Chapter 4

Z-TRANSFORMS

4.1 Introduction

$Z-$ Transform plays an important role in discrete analysis and may be seen as discrete analogue of Laplace transform. Role of $Z-$ Transforms in discrete analysis is the same as that of Laplace and Fourier transforms in continuous systems.

Definition: The Z–Transform of a sequence u_n defined for discrete values $n = 0,1,2,3,\ldots$ and ($u_n = 0$ for $n < 0$) is defined as $Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$. $Z-$ Transform of the sequence u_n i.e. $Z\{u_n\}$ is a function of z and may be denoted by $U(z)$

Remark:

- $Z-$ Transform exists only when the infinite series $\sum_{n=0}^{\infty} u_n z^{-n}$ is convergent.
- $Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$ is termed as one sided transform and for two sided $Z-$ transform $Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$

<u>Results on $Z-$ Transforms of standard sequences</u>

1. $$\boldsymbol{Z\{a^n\} = \frac{z}{z-a}}$$

$$Z\{a^n\} = \sum_{n=0}^{\infty} a^n z^{-n}$$
$$= 1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots + \frac{a^n}{z^n} + \cdots$$
$$= \frac{1}{1-\frac{a}{z}}, \quad \left|\frac{a}{z}\right| < 1$$
$$\therefore Z\{a^n\} = \frac{z}{z-a}, \quad \left|\frac{a}{z}\right| < 1$$

2. $$\boldsymbol{Z\{1\} = \frac{z}{z-1}}$$

$Z\{1\} = \frac{z}{z-1}$ Putting $a = 1$ in Result **1**

3. $$\boldsymbol{Z\{(-1)^n\} = \frac{z}{z+1}}$$

$Z\{(-1)^n\} = \frac{z}{z+1}$ Putting $a = -1$ in Result **1**

4. $$\boldsymbol{Z\{k\} = \frac{kz}{z-1}}$$

$$Z\{k\} = \sum_{n=0}^{\infty} k z^{-n} = k \sum_{n=0}^{\infty} z^{-n}$$
$$= k\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots + \frac{1}{z^n} + \cdots\right]$$
$$\therefore Z\{k\} = \frac{kz}{z-1}$$

5. Recurrence formula for n^p: $$\boldsymbol{Z\{n^p\} = -z\frac{d}{dz}Z\{n^{p-1}\}}$$

$Z\{n^p\} = \sum_{n=0}^{\infty} n^p z^{-n}$, p is a positive integer ... ①

$Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1} z^{-n}$ ... ②

Differentiating ② w.r.t. z, we get

$$\frac{d}{dz} Z\{n^{p-1}\} = \sum_{n=0}^{\infty} n^{p-1}(-n) z^{-n-1}$$

$$= -z^{-1} \sum_{n=0}^{\infty} n^p z^{-n}$$

$\Rightarrow \frac{d}{dz} Z\{n^{p-1}\} = -z^{-1} Z\{n^p\}$ using①

$\Rightarrow Z\{n^p\} = -z \frac{d}{dz} Z\{n^{p-1}\}$

6. **Multiplication by n:** $\boldsymbol{Z\{nu_n\} = -z \frac{d}{dz} Z\{u_n\}}$

$$Z\{nu_n\} = \sum_{n=0}^{\infty} nu_n z^{-n}$$

$$= -z \sum_{n=0}^{\infty} u_n (-n) z^{-n-1}$$

$$= -z \sum_{n=0}^{\infty} u_n \frac{d}{dz} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} \frac{d}{dz} (u_n z^{-n})$$

$$= -z \frac{d}{dz} \left(\sum_{n=0}^{\infty} u_n z^{-n}\right)$$

$$= -z \frac{d}{dz} Z\{u_n\}$$

7. $\boldsymbol{Z\{n\} = \frac{z}{(z-1)^2}}$

$Z\{n\} = -z \frac{d}{dz} Z\{n^0\}$ using Recurrence Result **5 or 6**

$= -z \frac{d}{dz} Z\{1\}$

$= -z \frac{d}{dz} \frac{z}{z-1}$ using result **2**

$\Rightarrow Z\{n\} = \boldsymbol{\frac{z}{(z-1)^2}}$

8. $\boldsymbol{Z\{n^2\} = \frac{z^2+z}{(z-1)^3}}$

$Z\{n^2\} = -z \frac{d}{dz} Z\{n\}$ using Recurrence Result **5 or 6**

$= -z \frac{d}{dz} \boldsymbol{\frac{z}{(z-1)^2}}$ using Result **7**

$\Rightarrow Z\{n^2\} = \frac{z^2+z}{(z-1)^3}$

9. $\boldsymbol{Z\left\{u(n) = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}\right\} = \frac{z}{z-1}}$ $\boldsymbol{u(n) = \begin{cases} 0, n < 0 \\ 1, n \geq 0 \end{cases}}$ **is Unit step sequence**

$Z\{u(n)\} = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} 1 z^{-n}$

$= 1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots + \frac{1}{z^n} + \cdots$

$\Rightarrow Z\{u(n)\} = \boldsymbol{\frac{z}{z-1}}$

10. $$\boldsymbol{Z\left\{\delta(n) = \begin{cases}1, n=0\\0, n\neq 0\end{cases}\right\} = 1}$$ $\boldsymbol{\delta(n) = \begin{cases}1, n=0\\0, n\neq 0\end{cases}}$ **is Unit impulse sequence**

$$Z\{\delta(n)\} = \sum_{n=0}^{\infty} \delta(n) z^{-n}$$
$$= 1 + 0 + 0 + \cdots$$
$$\Rightarrow Z\{\delta(n)\} = 1$$

4.2 Properties of Z-Transforms

1. **Linearity:** $\boldsymbol{Z\{au_n + bv_n\} = aZ\{u_n\} + bZ\{v_n\}}$

   Proof: $Z\{au_n + bv_n\} = \sum_{n=0}^{\infty}(au_n + bv_n)z^{-n}$
   $$= a\sum_{n=0}^{\infty} u_n z^{-n} + b\sum_{n=0}^{\infty} v_n z^{-n}$$
   $$= aZ\{u_n\} + bZ\{v_n\}$$

