$[x(u'v - uv')]_0^1 + (\alpha^2 - \beta^2)\int_0^1 uvx\,dx = 0$

$\Rightarrow (\alpha^2 - \beta^2)\int_0^1 uvx\,dx = [uv' - u'v]_{x=1}$

$\Rightarrow (\alpha^2 - \beta^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = [\beta J_n(\alpha x)J_n'(\beta x) - \alpha J_n'(\alpha x)J_n(\beta x)]_{x=1}$

$\because u = J_n(\alpha x)$, $v = J_n(\beta x)$, $u' = \alpha J_n'(\alpha x)$, $v' = \beta J_n'(\beta x)$

$\Rightarrow (\alpha^2 - \beta^2)\int_0^1 xJ_n(\alpha x)J_n(\beta x)\,dx = \beta J_n(\alpha)J_n'(\beta) - \alpha J_n'(\alpha)J_n(\beta)$ …..⑤

$\therefore \int_0^1 xJ_n(\alpha x)\,J_n(\beta x)dx = 0$ if $if\ \beta \neq \alpha$

$\because J_n(\alpha) = J_n(\beta) = 0$ From ①

Again if $\beta = \alpha$, $\int_0^1 J_{n+1}^2(\alpha)\,dx = \frac{\alpha J_n(\alpha)J_n'(\alpha) - \alpha J_n'(\alpha)J_n(\alpha)}{\alpha^2 - \alpha^2}$ which is $\frac{0}{0}$ form

To overcome this difficulty, let α be root of the equation $J_n(x) = 0$,

so that $J_n(\alpha) = 0$, also let $\beta = \alpha + h$

Substituting $J_n(\alpha) = 0$, $\beta = \alpha + h$ and taking limit $h \to 0$ in ⑤

$\lim_{h\to 0}\int_0^1 xJ_n(\alpha x)J_n(\alpha + h\ )x\,dx = \lim_{h\to 0}\frac{-\alpha J_n'(\alpha)J_n(\alpha+h)}{\alpha^2 - (\alpha+h)^2} = \lim_{h\to 0}\frac{\alpha J_n'(\alpha)J_n(\alpha+h)}{h^2+2\alpha h}$

It is still $\frac{0}{0}$ form, so applying L Hopital's rule on R.H.S.

$$\lim_{h\to 0}\int_0^1 xJ_n(\alpha x)J_n(\alpha+h\ )x\,dx = \lim_{h\to 0}\alpha\left[\frac{J_n'(\alpha)\,J_n'(\alpha+h)+J_n''(\alpha)J_n(\alpha+h\ )}{2h+2\alpha}\right]$$

$$=\frac{1}{2}J_n'^2(\alpha)=\frac{1}{2}J_{n+1}^2(\alpha)$$

12.3 Legendre's Equation

Another important differential equation used in problems showing spherical symmetry is Legendre's equation given by $(1-x^2)\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2}-2x\frac{\mathrm{dy}}{\mathrm{d}x}+n(n+1)y\ =0$ …… ①

Here n is a real number, though in most practical applications only non-negative integral values are required.

Series solution of ①in terms of Legendre's function $P_n(x)$ and $Q_n(x)$ is given by

$y = AP_n(x)+BQ_n(x)$,

where $P_n(x)=\frac{1.3.5\ldots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}-\cdots\right]$

$P_n(x)$ is a terminating series containing positive powers of x.

$Q_n(x)=\frac{n!}{1.3.5\ldots(2n+1)}\left[x^{-n-1}+\frac{(n+1)(n+2)}{2(2n+3)}x^{-n-3}+\frac{(n+1)(n+2)(n+3)(n+4)}{2.4(2n+3)(2n+5)}x^{-n-5}\ldots\right]$

$Q_n(x)$ is a non-terminating (infinite) series containing negative powers of x.

12.3.1 Generating function for $P_n(x)$

The function $(1-2xz+z^2)^{-\frac{1}{2}}$ is called the generating function of Legendre's polynomials

as $(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)z^n$

Proof: $(1-2xz+z^2)^{-\frac{1}{2}}=\ [1-(2xz-z^2)]^{-\frac{1}{2}}$

$$=1+\frac{1}{2}(2xz-z^2)+\frac{1}{2}\,\frac{3}{4}(2xz-z^2)^2+\cdots+\frac{1}{2}\,\frac{3}{4}\ldots\frac{2k-1}{2}(2xz-z^2)^k+\cdots$$

$$\because(1-t)^{-\frac{1}{2}}=1+\frac{1}{2}\,t+\frac{1}{2}\,\frac{3}{2}\,\frac{t^2}{2!}+\frac{1}{2}\,\frac{3}{2}\frac{5}{2}\,\frac{t^3}{3!}+\cdots$$

$$=1+\sum_{k=0}^{\infty}\frac{1}{2}\,\frac{3}{4}\ldots\frac{2k-1}{2k}(2xz-z^2)^k\ \ldots\ldots①$$

Again $(2xz-z^2)^k=\ z^k[2x-z]^k$

$$=\ z^k\left[(2x)^k-k(2x)^{k-1}z+\frac{k(k-1)}{2!}(2x)^{k-2}z^2-\cdots+(-1)^kz^k\right]\ldots ②$$

Using ② in ① , we get $(1-2xz+z^2)^{-\frac{1}{2}}=$

$$1+\sum_{k=0}^{\infty}\frac{1}{2}\,\frac{3}{4}\dots\frac{2k-1}{2k}\left[(2x)^k z^k - k(2x)^{k-1}z^{k+1}+\frac{k(k-1)}{2!}(2x)^{k-2}z^{k+2}-\dots+(-1)^k z^{2k}\right]$$

