Chapter 3

DIFFERENCE EQUATIONS

3.1 Introduction

Differential equations are applicable for continuous systems and cannot be used for discrete variables. Difference equations are the discrete equivalent of differential equations and arise whenever an independent variable can have only discrete values. They are of growing importance in engineering in view of their applications in discrete time- systems used in association with microprocessors.

Some Useful Results

- . 'Δ' is a Forward difference operator such that $\Delta f(x) = f(x+h) - f(x)$

$\therefore\ \Delta y_x = y_{x+1} - y_x$ Taking h as one unit

$$\Delta y_0 = y_1 - y_0$$
$$\Delta y_1 = y_2 - y_1$$
$$\vdots$$
$$\Delta y_n = y_{n+1} - y_n$$

Also $\Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$

$$\vdots$$
$$\Delta^n y_0 = y_n - {}^nC_1 y_{n-1} + {}^nC_2 y_{n-2} - \cdots + (-1)^{n-1}\, {}^nC_{n-1} y_1 + (-1)^n y_0$$

Generalizing $\Delta^n y_r = y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

Properties of operator 'Δ'

$\Delta C = 0$, C being a constant

$\Delta C\, f(x) = C f(x)$

$\Delta[af(x) \pm bg(x)] = a\Delta f(x) \pm b\Delta g(x)$

Example 1 Evaluate the following:

i. Δe^x ii. $\Delta^2 e^x$ iii. $\Delta tan^{-1}x$ iv. $\Delta\left(\frac{x+1}{x^2-3x+2}\right)$

Solution: i. $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

$\Delta e^x = e^x(e-1)$, if $h = 1$

ii. $\Delta^2 e^x = \Delta(\Delta e^x)$

$= \Delta[e^x(e^h - 1)]$

$= (e^h - 1)\, \Delta e^x$

$= (e^h - 1)\, [e^{x+h} - e^x]$

$= (e^h - 1)\, e^x(e^h - 1)$

$= e^x\ (e^h - 1)^2$

iii. $\Delta tan^{-1}x = tan^{-1}(x+h) - tan^{-1}x$

$= tan^{-1}\left(\frac{x+h-x}{1+(x+h)x}\right)$

$= \ tan^{-1}\frac{h}{1+(x+h)x}$

iv. $\Delta\left(\frac{x+1}{x^2-3x+2}\right) \ = \ \Delta\left(\frac{x+1}{(x-1)(x-2)}\right)$

$= \Delta\left(\frac{-2}{x-1} + \frac{3}{x-2}\right) = \Delta\left(\frac{-2}{x-1}\right) + \ \Delta\left(\frac{3}{x-2}\right)$

$= -2\left(\frac{1}{x+1-1} - \frac{1}{x-1}\right) + 3\left(\frac{1}{x+1-2} - \frac{1}{x-2}\right)$

$= -2\left(\frac{1}{x} - \frac{1}{x-1}\right) + 3\left(\frac{1}{x-1} - \frac{1}{x-2}\right)$

$= -\frac{(x+4)}{x(x-1)(x-2)}$

- The shift operator 'E' is defined as $Ef(x) = f(x+h)$
  $\therefore Ey_x = \ y_{x+h}$
  Clearly effect of the shift operator E is to shift the function value to the next higher value $f(x+h)$ or y_{x+h}
  Also $E^2 f(x) = E(Ef(x)) = Ef(x+h) = f(x+2h)$
  $\therefore E^n f(x) = \ f(x+nh)$
  Moreover $E^{-1}f(x) = \ f(x-h)$, where E^{-1} is the inverse operator.

- Relation between Δ and E is given by $E \equiv 1 + \Delta$
  Proof: we know that $\Delta y_n = y_{n+1} - y_n$

  $= Ey_n - y_n$

  $\Rightarrow \ \Delta y_n = (E-1)\ y_n$

  $\Rightarrow \Delta \equiv E - 1$ or $E \equiv 1 + \Delta$

- **Factorial Notation of a Polynomial**

A product of the form $x(x-1)(x-2) \ldots (x-r+1)$ is called a factorial and is denoted by $[x]^r$

$\therefore \quad [x] \ = x$

$[x]^2 = x(x-1)$

$[x]^3 = x(x-1)(x-2)$

$\vdots$

$[x]^n = x(x-1)(x-2) \ldots (x-n+1)$

In case, the interval of differencing is h, then

$[x]^n = x(x-h)(x-h) \ldots \left(x - \overline{n-1}\ \ h\right)$

The results of differencing $[x]^r$ are analogous to that differentiating x^r

$\therefore \Delta[x]^n = n[x]^{n-1}$

$\Delta^2[x]^n = n(n-1)[x]^{n-2}$

$\Delta^3[x]^n = n(n-1)(n-2)[x]^{n-3}$

$\vdots$

$\Delta^n[x]^n = n(n-1)(n-2)\ldots 3.2.1 = n!$

$\Delta^{n+1}[x]^n = 0$

Also $\frac{1}{\Delta}[n] = \frac{[n]^2}{2}$ and so on

Remark:

i. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice-versa.

ii. The coefficient of highest power of x and also the constant term remains unchanged while transforming a polynomial to factorial notation.