2. **Change of scale (or Damping rule):**

   If $\boldsymbol{Z\{u_n\} \equiv U(z)}$**, then** $\boldsymbol{Z\{a^{-n}u_n\} \equiv U(az)}$ **and** $\boldsymbol{Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)}$

   Proof: $Z\{a^{-n}u_n\} = \sum_{n=0}^{\infty} a^{-n} u_n z^{-n}$
   $$= \sum_{n=0}^{\infty} u_n (az)^{-n} \equiv U(az)$$

   Similarly $Z\{a^n u_n\} \equiv U\left(\frac{z}{a}\right)$

<u>Results from application of Damping rule</u>

i. $$\boldsymbol{Z\{a^n n\} = \frac{az}{(z-a)^2}}$$

Proof: $Z\{n\} = \frac{z}{(z-1)^2} \equiv U(z)$ say

$$\therefore Z\{a^n n\} \equiv U\left(\frac{z}{a}\right) = \frac{\frac{z}{a}}{\left(\frac{z}{a}-1\right)^2} = \frac{az}{(z-a)^2}$$

ii. $$\boldsymbol{Z\{a^n n^2\} = \frac{az^2 + a^2 z}{(z-a)^3}}$$

Proof: $Z\{n^2\} = \frac{z^2+z}{(z-1)^3} \equiv U(z)$ say

$$\therefore Z\{a^n n^2\} \equiv U\left(\frac{z}{a}\right) = \frac{\left(\frac{z}{a}\right)^2 + \left(\frac{z}{a}\right)}{\left(\left(\frac{z}{a}\right)-1\right)^3} = \frac{a(z^2+az)}{(z-a)^3}$$

iii. $$\boldsymbol{Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \quad Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}}$$

Proof: We have $Z\left\{e^{-in\theta}\right\} = Z\left\{\left(e^{i\theta}\right)^{-n}\right\} = Z\left\{\left(e^{i\theta}\right)^{-n}.1\right\}$

Now $Z\{1\} = \frac{z}{z-1}$

$$\therefore Z\left\{\left(e^{i\theta}\right)^{-n}.1\right\} = \frac{ze^{i\theta}}{ze^{i\theta}-1} \qquad \because Z\{a^{-n}u_n\} \equiv U(az)$$
$$= \frac{z}{z-e^{-i\theta}}$$
$$= \frac{z(z-e^{i\theta})}{\left(z-e^{-i\theta}\right)\left(z-e^{i\theta}\right)}$$

$$= \frac{z\,(z-\cos\theta - i\sin\theta\,)}{z^2 - z\,(e^{i\theta} + e^{-i\theta}) + 1} \quad \because e^{i\theta} = \cos\theta + i\sin\theta$$

$$= \frac{z\,(z-\cos\theta - i\sin\theta}{z^2 - 2z\cos\theta + 1} \qquad \because \cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$$

$$\therefore Z\left\{e^{-in\theta}\right\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\Rightarrow Z\{\cos n\theta - i\sin n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\therefore Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} \qquad \text{...③}$$

and $Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$ ...④

iv. $\boldsymbol{Z\{a^n \cos n\theta\} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}}$, $\boldsymbol{Z\{a^n \sin n\theta\} = \frac{az\sin\theta}{z^2-2z\cos\theta+a^2}}$

By Damping rule, replacing z by $\frac{z}{a}$ in ③and ④, we get

$Z\{a^n\cos n\theta\} = \frac{z(z-a\cos\theta)}{z^2-2az\cos\theta+a^2}$ and $Z\{a^n\sin n\theta\} = \frac{az\sin\theta}{z^2-2z\cos\theta+a^2}$

3. Right Shifting Property

For $n \geq k$, $Z\{u_{n-k}\} = z^{-k}Z\{u_n\}$, k is positive integer

Proof: $Z\{u_{n-k}\} = \sum_{n=0}^{\infty} u_{n-k} z^{-n}$

$$= u_{-k}z^0 + u_{1-k}z^{-1} + \cdots + u_{-1}z^{-k+1} + u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \cdots$$
$$= 0 + u_0 z^{-k} + u_1 z^{-(k+1)} + u_2 z^{-(k+2)} + \cdots \qquad \because u_n = 0 \text{ for } n < 0$$
$$= \sum_{n-k=0}^{\infty} u_{n-k} z^{-n}$$
$$= \sum_{m=0}^{\infty} u_m z^{-k-m}$$
$$= z^{-k}\sum_{m=0}^{\infty} u_m z^{-m}$$
$$= z^{-k}\sum_{n=0}^{\infty} u_n z^{-n}$$
$$= z^{-k} Z\{u_n\}$$

4. Left Shifting Property

If k is a positive integer $Z\{u_{n+k}\} = z^k\left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$

Proof: $Z\{u_{n+k}\} = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$

$$= z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$$
$$= z^k\left[u_k z^{-k} + u_{1+k} z^{-(1+k)} + u_{2+k} z^{-(2+k)} + \cdots\right]$$
$$= z^k\left[u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)} + u_k z^{-k} + \cdots\right]$$
$$-z^k\left[u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_{k-1} z^{-(k-1)}\right]$$
$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$
$$= z^k\left[\sum_{n=0}^{\infty} u_n z^{-n} - \sum_{n=0}^{k-1} u_n z^{-n}\right]$$
$$= z^k\left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$$

In particular for $k = 1,2,3$

$$Z\{u_{n+1}\} = z[Z\{u_n\} - u_0]$$
$$Z\{u_{n+2}\} = z^2\left[Z\{u_n\} - u_0 - \frac{u_1}{z}\right]$$

$$Z\{u_{n+3}\} = z^3\left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

5. **Initial Value theorem**:
   If $Z\{u_n\} = U(z)$, then $u_0 = \lim_{z\to\infty} U(z)$

$$u_1 = \lim_{z\to\infty} z[U(z) - u_0]$$

$$u_2 = \lim_{z\to\infty} z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$$

$$\vdots$$

Proof: By definition $U(z) = Z\{u_n\} = \sum_{n=0}^{\infty} u_n z^{-n}$

$$\Rightarrow U(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots \qquad \ldots ⑤$$

$$\therefore u_0 = \lim_{z\to\infty} U(z) = \lim_{z\to\infty}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\right]$$

$$= u_0 + 0 + 0 + 0 + \cdots = u_0$$

Again from⑤, we get

$$U(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots$$

$$\Rightarrow z[U(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots$$

$$\Rightarrow \lim_{z\to\infty} z[U(z) - u_0] = \lim_{z\to\infty}\left[u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots\right] = u_1$$