……③

Coefficient of z^n in expression ③ is given by

$$\frac{1}{2}\,\frac{3}{4}\dots\left(\frac{2n-1}{2n}\right)(2x)^n-\frac{1}{2}\,\frac{3}{4}\dots\left(\frac{2n-3}{2n-2}\right)(n-1)(2x)^{n-2}+\frac{1}{2}\,\frac{3}{4}\dots\left(\frac{2n-5}{2n-4}\right)\frac{(n-2)(n-3)}{2!}(2x)^{n-4}-\dots$$

$$=\frac{1.3.5\dots(2n-1)}{n!}\left[x^n-\frac{n(n-1)}{2(2n-1)}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2.4.(2n-1)(2n-3)}x^{n-4}-\dots\right]$$

$$=P_n(x)$$

$$\therefore\ (1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)z^n$$

Corollary (i) $\boldsymbol{P_n(1)=1}$ (ii) $\boldsymbol{P_n(-1)=(-1)^n}$

Proof: $(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)z^n$ …..①

Putting $x=1$ on both sides of ①

$$(1-2z+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(1)z^n$$

$$\Rightarrow\ (1-z)^{-1}=\sum_{n=0}^{\infty}P_n(1)z^n$$

$$\Rightarrow 1+z+z^2+\dots+z^n+\dots=\sum_{n=0}^{\infty}P_n(1)z^n$$

$$\sum_{n=0}^{\infty}z^n=\sum_{n=0}^{\infty}P_n(1)z^n$$

Comparing the coefficients of z^n on both sides, we get $P_n(1)=1$

Putting $x=-1$ on both sides of ①

$$(1+2z+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(1)z^n$$

$$\Rightarrow\ (1+z)^{-1}=\sum_{n=0}^{\infty}P_n(1)z^n$$

$$\Rightarrow 1-z+z^2-\dots+(-1)^n z^n+\dots=\sum_{n=0}^{\infty}P_n(1)z^n$$

$$\sum_{n=0}^{\infty}(-1)^n z^n=\sum_{n=0}^{\infty}P_n(1)z^n$$

Comparing the coefficients of z^n on both sides, we get $P_n(-1)=(-1)^n$

Example 5 Prove that $P_n(-x)=(-1)^nP_n(x)$

Solution: Generating function of $P_n(x)$ is given by

$$(1-2xz+z^2)^{-\frac{1}{2}}=\sum_{n=0}^{\infty}P_n(x)z^n\text{…..①}$$

Replacing x by $-x$ on both sides of ①

$$(1+2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(-x)z^n \qquad ..... ②$$

Again replacing z by $-z$ on both sides of ①

$$(1+2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} P_n(x)(-z)^n = \sum_{n=0}^{\infty}(-1)^n P_n(x)z^n \qquad \cdots ③$$

Comparing ②and ③

$$\sum_{n=0}^{\infty} P_n(-x)z^n = \sum_{n=0}^{\infty}(-1)^n P_n(x)z^n$$

$$\Rightarrow P_n(-x) = (-1)^n P_n(x)$$

12.3.2 Recurrence Relations of Legendre's Function $P_n(x)$

(1) $(n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)$

Proof: From generating function $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x) \ldots\ldots ①$

Differentiating both sides of ① partially with respect to z, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2x+2z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)(1-2xz+z^2)^{-\frac{1}{2}-1} = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x) \text{ using ①}$$

Equating coefficient of z^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)$$

(2) $P_n(x) = \mathrm{P}'_{n+1}(x) - 2x\, \mathrm{P}'_n(x) + \mathrm{P}'_{n-1}(x)$

Differentiating both sides of ① partially with respect to x, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-1-\frac{1}{2}}(-2z) = \sum_{n=0}^{\infty} z^n\, \mathrm{P}'_n(x)$$

$$\Rightarrow z(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\, \mathrm{P}'_n(x)$$

$$\Rightarrow z\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\, \mathrm{P}'_n(x) \text{ using ①}$$

Equating coefficient of z^{n+1} on both sides

$$P_n(x) = \mathrm{P}'_{n+1}(x) - 2x\,\mathrm{P}'_n(x) + \mathrm{P}'_{n-1}(x)$$

(3) $\boldsymbol{nP_n(x) = x\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)}$

Differentiating recurrence relation (1) partially with respect to x, we get

$(n+1)\mathrm{P}'_{n+1}(x) = (2n+1)\,x\,\mathrm{P}'_n(x) + (2n+1)P_n(x) - n\mathrm{P}'_{n-1}(x)$…..②

Also from recurrence relation (2)

$\mathrm{P}'_{n+1}(x) = P_n(x) + 2x\,\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)$ …… ③

Using ③ in ②, we get

$$(n+1)[P_n(x) + 2x\,\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)] = (2n+1)\,x\,\mathrm{P}'_n(x) + (2n+1)P_n(x) - n\mathrm{P}'_{n-1}(x)$$

$\Rightarrow nP_n(x) = x\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)$

(4) $\boldsymbol{(n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - x\mathrm{P}'_n(x)}$

Adding recurrence relations (2) and (3), we get

$(n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - x\mathrm{P}'_n(x)$

(5) $\boldsymbol{(2n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - \mathrm{P}'_{n-1}(x)}$