Example2 Express the polynomial $2x^2 - 3x + 1$ in factorial notation.

Solution: $2x^2 - 3x + 1 = 2x^2 - 2x - x + 1$

$= 2x(x-1) - x + 1$

$= [x]^2 - [x] + 1$

Example3 Express the polynomial $3x^3 - x + 2$ in factorial notation.

Solution: $3x^3 - x + 2 = 3[x]^3 + A[x]^2 + B[x] + 2$ Using remarks i. and ii.

$= 3x(x-1)(x-2) + Ax(x-1) + Bx + 2$

$= 3x^3 + (A-9)x^2 + (-A+B+6)x + 2$

Comparing the coefficients on both sides

$\mathrm{A} - 9 = 0, \quad -A + \mathrm{B} + 6 = -1$

$\Rightarrow \mathrm{A} = 9, \ \mathrm{B} = 2$

$\therefore 3x^3 - x + 2 = 3[x]^3 + 9[x]^2 + 2[x] + 2$

or

We can also find factorial polynomial using synthetic division as shown below

Let $3x^3 - x + 2 = 3[x]^3 + A[x]^2 + B[x] + 2$

Now coefficients A and B can be found as remainders under x^2 and x columns

	x^3	x^2	x	
1	3	0	–1	2
	–	3	3	
2	3	3	$2 = B$	
	–	6		
	3	9 = A		

3.2 Order of Difference Equation

Order of a difference equation is the difference between the largest and the smallest argument occurring in the difference equation.

Order of the difference equation $x_{n+1} - x_n = 2$ is one $\because n + 1 - n = 1$

Similarly order of the difference equation $x_{n+2} - x_{n+1} + 2x_n = n$ is 2 $\because n + 2 - n = 2$

The equation $x_{n+1} . x_{n-1} = 2x_{n-2}$ is of order 3 as $(n + 1) - (n - 2) = 3$

Also the equation $y_{n+2} + y_n = 1$ is having order 2.

In general, the higher the order of an equation, the more difficult it is to solve.

3.2 Formation of Difference Equations

A difference equation is formed by eliminating the arbitrary constants from a given relation. The order of the difference equation is equal to the number of arbitrary constants in the given relation. Following examples illustrate the formation of difference equations.

Example 4 Write the given difference equation in the subscript notation

$\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$

Solution: Using the relation $\Delta^n y_r = y_{n+r} - {}^nC_1 y_{n+r-1} + {}^nC_2 y_{n+r-2} - \cdots + (-1)^r y_r$

$$\Rightarrow (y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x) + (y_{x+2} - 2y_{x+1} + y_x) + (y_{x+1} - y_x) - y_x = 0$$

$$\Rightarrow y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$$

Example 5 Find a difference equation satisfied by the relation $y = A2^n + n3^{n-1}$

Solution: Given that $y_n = A2^n + n3^{n-1}$ ... ①

Since there is only one arbitrary constant A, we need only first difference

$$\Rightarrow \quad y_{n+1} = A2^{n+1} + (n + 1)3^n$$

$$\Rightarrow \quad y_{n+1} = 2A2^n + 3(n + 1)3^{n-1} \text{ ... ②}$$

Subtracting 2 times ①from ②, we get the required difference equation

$$y_{n+1} - 2y_n = (n + 3)3^{n-1}$$

We can also form the difference equation by the method given below:

Given that $y_n = A2^n + n3^{n-1}$ ... ①

Since there is only one arbitrary constant A, taking the first difference

$$\therefore \; \Delta\, y_n = A\Delta 2^n + \Delta n3^{n-1}$$

$$\Rightarrow \quad y_{n+1} - y_n = A(2^{n+1} - 2^n) + (n + 1)3^n - n3^{n-1}$$

$$\Rightarrow \quad y_{n+1} - y_n = A2^n(2 - 1) + 3(n + 1)3^{n-1} - n3^{n-1}$$

$$\Rightarrow \quad y_{n+1} - y_n = A2^n + (2n + 3)3^{n-1} \text{ ... ②}$$

Subtracting ①from ②, we get the required difference equation

$$\Rightarrow y_{n+1} - 2y_n = (n + 3)3^{n-1}$$

Example 6 Find a difference equation satisfied by the relation $y = ax^2 - bx$

Solution: Since there are 2 arbitrary constants a and b , taking 1st and 2nd differences