Similarly $u_2 = \lim_{z\to\infty} z^2\left[U(z) - u_0 - \frac{u_1}{z}\right]$

Note: Initial value theorem may be used to determine the sequence u_n from the given function $U(z)$

6. **Final Value theorem**:
   If $Z\{u_n\} = U(z)$, then $\lim_{n\to\infty} u_n = \lim_{z\to 1} (z-1)U(z)$

Proof: $Z\{u_{n+1} - u_n\} = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$

$$\Rightarrow Z\{u_{n+1}\} - Z\{u_n\} = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

$$\Rightarrow z[Z\{u_n\} - u_0] - Z\{u_n\} = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

By using left shifting property for $k = 1$

$$\Rightarrow (z-1)Z\{u_n\} - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n}$$

or $(z-1)U(z) - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n)z^{-n} \quad \because Z\{u_n\} = U(z)$

Taking limits $z \to 1$ on both sides

$$\lim_{z\to 1} (z-1)U(z) - u_0 = \sum_{n=0}^{\infty}(u_{n+1} - u_n)$$

or $\lim_{z\to 1} (z-1)U(z) - u_0 = \lim_{n\to\infty} [(u_1 - u_0) + (u_2 - u_1) + \cdots + (u_{n+1} - u_n)]$

$$= \lim_{n\to\infty} [u_{n+1}] - u_0$$

$$\Rightarrow \lim_{z\to 1} (z-1)U(z) = u_\infty$$

or $\lim_{z\to 1} (z-1)U(z) = \lim_{n\to\infty} u_n$

Note: Initial value and final value theorems determine the value of u_n for $n = 0$ and for $n \longrightarrow \infty$ from the given function $U(z)$.

7. **Convolution theorem**
   Convolution of two sequences u_n and v_n is defined as $u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$
   Convolution theorem for Z-transforms states that
   If $U(z) = Z\{u_n\}$ and $V(z) = Z\{v_n\}$, then $Z\{u_n * v_n\} = U(z).V(z)$

Proof: $U(z).V(z) = Z\{u_n\}.Z\{v_n\}$

$$= \left[\sum_{n=0}^{\infty} u_n z^{-n}\right].\left[\sum_{n=0}^{\infty} v_n z^{-n}\right]$$

$$= \left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots + \frac{u_n}{z^n} + \cdots\right].\left[v_0 + \frac{v_1}{z} + \frac{v_2}{z^2} + \cdots + \frac{v_n}{z^n} + \cdots\right]$$
$$= (u_0v_0) + (u_0v_1 + u_1v_0)z^{-1} + (u_0v_2 + u_1v_1 + u_2v_0)z^{-2} + \cdots$$
$$= \sum_{n=0}^{\infty}(u_0v_n + u_1v_{n-1} + u_2v_{n-2} + \cdots + u_nv_0)z^{-n}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n} u_m v_{n-m}\right)z^{-n}$$
$$\Rightarrow U(z).V(z) = Z\{\sum_{m=0}^{n} u_m v_{n-m}\} \qquad \because \sum_{n=0}^{\infty} u_n z^{-n} = Z\{u_n\}$$
$$\Rightarrow U(z).V(z) = Z\{u_n * v_n\} \qquad \because u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$$

Example1 Find the Z–transform of $2n + 3\sin\frac{n\pi}{4} - 5a^4$

Solution: By linearity property

$$Z\left\{2n + 3\sin\frac{n\pi}{4} - 5a^4\right\} = 2Z\{n\} + 3Z\left\{\sin\frac{n\pi}{4}\right\} - 5Z\{a^4\}$$
$$= 2Z\{n\} + 3Z\left\{\sin\frac{n\pi}{4}\right\} - 5a^4Z\{1\}$$
$$= \frac{2z}{(z-1)^2} + \frac{3z\ Sin\frac{\pi}{4}}{z^2-2zcos\frac{\pi}{4}+1} - \frac{5a^4z}{z-1}$$
$$\because Z\{n\} = \frac{z}{(z-1)^2}, Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}, Z\{1\} = \frac{z}{z-1}$$
$$\therefore Z\left\{2n + 3\sin\frac{n\pi}{4} - 5a^4\right\} = \frac{2z}{(z-1)^2} + \frac{\frac{3z}{\sqrt{2}}}{z^2-\sqrt{2}z+1} - \frac{5a^4z}{z-1}$$

Example2 Find the Z–transform of the sequence $\{4, 8, 16, 32, \ldots\}$

Solution: $u_n = 2^{n+2}$, $n = 0,1,2,3 \ldots.$

$$Z\{2^{n+2}\} = Z\{2^2 2^n\}$$
$$= 4\,Z\{2^n\} = \frac{4z}{z-2}, \left|\frac{2}{z}\right| < 1 \quad \because Z\{a^n\} = \frac{z}{z-a}, \left|\frac{a}{z}\right| < 1$$

Example3 Find the Z–transform of $(n+1)^2$

Solution: $Z\{(n+1)^2\} = Z\{n^2 + 2n + 1\}$

$$= Z\{n^2\} + 2Z\{n\} + Z\{1\}$$
$$= \frac{z^2+z}{(z-1)^3} + \frac{2z}{(z-1)^2} + \frac{z}{z-1}$$
$$\because Z\{n^2\} = \frac{z^2+z}{(z-1)^3}, Z\{n\} = \frac{z}{(z-1)^2}, Z\{1\} = \frac{z}{z-1}$$

Example4 Find the Z–transform of

i. $n\cos n\theta$ ii. $\sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$ iii. e^{-2n} iv. $a^{|n|}$