Adding recurrence relations (3) and (4), we get

$(2n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - \mathrm{P}'_{n-1}(x)$

(6) $\boldsymbol{(1-x^2)\,\mathrm{P}'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]}$

Replacing n by $(n-1)$ in recurrence relation (4)

$nP_{n-1}(x) = \mathrm{P}'_n(x) - x\mathrm{P}'_{n-1}(x)$ …… ④

Also multiplying recurrence relation (3) by x

$n\,xP_n(x) = x^2\mathrm{P}'_n(x) - x\mathrm{P}'_{n-1}(x)$ ……. ⑤

Subtracting ⑤ from ④

$(1-x^2)\,\mathrm{P}'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]$

(7) $\boldsymbol{(1-x^2)\,\mathrm{P}'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]}$

Replacing n by $(n+1)$ in recurrence relation (3)

$\Rightarrow (n+1)P_{n+1}(x) = x\mathrm{P}'_{n+1}(x) - \mathrm{P}'_n(x)$ ……⑥

Also multiplying recurrence relation (4) by x

$$(n+1)xP_n(x) = x\mathrm{P}'_{n+1}(x) - x^2\mathrm{P}'_n(x) \quad \text{........} \ ⑦$$

Subtracting ⑥ from ⑦, we get

$$(1-x^2)\ \mathrm{P}'_n(x) = (n+1)\ [xP_n(x) - P_{n+1}(x)]$$

Example 6 Prove that $\int P_n(x)dx + \mathrm{C} = \frac{P_{n+1}(x)-P_{n-1}(x)}{2n+1}$

Solution: From recurrence relation (5)

$$(2n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - \mathrm{P}'_{n-1}(x)$$

$$\Rightarrow P_n(x) = \frac{\mathrm{P}'_{n+1}(x) - \mathrm{P}'_{n-1}(x)}{(2n+1)}$$

Integrating both sides, we get

$$\int P_n(x)dx + \mathrm{C} = \frac{P_{n+1}(x)-P_{n-1}(x)}{2n+1}$$

Example 7 Show that $\mathrm{P}'_{n+1} + \mathrm{P}'_{n-1} = \mathrm{P}_0 + 3\mathrm{P}_1 + \cdots + (2n+1)\mathrm{P}_n$

Solution: From recurrence relation (5)

$$(2n+1)P_n = \mathrm{P}'_{n+1} - \mathrm{P}'_{n-1}$$

Putting $n = 1,2,3 \ldots .n$, we get

$$3P_1 = \mathrm{P}'_2 - \mathrm{P}'_0$$

$$5P_2 = \mathrm{P}'_3 - \mathrm{P}'_1$$

$$7P_3 = \mathrm{P}'_4 - \mathrm{P}'_2$$

... .

$$(2n-1)P_{n-1} = \mathrm{P}'_n - \mathrm{P}'_{n-2}$$

$$(2n+1)P_n = \mathrm{P}'_{n+1} - \mathrm{P}'_{n-1}$$

Adding all these relations, we get

$$3P_1 + 5P_2 + 7P_3 + \cdots + (2n+1)P_n = -\mathrm{P}'_0 - \mathrm{P}'_1 + \mathrm{P}'_n + \mathrm{P}'_{n+1}$$

$$= 0 - P_0 + \mathrm{P}'_n + \mathrm{P}'_{n+1}$$

$$\because \mathrm{P}'_0 = 0, \mathrm{P}'_1 = 1 = P_0$$

$$\Rightarrow P'_{n+1} + P'_{n-1} = P_0 + 3P_1 + \cdots + (2n+1)P_n$$

12.3.3 Rodrigue's Formula

Rodrigue's formula is helpful in producing Legendre's polynomials of various orders and is given by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Proof: Let $y = (x^2-1)^n$

$$\therefore \frac{dy}{dx} = n(x^2-1)^{n-1}2x = 2nx\frac{(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow y_1(x^2-1) - 2nxy = 0\ ,\ y_1 \equiv \frac{dy}{dx} \quad \ldots\ldots\ldots ②$$

Differentiating ② $(n+1)$ times using Leibnitz's theorem:

$$\Rightarrow y_{n+2}(x^2-1) + (n+1)y_{n+1}(2x) + \frac{(n+1).n}{2!}y_n(2) - 2n[y_{n+1}(x) + (n+1)y_n(1)] = 0$$

$$\Rightarrow y_{n+2}(x^2-1) + 2xy_{n+1} - (n^2+n)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0 \ \ldots\ldots\ldots ③$$

Putting $y_n = V$, so that $y_{n+1} = \frac{dV}{dx}$ and $y_{n+2} = \frac{d^2V}{dx}$

$$③ \Rightarrow (1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + n(n+1)V = 0$$

which is Legendre's equation with the solution $V = AP_n(x) + BQ_n(x)$

But since $V = y_n = \frac{d^n}{dx^n}(x^2-1)^n$ contains only positive powers of x, solution can only be a constant multiple of $P_n(x)$.

$$\therefore P_n(x) = CV = Cy_n$$

$$= C\frac{d^n}{dx^n}(x^2-1)^n \ \ldots\ldots\ ④$$

$$= CD^n[(x-1)^n(x+1)^n] \quad , \frac{d^n}{dx^n} \equiv D^n$$

$$= CD^n[(x-1)^n(x+1)^n]$$

$$= C\left[D^n(x-1)^n(x+1)^n + n_{C_1}D^{n-1}(x-1)^n n(x+1)^{n-1} + \cdots + (x-1)^n D^n(x+1)^n\right]$$

$$= C[n!\,(x+1)^n + n.n(n-1)\ldots 3.2.(x-1)n(x+1)^{n-1} + \cdots + (x-1)^n n!]$$