$$y_x = ax^2 - bx$$

$$\Rightarrow \; y_{x+1} = a(x+1)^2 - b(x+1)$$

$$\text{and} \quad y_{x+2} = a(x+2)^2 - b(x+2)$$

Eliminating arbitrary constants a and b from the given set of equations

$$\Rightarrow \begin{vmatrix} y_x & x^2 & x \\ y_{x+1} & (x+1)^2 & (x+1) \\ y_{x+2} & (x+2)^2 & (x+2) \end{vmatrix} = 0$$

$$\Rightarrow y_x[(x+1)^2(x+2) - (x+2)^2(x+1)] - y_{x+1}[x^2(x+2) - (x+2)^2 x]$$
$$+ y_{x+2}[x^2(x+1) - (x+1)^2 x] = 0$$

$$\Rightarrow y_x[(x+1)(x+2)(x+1-x-2)] - y_{x+1}[x(x+2)(x-x-2)]$$
$$+ y_{x+2}[x(x+1)(x-x-1)] = 0$$

∴ The required difference equation is given by:

$$y_{x+2}(x^2+x) - 2y_{x+1}(x^2+2x) + y_x(x^2+3x+2) = 0$$

We can also form the difference equation by the method given below:

Given that $y = ax^2 - bx$ ... ①

Taking the first difference

$$\Delta y = a\Delta x^2 - b\Delta x$$
$$= a[(x+1)^2 - x^2] - b[x+1-x]$$
$$\Rightarrow \; \Delta y = a(2x+1) - b \; ... \; ②$$

Again taking the second difference

$$\Delta^2 y = 2a\Delta x + 0$$
$$= 2a(x+1-x) = 2a$$
$$\Rightarrow a = \frac{1}{2}\Delta^2 y \; ... \; ③$$

Also from ②, $b = a(2x+1) - \Delta y = \frac{1}{2}\Delta^2 y(2x+1) - \Delta y$ using ③

$$\Rightarrow \; b = \frac{1}{2}\Delta^2 y(2x+1) - \Delta y \; ... \; ④$$

Using ③and ④ in ①, we get

$$y = \frac{1}{2}\Delta^2 y(x^2) - \left(\frac{1}{2}\Delta^2 y(2x+1) - \Delta y\right)x$$

$$\Rightarrow 2y = \Delta^2 y(x^2 - 2x^2 - x) + 2x\Delta y$$

$$\Rightarrow (x^2+x)\Delta^2 y - 2x\Delta y + 2y = 0$$

Writing in subscript notation, we get

$$(x^2+x)(y_{x+2} - 2y_{x+1} + y_x) - 2x(y_{x+1} - y_x) + 2y_x = 0$$

$$\Rightarrow y_{x+2}(x^2+x) - 2y_{x+1}(x^2+2x) + y_x(x^2+3x+2) = 0$$

Example7 Find a difference equation satisfied by the relation $y = \frac{a}{x} + b$

Solution: Since there are 2 arbitrary constants a and b , taking 1st and 2nd differences

$$y_x = \frac{a}{x} + b$$

$$\Rightarrow \; y_{x+1} = \frac{a}{x+1} + b$$

$$\text{and} \quad y_{x+2} = \frac{a}{x+2} + b$$

Eliminating arbitrary constants a and b from the given set of equations

$$\Rightarrow \begin{vmatrix} y_x & \frac{1}{x} & 1 \\ y_{x+1} & \frac{1}{x+1} & 1 \\ y_{x+2} & \frac{1}{x+2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow y_x \left[\frac{1}{x+1} - \frac{1}{x+2}\right] - y_{x+1}\left[\frac{1}{x} - \frac{1}{x+2}\right] + y_{x+2}\left[\frac{1}{x} - \frac{1}{x+1}\right] = 0$$

$$\Rightarrow y_x \left[\frac{x+2-x-1}{(x+1)(x+2)}\right] - y_{x+1}\left[\frac{x+2-x}{x(x+2)}\right] + y_{x+2}\left[\frac{x+1-x}{x(x+1)}\right] = 0$$

∴ The required difference equation is given by:

$$(x+2)y_{x+2} - 2(x+1)y_{x+1} - xy_x = 0$$

Example 8 Find a difference equation satisfied by the relation $y = A2^n + B3^n + \frac{1}{2}$