Solution: i. $Z\{\cos n\theta\} = \frac{z\,(z-cos\theta)}{z^2-2zcos\theta+1}$

$$\therefore Z\{n\cos n\theta\} = -z\frac{d}{dz}\left[\frac{z\,(z\text{-}cos\theta)}{z^2-2zcos\theta+1}\right]$$
$$\because Z\{nu_n\} = -z\frac{d}{dz}Z\{u_n\}$$
$$= -z\left[\frac{-z^2cos\theta+2z-cos\theta}{(z^2-2zcos\theta+1)^2}\right]$$
$$= \frac{z^3cos\theta-2z^2+zcos\theta}{(z^2-2zcos\theta+1)^2}$$

ii. $Z\left\{\sin\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)\right\} = Z\left\{\sin\frac{n\pi}{2}\cos\frac{\pi}{4} + \cos\frac{n\pi}{2}\sin\frac{\pi}{4}\right\}$

$$= \cos\frac{\pi}{4}Z\left\{\sin\frac{n\pi}{2}\right\} + \sin\frac{\pi}{4}Z\left\{\cos\frac{n\pi}{2}\right\}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z\sin\frac{\pi}{2}}{z^2-2z\cos\frac{\pi}{2}+1} + \frac{z\left(z-cos\frac{\pi}{2}\right)}{z^2-2z\ cos\frac{\pi}{2}+1}\right]$$

$$\because Z\{\sin n\theta\} = \frac{z\sin\theta}{z^2-2z\cos\theta+1}, Z\{\cos n\theta\} = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z}{z^2+1} + \frac{z^2}{z^2+1}\right] = \frac{1}{\sqrt{2}}\left[\frac{z+z^2}{z^2+1}\right]$$

iii. $Z\{e^{-2n}\} = Z\{(e^{-2})^n\}$

$$= \frac{z}{z-e^{-2}} \qquad \because Z\{a^n\} = \frac{z}{z-a}$$

$$= \frac{ze^2}{ze^2-1}$$

iv. $a^{|n|} = \begin{cases} a^{-n}, & n < 0 \\ a^n, & n \geq 0 \end{cases}$

Taking two sided Z– transform: $Z\left\{a^{|n|}\right\} = \sum_{n=-\infty}^{\infty} a^{|n|}z^{-n}$

$$\therefore Z\left\{a^{|n|}\right\} = \sum_{-\infty}^{-1} a^{-n}z^{-n} + \sum_{0}^{\infty} a^n z^{-n}$$

$$= [\ldots + a^3z^3 + a^2z^2 + az] + \left[1 + \frac{a}{z} + \frac{a^2}{z^2} + \frac{a^3}{z^3} + \cdots\right]$$

$$= \frac{az}{1-az} + \frac{1}{1-\frac{a}{z}}, \qquad |az| < 1 \text{ and } \left|\frac{a}{z}\right| < 1$$

$$= \frac{az}{1-az} + \frac{z}{z-a}, \qquad |z| < \frac{1}{|a|} \text{ and } |a| < |z|$$

$$= \frac{z(1-az)}{(1-az)(z-a)}, \qquad |a| < |z| < \frac{1}{|a|}$$

$$\Rightarrow Z\left\{a^{|n|}\right\} = \frac{z}{(z-a)}, \qquad |a| < |z| < \frac{1}{|a|}$$

Example5 Find the Z–transform of $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$

Taking two sided Z– transform: $Z\{u_n\} = \sum_{n=-\infty}^{\infty} u_n z^{-n}$

$$\therefore Z\{u_n\} = \sum_{-\infty}^{-1} 2^n z^{-n} + \sum_{0}^{\infty} 3^n z^{-n}$$

$$= \left[\ldots + \frac{z^3}{2^3} + \frac{z^2}{2^2} + \frac{z}{2}\right] + \left[1 + \frac{3}{z} + \frac{3^2}{z^2} + \frac{3^3}{z^3} + \cdots\right]$$

$$= \frac{\frac{z}{2}}{1-\frac{z}{2}} + \frac{1}{1-\frac{3}{z}}, \qquad \left|\frac{z}{2}\right| < 1 \text{ and } \left|\frac{3}{z}\right| < 1$$

$$= \frac{z}{2-z} + \frac{z}{z-3}, \qquad |z| < |2| \text{ and } |3| < |z|$$

$$= \frac{z(z-3)+z(2-z)}{(2-z)(z-3)}, \qquad |3| < |z| < |2|$$

$$\Rightarrow Z\{u_n\} = \frac{z}{z^2-5z+6}, \qquad 3 < |z| < 2$$

$\therefore$ Z–transform does not exist for $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$ as the set $3 < |z| < 2$ is infeasible.

Example6 Find the Z–transform of

i. ${}^nC_r\ 0 \leq r \leq n$ ii. ${}^{n+r}C_r$ iii. $\frac{1}{(n+r)!}$ iv. $\frac{1}{(n-r)!}$

Solution: i. $Z\{{}^nC_r\} = \sum_{r=0}^{n} {}^nC_r\, z^{-r}$

$$= 1 + {}^nC_1\, z^{-1} + {}^nC_2\, z^{-2} + \cdots + {}^nC_n\, z^{-n}$$

$$= (1 + z^{-1})^n$$

ii. $Z\{{}^{n+r}C_r\} = \sum_{r=0}^{\infty} {}^{n+r}C_r\, z^{-r}$

$$= 1 + {}^{n+1}C_1\, z^{-1} + {}^{n+2}C_2\, z^{-2} + {}^{n+3}C_3\, z^{-3} + \cdots$$

$$= 1 + (n+1)\, z^{-1} + \frac{(n+2)(n+1)}{2!}\, z^{-2} + \frac{(n+3)(n+2)(n+1)}{3!}\, z^{-3} + \cdots$$

$$= 1 + (-n-1)(-z^{-1}) + \frac{(-n-1)(-n-2)}{2!}(-z^{-1})^{-2}$$
$$+ \frac{(-n-1)(-n-2)(-n-3)}{3!}(-z^{-1})^{-3} + \cdots$$
$$= (1 - z^{-1})^{-n-1}$$