Taking $x = 1$ on both sides

$$\Rightarrow 1 = Cn!\,2^n + 0 \quad \because P_n(1) = 1$$

$$\Rightarrow C = \frac{1}{2^n n!} \quad \ldots\ldots\ldots ⑤$$

Using ⑤ in ④ , we get

$P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$

Putting $n = 0$, $P_0(x) = 1$

Putting $n = 1$, $P_1(x) = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1)^1 = \frac{1}{2}2x = x$

Putting $n = 2$, $P_2(x) = \frac{1}{2^2 2!}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1)$

Putting $n = 3$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

Putting $n = 4$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Putting $n = 5$, $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$ etc…

Example 8 Expand the following functions in series of Legendre's polynomials.

(i) $(1+2x-x^2)$

(ii) (x^3-5x^2+x+1)

Solution: $1 = P_0(x)$, $x = P_1(x)$,

$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{1}{3}(2P_2(x)+1) = \frac{1}{3}\left(2P_2(x)+P_0(x)\right)$

$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow x^3 = \frac{1}{5}(2P_3(x)+3x) = \frac{1}{5}\left(2P_3(x)+3P_1(x)\right)$

(i) Let $E = (1+2x-x^2)$

Substituting values of 1, x and x^2 in terms of Legendre's polynomials, we get

$E = \left(P_0(x)+2P_1(x)-\frac{1}{3}\left(2P_2(x)+P_0(x)\right)\right)$

$= \frac{1}{3}\left(3P_0(x)+6P_1(x)-2P_2(x)-P_0(x)\right)$

$= \frac{2}{3}\left(P_0(x)+3P_1(x)-P_2(x)\right)$

(ii) Let $F = (x^3-5x^2+x+1)$

Substituting values of 1, x, x^2 and x^3 in terms of Legendre's polynomials, we get

$F = \left[\frac{1}{5}\left(2P_3(x)+3P_1(x)\right)-\frac{5}{3}\left(2P_2(x)+P_0(x)\right)+P_1(x)+P_0(x)\right]$

$= \frac{2}{5}P_3(x)-\frac{10}{3}P_2(x)+\frac{8}{5}P_1(x)-\frac{2}{3}P_0(x)$

Example 9 Prove that (i) $\int_{-1}^{1} P_n(x)dx = \begin{cases} 0 & n \neq 0 \\ 2 & n = 0 \end{cases}$

(ii) $\int_{-1}^{1} x^m P_n(x)dx = 0$, if $m < n$ where m and n are positive integers

Solution: (i) $\int_{-1}^{1} P_n(x) = \frac{1}{2^n n!} \int_{-1}^{1} D^n (x^2-1)^n dx$

$$= \frac{1}{2^n n!} [D^{n-1}(x^2-1)^n]_{-1}^{1}$$

$$= \frac{1}{2^n n!} \left[D^{n-1}[(x-1)^n(x+1)^n]\right]_{-1}^{1}$$

Expanding using Leibnitz's theorem

$$= \frac{1}{2^n n!} [D^{n-1}(x-1)^n(x+1)^n + \cdots + (x-1)^n D^{n-1}(x+1)^n]_{-1}^{1}$$

$$= \frac{1}{2^n n!} [n.(n-1) \ldots 3.2(x-1)(x+1)^n + \cdots + (x-1)^n n.(n-1) \ldots 3.2(x+1)]_{-1}^{1}$$

$= 0$ on putting $x = 1$ or -1

Now when $n = 0$, $P_0(x) = 1$

$$\therefore \int_{-1}^{1} P_0(x)dx = \int_{-1}^{1} 1dx = 2$$

(ii) $\int_{-1}^{1} x^m P_n(x)dx = \frac{1}{2^n n!} \int_{-1}^{1} x^m D^n (x^2-1)^n dx$

$$= \frac{1}{2^n n!} \left\{[x^m D^{n-1}(x^2-1)^n]_{-1}^{1} - \int_{-1}^{1} m x^{m-1} D^{n-1}(x^2-1)^n dx\right\}$$

$$= \frac{1}{2^n n!} \left\{0 - \int_{-1}^{1} m x^{m-1} D^{n-1}(x^2-1)^n dx\right\} \quad \text{By part (i)}$$

$$= \frac{-m}{2^n n!} \left\{\int_{-1}^{1} x^{m-1} D^{n-1}(x^2-1)^n dx\right\}$$

Continuing the process $(m-1)$ times, we get

$$\int_{-1}^{1} x^m P_n(x)dx = \frac{-1^m m!}{2^n n!} \left\{\int_{-1}^{1} x^0 D^{n-m}(x^2-1)^n dx\right\}$$

$$= \frac{-1^m m!}{2^n n!} [D^{m-n-1}(x^2-1)^n]_{-1}^{1}$$

$= 0$ if $n \geq m+1$ i.e. $n > m$

12.3.4 Orthogonality of Legendre's polynomial

Orthogonality property of Legendre's polynomials is given by the relations

$$\int_{-1}^{1} P_m(x)P_n(x)dx = 0, \text{ when } m \neq n$$

and $\int_{-1}^{1} {P_n}^2(x)dx = \frac{2}{2n+1}$, when $m = n$

where m and n are positive integers

Proof: By Rodrigue's formula

$$P_m(x) = \frac{1}{2^m m!} \frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^m \quad \text{and} \quad P_n(x) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

$$\therefore \mathrm{I} = \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{1}{2^m 2^n m!\, n!} \int_{-1}^{1} D^n(x^2-1)^n D^m(x^2-1)^m dx$$