Example 7 Find the Z–transform of

i. $\frac{1}{n!}$ ii. $\frac{1}{(n+r)!}$ iii. $\frac{1}{(n-r)!}$

i. $Z\left\{\frac{1}{n!}\right\} = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} = 1 + \frac{1}{1!} z^{-1} + \frac{1}{2!} z^{-2} + \frac{1}{3!} z^{-3} + \cdots$

$$= 1 + \frac{1}{1!}\frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots = e^{\frac{1}{z}}$$

$\Rightarrow Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

ii. $Z\left\{\frac{1}{(n+r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n+r)!} z^{-n}$

Now $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from left shifting property $Z\{u_{n+k}\} = z^k \left[Z\{u_n\} - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} - \cdots - \frac{u_{k-1}}{z^{k-1}}\right]$

$\therefore Z\left\{\frac{1}{(n+r)!}\right\} = z^r \left[e^{\frac{1}{z}} - 1 - \frac{1}{z} - \frac{1}{2!\,z^2} - \cdots - \frac{1}{(r-1)!z^{r-1}}\right]$

In particular $\left\{\frac{1}{(n+1)!}\right\} = z^1 \left[e^{\frac{1}{z}} - 1\right]$

$\left\{\frac{1}{(n+2)!}\right\} = z^2 \left[e^{\frac{1}{z}} - 1 - \frac{1}{z}\right]$

⋮

iii. $Z\left\{\frac{1}{(n-r)!}\right\} = \sum_{n=0}^{\infty} \frac{1}{(n-r)!} z^{-n}$

Now $Z\left\{\frac{1}{n!}\right\} = e^{\frac{1}{z}}$

Also from right shifting property, $Z\{u_{n-k}\} = z^{-k} Z\{u_n\}$, k is positive integer

$\therefore Z\left\{\frac{1}{(n-r)!}\right\} = z^{-r} e^{\frac{1}{z}}$

In particular $\left\{\frac{1}{(n-1)!}\right\} = z^{-1} e^{\frac{1}{z}}$

$\left\{\frac{1}{(n+2)!}\right\} = z^{-2} e^{\frac{1}{z}}$

⋮

Example8 Find $Z\{u_{n+2}\}$ if $Z\{u_n\} = \frac{z}{z-1} + \frac{z}{z^2+1}$

Solution: Given $Z\{u_n\} = U(z) = \frac{z}{z-1} + \frac{z}{z^2+1}$

From left shifting property $Z\{u_{n+2}\} = z^2 \left[Z\{u_n\} - u_0 - \frac{u_1}{z}\right]$ ...①

Now from initial value theorem $u_0 = \lim_{z\to\infty} U(z)$

$$= \lim_{z\to\infty} \left[\frac{z}{z-1} + \frac{z}{z^2+1}\right]$$

$$= \lim_{z\to\infty}\left[\frac{1}{1-\frac{1}{z}} + \frac{\frac{1}{z}}{1+\frac{1}{z^2}}\right] = 1 + 0$$

$$\therefore u_0 = 1 \qquad \ldots ②$$

Also from initial value theorem $u_1 = \lim_{z\to\infty} z[U(z) - u_0]$

$$= \lim_{z\to\infty} z\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1\right]$$

$$= \lim_{z\to\infty} z\left[\frac{2z^2-z+1}{(z^2+1)(z-1)}\right] = 2$$

$$\therefore u_1 = 2 \qquad \ldots ③$$

Using ②and ③ in ①, we get $Z\{u_{n+2}\} = z^2\left[\frac{z}{z-1} + \frac{z}{z^2+1} - 1 - \frac{2}{z}\right]$

$$\Rightarrow Z\{u_{n+2}\} = \frac{z(z^2-z+2)}{(z-1)(z^2+1)}$$

Example9 Verify convolution theorem for $u_n = n$ and $v_n = 1$

Solution: Convolution theorem states that $Z\{u_n * v_n\} = U(z).V(z)$

We know that $u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$

$$= \sum_{m=0}^{n} m\,.\,1$$

$$= 0 + 1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$$

$$\Rightarrow Z\{u_n * v_n\} = \sum_{n=0}^{\infty} \frac{n(n+1)}{2} z^{-n} = \frac{1}{2}\left[\sum_{n=0}^{\infty} n^2 z^{-n} + \sum_{n=0}^{\infty} n\, z^{-n}\right]$$

$$= \frac{1}{2}[Z\{n^2\} + Z\{n\}]$$

$$= \frac{1}{2}\left[\frac{z(z+1)}{(z-1)^3} + \frac{z}{(z-1)^2}\right] = \frac{1}{2}\left[\frac{z(z+1)+z(z-1)}{(z-1)^3}\right] = \frac{1}{2}\left[\frac{2z^2}{(z-1)^3}\right]$$

$$\therefore Z\{u_n * v_n\} = \frac{z^2}{(z-1)^3} \qquad \ldots ①$$

Also $U(z) = Z\{n\} = \frac{\mathbf{z}}{(\mathbf{z-1})^2}$ and $V(z) = Z\{1\} = \frac{z}{z-1}$

$$\Rightarrow U(z).V(z) = \frac{z^2}{(z-1)^3} \qquad \ldots ②$$

From ①and ② $Z\{u_n * v_n\} = U(z).V(z)$

Example10 If $u_n = \delta(n) - \delta(n-1)$, $v_n = 2\delta(n) + \delta(n-1)$, Find the Z-transform of their convolution.