Integrating by parts

$$= \frac{1}{2^{m+n}\, m!\, n!} \left\{[D^n(x^2-1)^n D^{m-1}(x^2-1)^m]_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n D^{m-1}(x^2-1)^m dx\right\}$$

$$= \frac{1}{2^{m+n} m!\, n!} \left\{0 - \int_{-1}^{1} D^{n+1}(x^2-1)^n D^{m-1}(x^2-1)^m dx\right\}$$

$$= \frac{-1}{2^{m+n} m!\, n!} \left\{\int_{-1}^{1} D^{n+1}(x^2-1)^n D^{m-1}(x^2-1)^m dx\right\}$$

Continuing the process $(n-1)$ times

$$= \frac{(-1)^n}{2^{m+n} m!\, n!} \left\{\int_{-1}^{1} D^{2n}(x^2-1)^n D^{m-n}(x^2-1)^m dx\right\}$$

$$= \frac{(-1)^n}{2^{m+n} m!\, n!} \left\{\int_{-1}^{1} D^{2n}(x^2-1)^n D^{m-n}(x^2-1)^m dx\right\}$$

$$= \frac{(-1)^n\, 2n!}{2^{m+n} m!\, n!} \left\{\int_{-1}^{1} D^{m-n}(x^2-1)^m dx\right\} \qquad \ldots\ldots ①$$

$\because D^{2n}(x^2-1)^n = 2n!$ using Leibnitz's theorem

$$= \frac{(-1)^n\, 2n!}{2^{m+n} m!\, n!} [D^{m-n-1}(x^2-1)^m]_{-1}^{1} = 0$$

$\therefore \mathrm{I} = \int_{-1}^{1} P_m(x)P_n(x)dx = 0$, when $m \neq n$

Again putting $m = n$ in ①

$$\mathrm{I} = \int_{-1}^{1} {P_n}^2(x)dx = \frac{(-1)^n\, 2n!}{2^{2n}\,(n!)^2} \left\{\int_{-1}^{1}(x^2-1)^n dx\right\}$$

$$= \frac{2n!}{2^{2n}\,(n!)^2} \left\{\int_{-1}^{1}(1-x^2)^n dx\right\}$$

Put $x = sin\theta \;\therefore dx = cos\theta\; d\theta$

$$\mathrm{I} = \frac{2n!}{2^{2n}\,(n!)^2} \left\{\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1}\theta\, d\theta\right\}$$

$$= \frac{2n!}{2^{2n}\,(n!)^2} 2 \left\{\int_{0}^{\frac{\pi}{2}} \cos^{2n+1}\theta\, d\theta\right\}$$

$$= \frac{2n!}{2^{2n}\,(n!)^2}\; 2\,\frac{\Gamma\frac{(2n+1+1)}{2}\,\Gamma\frac{1}{2}}{2\,\Gamma\frac{(2n+1+2)}{2}} = \frac{2n!}{2^{2n}\,(n!)^2}\frac{\Gamma(n+1)\Gamma\frac{1}{2}}{\Gamma\frac{(2n+3)}{2}} = \frac{2n!}{2^{2n}\,(n!)^2}\;\frac{n!\,\Gamma\frac{1}{2}}{\frac{(2n+1)}{2}\frac{(2n-1)(2n-3)}{2\quad\;\; 2}\cdots\frac{31}{22}\Gamma\frac{1}{2}}.$$

$= \frac{2n!}{2^{2n}n!}\ \frac{2^{n+1}}{(2n+1)(2n-1)(2n-3)\ldots3.1} = \frac{2(2n)(2n-1)(2n-2)\ldots3.2.1}{2^n n!\,(2n+1)(2n-1)(2n-3)\ldots3.1}$

$= \frac{2[(2n)(2n-2)\ldots.2]\,[(2n-1)(2n-3)\ldots3.1]}{2^n\,n!\,(2n+1)(2n-1)(2n-3)\ldots3.1} = \frac{2\,.2^n\,n!}{2^n\,n!\,(2n+1)}$

$= \frac{2}{(2n+1)}$

Example 10 Prove that

(i) $P_n'(1) = \frac{n(n+1)}{2}$

(ii) $P_n'(-1) = (-1)^{(n+1)}\frac{n(n+1)}{2}$

Solution: $P_n(x)$ is the solution of Legendre's equation given by:

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0 \quad \ldots\ldots① $$

$\therefore y = P_n(x)$ will satisfy equation ①

$\Rightarrow (1 - x^2)P_n''(x) - 2x\,P_n'(x) + n(n+1)P_n(x) = 0 \ \ldots\ldots②$

Putting $x = 1$ in ② we get

$-2P_n'(1) + n(n+1)P_n(1) = 0$

$\Rightarrow P_n'(1) = \frac{n(n+1)}{2} \because P_n(1) = 1$

Putting $x = -1$ in ② we get

$2P_n'(-1) + n(n+1)P_n(-1) = 0$

$\Rightarrow P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$

$= (-1)^{(n+1)}\frac{n(n+1)}{2} \because P_n(-1) = (-1)^n$

Exercise 12 A

Q1. Prove the following relations for Bessel's Function $J_n(x)$

i. $J_0^2 + 2(J_1^2 + J_2^2 + J_3^2 + \cdots) = 1$

ii. $\frac{1}{2}xJ_n = (n+1)J_{n+1} - (n+3)J_{n+3} + (n+5)J_{n+5} - \cdots$

iii. $J_2 - J_0 = 2J_0''$

iv. $J_2 = J_0'' - \frac{1}{x}J_0'$

v. $\int_0^\infty x^{n+1}J_n(x)\,dx = x^{n+1}J_{n+1}(x)\,,\, n \geq -1$

Q2. Prove the following relations for Legendre's Function $P_n(x)$

i. $(1-x^2)P_n'(x) = \frac{n\,(n+1)}{2n+1}[P_{n-1}(x) - P_{n+1}(x)]$
ii. $P_n'(x) - P_{n-2}'(x) = (2n-1)P_{n-1}(x)$
iii. $\int_0^1 P_n(x)\,dx = \frac{1}{n+1}P_{n-1}(0)$
iv. $\int_{-1}^1 (1-x^2)P_m'P_n'\,dx = 0$, where m and n are distinct integers

Q3. Express the following into Legendre's polynomial:

i. $3x^3 - 2x^2 + 1$ Ans. $\langle \frac{6}{5}P_3 + \frac{4}{3}P_2 + \frac{9}{5}P_1 + \frac{5}{3}P_0 \rangle$
ii. $5x^3 + x^2 - 2x + 1$ Ans. $\langle\, 2P_3 + \frac{2}{3}P_2 + P_1 + \frac{4}{3}P_0 \rangle$

12.4 Previous Years Solved Questions

Q1. Prove that $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$

$\langle \boldsymbol{Q1(g), GGSIPU, December\ 2013} \rangle$

Solution: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$\Rightarrow\ x^n J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)} \frac{x^{2n+2r}}{2^{n+2r}}$$

$$\Rightarrow\ \frac{\boldsymbol{d}}{\boldsymbol{dx}}[x^n J_n(x)] = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,(n+r)\Gamma(n+r)} \frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}}$$

$$\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)$$

$$= x^n \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma((n-1)+r+1)}\left(\frac{x}{2}\right)^{(n-1)+2r}$$

$$= x^n J_{n-1}(x)$$

Q2. Express the following into Legendre's polynomial:

$(x^3 + 2x^2 - x - 3)$ $\langle \boldsymbol{Q1(h), GGSIPU, December\ 2013} \rangle$

Solution: $1 = P_0(x)\ ,\ x = P_1(x),$

$$P_2(x) = \frac{1}{2}(3x^2 - 1) \Rightarrow x^2 = \frac{1}{3}(2P_2(x) + 1) = \frac{1}{3}\big(2P_2(x) + P_0(x)\big)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x) \Rightarrow x^3 = \frac{1}{5}(2P_3(x) + 3x) = \frac{1}{5}\big(2P_3(x) + 3P_1(x)\big)$$

Let $E = (x^3 + 2x^2 - x - 3)$

Substituting values of 1, x, x^2 and x^3 in terms of Legendre's polynomials, we get

$$E = \left[\tfrac{1}{5}\left(2P_3(x) + 3P_1(x)\right) + \tfrac{2}{3}\left(2P_2(x) + P_0(x)\right) - P_1(x) - 3P_0(x)\right]$$

$$= \tfrac{2}{5}P_3(x) + \tfrac{4}{3}P_2(x) + \left(\tfrac{3}{5} - 1\right)\, P_1(x) + \left(\tfrac{2}{3} - 3\right)P_0(x)$$

$$= \tfrac{2}{5}P_3(x) + \tfrac{4}{3}P_2(x) - \tfrac{2}{5}\, P_1(x) - \tfrac{7}{3}P_0(x)$$

Q3. Prove that $J_4(x) = \left(\frac{48}{x^3} - \frac{8}{x}\right) J_1(x) + \left(1 - \frac{24}{x^2}\right) J_0(x)$

$\langle \boldsymbol{Q9(a), GGSIPU, December\ 2013} \rangle$

Solution: From recurrence relation (6)

$$2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)] \;\Rightarrow J_{n+1}(x) = \tfrac{2n}{x}\, J_n(x) - J_{n-1}(x) \;\ldots.①$$

Putting $n = 3$ in ①, we get $\quad J_4(x) = \frac{6}{x}\, J_3(x) - J_2(x) \;\ldots\ldots②$

Putting $n = 2$ in ①, we get $\quad J_3(x) = \frac{4}{x}\, J_2(x) - J_1(x) \;\ldots\ldots③$

Putting $n = 1$ in ①, we get $\quad J_2(x) = \frac{2}{x}\, J_1(x) - J_0(x) \;\ldots\ldots④$

Using ④ in ③, we get $J_3(x) = \frac{4}{x}\left[\frac{2}{x}\, J_1(x) - J_0(x)\right] - J_1(x)$

$\Rightarrow\; J_3(x) = \left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x}J_0(x) \;\ldots\ldots⑤$

$\therefore J_3(x)$ in terms of $J_0(x)$ and $J_1(x)$ = $\left(\frac{8}{x^2} - 1\right) J_1(x) - \frac{4}{x}J_0(x)$

Using ⑤ and ④ in ② , we get

$$J_4(x) = \tfrac{6}{x}\left[\left(\tfrac{8}{x^2} - 1\right) J_1(x) - \tfrac{4}{x}J_0(x)\right] - \left[\tfrac{2}{x}\, J_1(x) - J_0(x)\right]$$

$$= \left[\left(\tfrac{48}{x^3} - \tfrac{6}{x} - \tfrac{2}{x}\right) J_1(x)\right] + \left[\left(-\tfrac{24}{x^2} + 1\right)J_0(x)\right]$$

$$= \left(\tfrac{48}{x^3} - \tfrac{8}{x}\right) J_1(x) + \left(1 - \tfrac{24}{x^2}\right)J_0(x)$$

Q4. Prove that $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}$, $m = n$ where m and n are positive integers

$\langle \boldsymbol{Q9(b), GGSIPU, December\ 2013} \rangle$

Solution: This is orthogonality property of Legendre's polynomials

By Rodrigue's formula

$$P_m(x) = \frac{1}{2^m m!}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(x^2-1)^m \quad \text{and} \quad P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n$$

$$\therefore \mathrm{I} = \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{1}{2^m 2^n m!\,n!}\int_{-1}^{1} D^n(x^2-1)^n D^m(x^2-1)^m dx$$

Integrating by parts

$$=\frac{1}{2^{m+n}\,m!\,n!}\left\{[D^n(x^2-1)^n D^{m-1}(x^2-1)^m]_{-1}^{1} - \int_{-1}^{1} D^{n+1}(x^2-1)^n D^{m-1}(x^2-1)^m dx\right\}$$

$$=\frac{1}{2^{m+n}m!\,n!}\left\{0 - \int_{-1}^{1} D^{n+1}(x^2-1)^n D^{m-1}(x^2-1)^m dx\right\}$$

$$=\frac{-1}{2^{m+n}m!\,n!}\left\{\int_{-1}^{1} D^{n+1}(x^2-1)^n D^{m-1}(x^2-1)^m dx\right\}$$

Continuing the process $(n-1)$ times

$$=\frac{(-1)^n}{2^{m+n}m!\,n!}\left\{\int_{-1}^{1} D^{2n}(x^2-1)^n D^{m-n}(x^2-1)^m dx\right\}$$

$$=\frac{(-1)^n}{2^{m+n}m!\,n!}\left\{\int_{-1}^{1} D^{2n}(x^2-1)^n D^{m-n}(x^2-1)^m dx\right\}$$

$$=\frac{(-1)^n\,2n!}{2^{m+n}m!\,n!}\left\{\int_{-1}^{1} D^{m-n}(x^2-1)^m dx\right\} \qquad \ldots\ldots①$$

$\because D^{2n}(x^2-1)^n = 2n!$ using Leibnitz's theorem

$$=\frac{(-1)^n\,2n!}{2^{m+n}m!\,n!}\,[D^{m-n-1}(x^2-1)^m]_{-1}^{1} = 0$$

Putting $m = n$ in ①

$$\mathrm{I} = \int_{-1}^{1} P_m(x)P_n(x)dx = \frac{(-1)^n\,2n!}{2^{2n}\,(n!)^2}\left\{\int_{-1}^{1}(x^2-1)^n dx\right\}$$

$$= \frac{2n!}{2^{2n}\,(n!)^2}\left\{\int_{-1}^{1}(1-x^2)^n dx\right\}$$

Put $x = sin\theta \;\therefore dx = \; cos\theta\; d\theta$

$$\mathrm{I} = \frac{2n!}{2^{2n}\,(n!)^2}\left\{\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \cos^{2n+1}\theta\, d\theta\right\}$$

$$= \frac{2n!}{2^{2n}\,(n!)^2}\,2\left\{\int_{0}^{\frac{\pi}{2}} \cos^{2n+1}\theta\, d\theta\right\}$$

$$= \frac{2n!}{2^{2n}\,(n!)^2}\;2\,\frac{\left\lceil\frac{(2n+1+1)}{2}\right\lceil\frac{1}{2}}{2\left\lceil\frac{(2n+1+2)}{2}\right.} = \frac{2n!}{2^{2n}\,(n!)^2}\,\frac{\lceil(\mathrm{n}+1)\lceil\frac{1}{2}}{\left\lceil\frac{(2n+3)}{2}\right.} = \frac{2n!}{2^{2n}\,(n!)^2}\;\frac{\mathrm{n}!\left\lceil\frac{1}{2}\right.}{\frac{(2n+1)}{2}\,\frac{(2n-1)}{2}\frac{(2n-3)}{2}\cdots\frac{3}{2}\frac{1}{2}\left\lceil\frac{1}{2}\right.}$$

$$= \frac{2n!}{2^{2n}n!}\;\frac{2^{n+1}}{(2n+1)(2n-1)(2n-3)\ldots3.1} = \frac{2(2n)(2n-1)(2n-2)\ldots3.2.1}{2^n n!\,(2n+1)(2n-1)(2n-3)\ldots3.1}$$

$$= \frac{2[(2n)(2n-2)\ldots.2]\,[(2n-1)(2n-3)\ldots3.1]}{2^n\,n!\,(2n+1)(2n-1)(2n-3)\ldots3.1} = \frac{2\,.2^n\,n!}{2^n\,n!\,(2n+1)}$$

$= \frac{2}{(2n+1)}$

Q4. Show that $J_{\frac{3}{2}}(x) = \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right)$

$\langle \boldsymbol{Q1(j), GGSIPU, December\ 2014} \rangle$

Solution: From recurrence relation (6)

$$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x) \quad \ldots\ldots\ldots\ldots ①$$