Solution: $U(z) = Z\{\delta(n) - \delta(n-1)\}$, $V(z) = Z\{2\delta(n) + \delta(n-1)\}$

Now $Z\{\delta(n)\} = 1$ and $Z\{\delta(n-1)\} = z^{-1}$ $\quad\because Z\{u_{n-k}\} = z^{-k}Z\{u_n\}$

$\therefore U(z) = 1 - z^{-1}$ and $V(z) = 2 + z^{-1}$

Also $Z\{u_n * v_n\} = U(z).V(z)$

$\Rightarrow Z\{u_n * v_n\} = (1 - z^{-1})(2 + z^{-1}) = 2 - z^{-1} - z^{-2}$

4.3 Inverse Z-Transform

Given a function $U(z)$, we can find the sequence u_n by one of the following methods

- Inspection (Direct inversion)
- Direct division
- Partial fractions

- Residues (Inverse integral)
- Power-Series
- Convolution theorem

4.3.1 Inspection (Direct inversion) method

Sometimes by observing the coefficients in the given series $U(z)$, it is possible to find the sequence u_n as illustrated in the given examples.

Example 11 Find u_n if $U(z) = 1 + \frac{1}{2}z^{-1} + \frac{1}{4}z^{-2} + \frac{1}{8}z^{-3} + \cdots$

Solution: Given that $U(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$ ... ①

Also by the definition of Z-transform $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$ ...②

.Comparing ①and ②, we get $u_n = \left(\frac{1}{2}\right)^n$

Example12 Find u_n if $U(z) = \frac{z^3}{(z-1)^3}$

Solution: $U(z) = \frac{z^3}{(z-1)^3} = \left(\frac{z-1}{z}\right)^{-3} = \left(1 - \frac{1}{z}\right)^{-3}$

$\therefore U(z) = 1 + \frac{3}{z} + \frac{6}{z^2} + \frac{10}{z^3} + \cdots$

$\because (1-x)^{-n} = 1 + nx + \frac{n(n+1)}{2!}x^2 + \frac{n(n+1)(n+2)}{3!}x^3 + \cdots$

$\Rightarrow U(z) = \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} z^{-n}$

Comparing with $U(z) = \sum_{n=0}^{\infty} u_n z^{-n}$, we get $u_n = \frac{(n+1)(n+2)}{2}$

Example13 Find inverse Z-transform of $3 + \frac{2z}{z-1} - \frac{z}{2z-1}$

Solution: Given that $U(z) = 3 + \frac{2z}{z-1} - \frac{z}{2z-1}$

$\therefore u_n = 3Z^{-1}[1] + 2Z^{-1}\left[\frac{z}{z-1}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$

$\Rightarrow u_n = 3\delta(n) + 2u(n) - \frac{1}{2}\left(\frac{1}{2}\right)^n = 3\delta(n) + 2u(n) - \left(\frac{1}{2}\right)^{n+1}$

$\because Z^{-1}[1] = \delta(n)$, $Z^{-1}\left[\frac{z}{z-1}\right] = u(n)$, $Z^{-1}\left[\frac{z}{z-a}\right] = a^n$ where $\delta(n)$ and $u(n)$ are unit impulse and unit step sequences respectively.

Example14 Find inverse Z-transform of $2z^{-2} - \frac{z^{-3}}{z-1} + \frac{2z^{-5}}{2z-1}$

Solution: Given that $U(z) = 2z^{-2} - \frac{z^{-3}}{z-1} + \frac{2z^{-5}}{2z-1}$

$\therefore u_n = 2Z^{-1}[z^{-2}.1] - Z^{-1}\left[z^{-4}.\frac{z}{z-1}\right] + Z^{-1}\left[z^{-6}\frac{z}{z-\frac{1}{2}}\right]$

$\Rightarrow u_n = 2\delta(n-2) - u(n-4) + \left(\frac{1}{2}\right)^{n-6} u(n-6)$

$\because$ From Right shifting property $Z^{-1}[z^{-k}U(Z)] = u_{n-k}$

4.3.2 Direct division method

Direct division is one of the simplest methods for finding inverse Z -transform and can be used for almost every type of expression given in fractional form.

Example15 Find the inverse Z-transform of $\frac{z}{z^2-3z+2}$

Solution: Given that $U(z) = \frac{z}{z^2-3z+2}$

By actual division, we get

$$
\begin{array}{r|l}
 & z^{-1} + 3z^{-2} + 7z^{-3} \\
\hline
z^2 - 3z + 2 & z \\
 & \underline{z - 3 + 2z^{-1}} \\
 & 3 - 2z^{-1} \\
 & \underline{3 - 9z^{-1} + 6z^{-2}} \\
 & 7z^{-1} - 6z^{-2} \\
 & \underline{7z^{-1} - 21z^{-2} + 14z^{-3}} \\
 & 15z^{-2} - 14z^{-3} \\
 & \vdots
\end{array}
$$

$$\Rightarrow U(z) = z^{-1} + 3z^{-2} + 7z^{-3} + \cdots$$
$$= \sum_{n=0}^{\infty} (2^n - 1) z^{-n}$$
$$\therefore\ u_n = 2^n - 1$$

Example16 Find the inverse Z -transform of $\frac{4z^2+2z}{2z^2-3z+1}$

Solution: Given that $U(z) = \frac{4z^2+2z}{2z^2-3z+1}$, by actual division, we get

$$
\begin{array}{r|l}
 & 2 + 4z^{-1} + 5z^{-2} \\
\hline
2z^2 - 3z + 1 & 4z^2 + 2z \\
 & \underline{4z^2 - 6z + 2} \\
 & 8z - 2 \\
 & \underline{8z - 12 + 4z^{-1}} \\
 & 10 - 4z^{-1} \\
 & \underline{10 - 15z^{-1} + 5z^{-2}} \\
 & 11z^{-1} - 5z^{-2} \\
 & \vdots
\end{array}
$$

$$\Rightarrow U(z) = 2 + 4z^{-1} + 5z^{-2} + \cdots$$
$$= \sum_{n=0}^{\infty} (6 - 2^{2-n}) z^{-n}$$
$$\therefore\ u_n = 6 - 2^{2-n}$$
$$\therefore\ u_n = 6 - 4\left(\frac{1}{2}\right)^n$$

4.3.3 Partial fractions method

Partial fractions method can be used only if order of expression in the numerator is less than or equal to that in the denominator. If order of expression in the numerator is greater, then the fraction may be brought to desired form by direct division. Partial fractions are formed of the expression $\frac{U(z)}{z}$ as demonstrated in the examples below.