Putting $n = \frac{1}{2}$ in ①

$$\Rightarrow \frac{1}{x} J_{\frac{1}{2}}(x) = J_{\frac{-1}{2}}(x) + J_{\frac{3}{2}}(x)$$

$$\Rightarrow J_{\frac{3}{2}}(x) = \frac{1}{x}J_{\frac{1}{2}}(x) - J_{\frac{-1}{2}}(x)$$

$$= \sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x} - \cos x\right) \qquad \because J_{\frac{1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\sin x\,,\ J_{\frac{-1}{2}}(x) = \sqrt{\frac{2}{\pi x}}\cos x$$

Q5. Show that $\frac{d}{dx}[x\,J_n(x)\,J_{n+1}(x)] = x\left[{J_n}^2(x) - {J_{n+1}}^2(x)\right]$

$\langle \boldsymbol{Q9(a), GGSIPU, December\ 2014} \rangle$

Solution: $\frac{d}{dx}[x\,J_n(x)\,J_{n+1}(x)] = J_n(x)\,J_{n+1}(x) + x\,{J_n}'(x)\,J_{n+1}(x) + x\,J_n(x)\,{J_{n+1}}'(x) \ldots ①$

From recurrence relation (4)

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x) \qquad \ldots\ldots\ldots ②$$

Also from recurrence relation (3)

$$J_n'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

$$\Rightarrow J_{n+1}'(x) = J_n(x) - \frac{n+1}{x}J_{n+1}(x) \qquad \ldots\ldots ③$$

Using ② and ③ in ① we get

$$\frac{d}{dx}[x\,J_n(x)\,J_{n+1}(x)] = J_n(x)\,J_{n+1}(x) + x\left[-J_{n+1}(x) + \frac{n}{x}J_n(x)\right]J_{n+1}(x) +$$

$$x\,J_n(x)\left[J_n(x) - \frac{n+1}{x}J_{n+1}(x)\right]$$

$$= J_n(x)\,J_{n+1}(x) - x{J_{n+1}}^2(x) + nJ_n(x)J_{n+1}(x) +$$

$$xJ_n{}^2(x) - (n+1)J_n(x)J_{n+1}(x)$$

$$= x\left[J_n{}^2(x) - J_{n+1}{}^2(x)\right]$$

Q6. Prove that $\int_{-1}^{1} x^2\, P_{n+1}(x)\, P_{n-1}(x)dx = \frac{2n(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$

⟨Q9(b), GGSIPU, December 2014⟩

Solution: From recurrence relation (1) for Legendre's polynomials

$$(n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)$$

$$\Rightarrow (2n+1)x\, P_n(x) = (n+1)P_{n+1}(x) + nP_{n-1}(x) \ldots ①$$

Replacing n by $(n+1)$ in ①

$$(2n+3)x\, P_{n+1}(x) = (n+2)P_{n+2}(x) + (n+1)P_n(x)$$

$$\Rightarrow x\, P_{n+1}(x) = \frac{1}{2n+3}\left[(n+2)P_{n+2}(x) + (n+1)P_n(x)\right] \cdots ②$$

Replacing n by $(n-1)$ in ①

$$(2n-1)x\, P_{n-1}(x) = nP_n(x) + (n-1)P_{n-2}(x)$$

$$\Rightarrow x\, P_{n-1}(x) = \frac{1}{2n-1}\left[nP_n(x) + (n-1)P_{n-2}(x)\right] \ldots ③$$

Multiplying ②and ③, we get

$$x^2P_{n+1}(x)\, P_{n-1}(x) = \frac{1}{(2n+3)(2n-1)}\left[n(n+2)P_n(x)P_{n+2}(x) + n(n+1)P_n^2(x)\right.$$

$$\left. + (n-1)(n+2)P_{n-2}(x)P_{n+2}(x) + (n-1)(n+1)P_{n-2}(x)P_n(x)\right]$$

Integrating both sides w.r.t x within the limits 0 to 1

$$\int_{-1}^{1} x^2\, P_{n+1}(x)\, P_{n-1}(x)dx = \frac{n(n+2)}{(2n+3)(2n-1)}\int_{-1}^{1} P_n(x)P_{n+2}(x)\, dx + \frac{n(n+1)}{(2n+3)(2n-1)}\int_{-1}^{1} P_n^2(x)\, dx$$

$$+ \frac{(n-1)(n+2)}{(2n+3)(2n-1)}\int_{-1}^{1} P_{n-2}(x)P_{n+2}(x)\, dx + \frac{(n-1)(n+1)}{(2n+3)(2n-1)}\int_{-1}^{1} P_{n-2}(x)P_n(x)\, dx$$

$$\int_{-1}^{1} x^2\, P_{n+1}(x)\, P_{n-1}(x)dx = \left[0 + \frac{n(n+1)}{(2n+3)(2n-1)}\,\frac{2}{(2n+1)} + 0 + 0\right]$$

$$\because \text{By orthogonality property} \begin{cases} \int_{-1}^{1} P_m(x)P_n(x)dx = 0 & m \neq n \\ \int_{-1}^{1} P_n{}^2(x)dx = \frac{2}{2n+1} & m = n \end{cases}$$

$$\Rightarrow \int_{-1}^{1} x^2\, P_{n+1}(x)\, P_{n-1}(x)dx = \frac{2n(n+1)}{(2n-1)\,(2n+1)\,(2n+3)}$$