Example17 Find the inverse Z -transform of $\frac{z}{6z^2-5z+1}$

Solution: Given that $U(z) = \frac{z}{6z^2-5z+1}$

$$\therefore \frac{U(z)}{z} = \frac{1}{6z^2-5z+1} = \frac{1}{(2z-1)\,(3z-1)}$$
$$\Rightarrow \frac{U(z)}{z} = \frac{2}{2z-1} - \frac{3}{3z-1}$$
$$\Rightarrow U(z) = \frac{z}{z-\frac{1}{2}} - \frac{z}{z-\frac{1}{3}}$$
$$\therefore\ u_n = \left(\frac{1}{2}\right)^n - \left(\frac{1}{3}\right)^n \quad \because Z\{a^n\} = \frac{z}{z-a} \text{ or } Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

i.e. $u_n = 2^{-n} - 3^{-n}$

Example18 Find the inverse Z -transform of $\frac{4z^2+2z}{2z^2-3z+1}$

Solution: Given that $U(z) = \frac{4z^2-2z}{2z^2-3z+1} = \frac{2z(2z-1)}{(2z-1)(z-1)}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z+1)}{(2z-1)(z-1)}$$

By partial fractions, we get

$$\frac{U(z)}{z} = \frac{-8}{2z-1} + \frac{6}{z-1}$$

$$\Rightarrow U(z) = \frac{-8z}{2z-1} + \frac{6z}{z-1}$$

$$\therefore u_n = -4Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right] + 6Z^{-1}\left[\frac{z}{z-1}\right]$$

$$\Rightarrow u_n = -4\left(\frac{1}{2}\right)^n + 6(1)^n \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

i.e. $u_n = -4\left(\frac{1}{2}\right)^n + 6$

Example19 Find the inverse Z -transform of $\frac{1}{(1-z^{-1})(2-z^{-1})}$

Solution: Given that $U(z) = \frac{1}{(1-z^{-1})(2-z^{-1})}$

Multiplying and dividing by z^2, we get

$$U(z) = \frac{z^2}{z(1-z^{-1})z(2-z^{-1})} = \frac{z^2}{(z-1)(2z-1)}$$

$$\therefore \frac{U(z)}{z} = \frac{z}{(z-1)(2z-1)}$$

By partial fractions, we get

$$\frac{U(z)}{z} = \frac{1}{(z-1)} - \frac{1}{(2z-1)}$$

$$\Rightarrow U(z) = \frac{z}{(z-1)} - \frac{z}{(2z-1)}$$

$$\therefore u_n = Z^{-1}\left[\frac{z}{(z-1)}\right] - \frac{1}{2}Z^{-1}\left[\frac{z}{z-\frac{1}{2}}\right]$$

$$\Rightarrow u_n = (1)^n - \frac{1}{2}\left(\frac{1}{2}\right)^n \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$$

i.e. $u_n = 1 - \left(\frac{1}{2}\right)^{n+1}$

Example20 Find the inverse Z -transform of $\frac{4z^2-2z}{z^3-5z^2+8z-4}$

Solution: Given that $U(z) = \frac{4z^2-2z}{z^3-5z^2+8z-4} = \frac{2z(2z-1)}{(z-1)(z-2)^2}$

$$\therefore \frac{U(z)}{z} = \frac{2(2z-1)}{(z-1)(z-2)^2}$$

By partial fractions, we get

$$\frac{U(z)}{z} = \frac{2}{z-1} - \frac{2}{z-2} + \frac{6}{(z-2)^2}$$

$$\Rightarrow U(z) = \frac{2z}{z-1} - \frac{2z}{z-2} + \frac{6z}{(z-2)^2}$$

$$\therefore\ u_n = 2Z^{-1}\left[\frac{z}{z-1}\right] - 2Z^{-1}\left[\frac{z}{z-2}\right] + 3Z^{-1}\left[\frac{2z}{(z-2)^2}\right]$$

$$\Rightarrow u_n = 2(1)^n - 2(2)^n + 3n(2)^n \quad \because Z^{-1}\left[\frac{z}{z-a}\right] = a^n \text{ and } Z^{-1}\left[\frac{az}{(z-a)^2}\right] = na^n$$

i.e. $u_n = 2 - 2^{n+1} + 3n.2^n$

4.3.4 Method of residues (Inverse integral)

By using the theory of complex variables, it can be shown that the inverse Z-transform is given by $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$ = sum of residues of $U(z)$

where c is the closed contour which contains all the insolated singularities of $U(z)$in the region of convergence.

Method of residues is one of the most efficient methods and can be used to find the inverse Z- transform where partial fractions are tedious to find.

Example21 Find the inverse z-transform of $\frac{z}{z^2+7z+10}$

Solution: $U(z) = \frac{z}{z^2+7z+10}$

Now $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$

$$\Rightarrow u_n = \frac{1}{2\pi i}\oint_c \frac{z}{z^2+7z+10}z^{n-1}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{z^n}{z^2+7z+10}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{z^n}{(z+2)(z+5)}dz$$

There are two simple poles at $z = -2$ and $z = -5$

Residue at $z = -2$ is given by $\lim_{z\to-2}(z+2)\frac{z^n}{(z+2)(z+5)} = \frac{(-2)^n}{3}$

Residue at $z = -5$ is given by $\lim_{z\to-5}(z+5)\frac{z^n}{(z+2)(z+5)} = \frac{(-5)^n}{-3}$

$\therefore\ u_n$ = sum of residues = $\frac{(-2)^n}{3} + \frac{(-5)^n}{-3} = \frac{1}{3}\left\{(-2)^n - (-5)^n\right\}$

Example22 Find the inverse z-transform of $\frac{z^2+z}{(z-1)(z^2+1)}$

Solution: $U(z) = \frac{z^2+z}{(z-1)(z^2+1)}$

Now $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$

$$\Rightarrow u_n = \frac{1}{2\pi i}\oint_c \frac{z^2+z}{(z-1)(z^2+1)}z^{n-1}dz$$

$$= \frac{1}{2\pi i}\oint_c \frac{z^n(z+1)}{(z-1)(z+i)(z-i)}dz$$

There are three simple poles at $z = 1$, $z = -i$ and $z = i$

Residue at $z = 1$ is given by $\lim_{z\to1}(z-1)\frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = 1$

Residue at $z = -i$ is given by $\lim_{z\to-i}(z+i)\frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}(-i)^n$

Residue at $z = i$ is given by $\lim_{z\to i}(z-i)\frac{z^n(z+1)}{(z-1)(z+i)(z-i)} = -\frac{1}{2}i^n$

$\therefore\ u_n$ = sum of residues = $1 - \frac{1}{2}(-i)^n - \frac{1}{2}i^n = 1 - \frac{1}{2}\{(-i)^n + i^n\}$

Example23 Find the inverse z-transform of $\frac{z(z+1)}{(z-1)^3}$

Solution: $U(z) = \frac{z(z+1)}{(z-1)^3}$

Now $u_n = \frac{1}{2\pi i}\oint_c U(z)z^{n-1}dz$

$\Rightarrow u_n = \frac{1}{2\pi i}\oint_c \frac{z^n(z+1)}{(z-1)^3}dz$

Here $z = 1$ is a pole of order 3

Residue at $z = 1$ is given by $\frac{1}{2!}\lim_{z\to1}\frac{d^2}{dz^2}\left[\frac{(z-1)^3 z^n(z+1)}{(z-1)^3}\right]$

$$= \frac{1}{2!}\lim_{z\to1}\frac{d^2}{dz^2}[z^n(z+1)]$$
$$= \frac{1}{2!}\lim_{z\to1}\frac{d}{dz}[(n+1)z^n + nz^{n-1}]$$
$$= \frac{1}{2!}\lim_{z\to1}[(n+1)nz^{n-1} + n(n-1)z^{n-2}]$$
$$= [n^2 + n + n^2 - n] = n^2$$

$\therefore u_n =$ Sum of residues $= n^2$

4.3.5 Power series method

In this method, we find the inverse Z - transform by expanding $U(z)$ in power series.

Example 24 Find u_n if $U(z) = \log\frac{z}{z+1}$

Solution: Given $U(z) = \log\frac{z}{z+1} = \log\left(\frac{z+1}{z}\right)^{-1} = -\log\frac{z+1}{z} = -\log\left(1+\frac{1}{z}\right)$

$\therefore U(z) = -\log(1+y)$ Putting $\frac{1}{z} = y$

$$= -y + \frac{y^2}{2} - \frac{y^3}{3} + \frac{y^4}{4} - \cdots$$

$$\because \log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$\Rightarrow U(z) = -\frac{1}{z} + \frac{1}{2z^2} - \frac{1}{3z^3} + \frac{1}{4z^4} - \cdots$ $\because y = \frac{1}{z}$

$\Rightarrow U(z) = 0 + \sum_{n=1}^{\infty}\frac{(-1)^n}{n}z^{-n}$

Comparing with $U(z) = \sum_{n=0}^{\infty}u_n z^{-n}$, we get

$$u_n = \begin{cases} 0 & \text{for } n = 0 \\ \frac{(-1)^n}{n}, & otherwise \end{cases}$$

4.3.6 Convolution theorem method

Convolution theorem for Z-transforms states that:

If $U(z) = Z\{u_n\}$ and $V(z) = Z\{v_n\}$, then $Z\{u_n * v_n\} = U(z).V(z)$

$\Rightarrow Z^{-1}[U(z).V(z)] = u_n * v_n$

Example25 Find the inverse z-transform of $\frac{z^2}{(z-1)(2z-1)}$ using convolution theorem.

Solution: Let $U(z) = Z\{u_n\} = \frac{z}{(z-1)}$ and $V(z) = Z\{v_n\} = \frac{z}{(2z-1)} = \frac{1}{2}\left(\frac{z}{z-\frac{1}{2}}\right)$

Clearly $u_n = (1)^n$ and $u_n = \frac{1}{2}\left(\frac{1}{2}\right)^n$ $\because Z^{-1}\left[\frac{z}{z-a}\right] = a^n$

Now by convolution theorem $Z^{-1}[U(z).V(z)] = u_n * v_n$

$$\Rightarrow Z^{-1}\left[\frac{z^2}{(z-1)(2z-1)}\right] = (1)^n * \left(\frac{1}{2}\right)^{n+1}$$

We know that $u_n * v_n = \sum_{m=0}^{n} u_m v_{n-m}$

$$= \sum_{m=0}^{n}(1)^m \left(\frac{1}{2}\right)^{n+1-m}$$

$$= \left(\frac{1}{2}\right)^{n+1} + \left(\frac{1}{2}\right)^{n} + \left(\frac{1}{2}\right)^{n-1} + \cdots + \frac{1}{2}$$

$$= \frac{1}{2}\left[1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \cdots + \left(\frac{1}{2}\right)^n\right]$$

$$= \frac{1}{2}\left[\frac{1}{1-\frac{1}{2}}\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$\because S_n = \frac{a}{1-r}(1-r^n)$$

$$= \frac{1}{2}\left[2\left(1 - \left(\frac{1}{2}\right)^{n+1}\right)\right]$$

$$= 1 - \left(\frac{1}{2}\right)^{n+1}$$

Exercise 4A

1. Find the Z–transform of $u_n = \begin{cases} 2^n, & n < 0 \\ 3^n, & n \geq 0 \end{cases}$
2. Find the Z–transform of $u_n = \frac{1}{(n-p)!}$
3. Find the inverse Z–transform of $u_n = \frac{2z}{(z-1)(z^2+1)}$
4. Solve the difference equation $y_{x+2} + 4y_{x+1} + 3y_x = 3^x$, $y_0 = 0, y_1 = 1$ using Z–transforms

Answers

1. $\frac{2z}{z^2-8z+15}$, $3 < |z| < 5$
2. $z^{-p} e^{\frac{1}{z}}$
3. $1 - cos\frac{n\pi}{2} - isin\frac{n\pi}{2}$
4. $y_x = \frac{1}{24}3^x - \frac{5}{12}(-3)^x + \frac{3}{8}(-1)^x$