Solution: Since there are 2 arbitrary constants A and B , taking 1st and 2nd differences

$$y_n = A2^n + B3^n + \frac{1}{2}$$

$$\Rightarrow \quad y_{n+1} = A2^{n+1} + B3^{n+1} + \frac{1}{2}$$

$$\text{and} \quad y_{n+2} = A2^{n+2} + B3^{n+2} + \frac{1}{2}$$

Rewriting the given equations as

$$y_n - \frac{1}{2} = A2^n + B3^n$$

$$\Rightarrow \quad y_{n+1} - \frac{1}{2} = 2A2^n + 3B3^n$$

$$\text{and} \quad y_{n+2} - \frac{1}{2} = 4A2^n + 9B3^n$$

Eliminating arbitrary constants A and B from the given set of equations

$$\Rightarrow \begin{vmatrix} y_n - \frac{1}{2} & 1 & 1 \\ y_{n+1} - \frac{1}{2} & 2 & 3 \\ y_{n+2} - \frac{1}{2} & 4 & 9 \end{vmatrix} = 0$$

$$\Rightarrow \left(y_n - \frac{1}{2}\right)(18 - 12) - \left(y_{n+1} - \frac{1}{2}\right)(9 - 4) + \left(y_{n+2} - \frac{1}{2}\right)(3 - 2) = 0$$

∴ The required difference equation is given by:

$$y_{n+2} - 5y_{n+1} + 6y_n = 1$$

Example 9 Find a difference equation satisfied by the relation

$$y = A\cos n\theta \; + B\sin n\theta$$

Solution: Since there are 2 arbitrary constants A and B , taking 1st and 2nd differences

$$y_n = A\cos n\theta \; + B\sin n\theta$$

$$\Rightarrow \quad y_{n+1} = A\cos(n+1)\theta \; + B\sin(n+1)\theta$$

and $$y_{n+2} = A\cos(n+2)\theta \; + B\sin(n+2)\theta$$

Eliminating arbitrary constants A and B from the given set of equations

$$\Rightarrow \begin{vmatrix} y_n & \cos n\theta & \sin n\theta \\ y_{n+1} & \cos(n+1)\theta & \sin(n+1)\theta \\ y_{n+2} & \cos(n+2)\theta & \sin(n+2)\theta \end{vmatrix} = 0$$

$$\Rightarrow y_n[\sin(n+2)\theta\,.\cos(n+1)\theta - \cos(n+2)\theta\,.\sin(n+1)\theta] -$$
$$y_{n+1}[\sin(n+2)\theta\,.\cos n\theta - \cos(n+2)\theta\,.\sin n\theta]$$
$$+y_{n+2}[\sin(n+1)\theta\,.\cos n\theta - \cos(n+1)\theta\,.\sin n\theta] = 0$$

$$\Rightarrow y_n\sin(n+2-n-1)\theta - y_{n+1}\sin(n+2-n)\theta + y_{n+1}\sin(n+1-n)\theta = 0$$

∴ The required difference equation is given by:

$$y_{n+2}\sin\theta - y_{n+1}\sin 2\theta + y_n\sin\theta = 0$$

$$\Rightarrow y_{n+2} - 2\cos\theta\, y_{n+1} + y_n = 0$$

Example10 Form the difference equation from the given relation

$$\frac{\log(1+z)}{(1-z)} = y_0 + y_1 z + y_2 z^2 + \cdots + y_n z^n + \cdots$$

Solution: Rewriting given relation as

$$\log(1+z) = (1-z)(y_0 + y_1 z + y_2 z^2 + \cdots + y_n z^n + \cdots)$$

$$\Rightarrow z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots = y_0 + (y_1 - y_0)z + (y_2 - y_1)z^2 + (y_3 - y_2)z^3 + \cdots +$$

Equating the powers of z on both sides, we get

$$(y_1 - y_0) = 1$$

$$(y_2 - y_1) = -\frac{1}{2}$$

$$(y_3 - y_2) = \frac{1}{3}$$

$\vdots$

$$(y_{n+1} - y_n) = \frac{(-1)^n}{n+1}$$

$\vdots$

$\therefore$ The required difference equation is given by $(y_{n+1} - y_n) = \frac{(-1)^n}{n+1}$

Or $\Delta y_n = \frac{(-1)^n}{n+1}$

3.3 Solution of Difference Equations

A general linear difference equation of order 'r' with constant coefficients is given by:

$(a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r) y_n = F(n)$

where $a_0, a_1, a_2, \ldots a_r$ are constants and $F(n)$ is a function of 'n' alone or constant.

Note:

- If the difference equation involves 'Δ' instead of 'E' use $\Delta \equiv E - 1$
- If it involves 'y_{n+r}' instead of 'y_n' use $y_{n+r} = E^r y_n$

Hence any linear difference equation may be written in symbolic form as:

$$\boldsymbol{f(E) y_n = F(n)}$$

Complete solution of equation $\boldsymbol{f(E) y_n = F(n)}$ is given by $\boldsymbol{y}$ **= C.F + P.I.**

where C.F. denotes complimentary function and P.I. is particular integral.

When $\boldsymbol{F(n) = 0}$, then solution of equation $\boldsymbol{f(E) y_n = 0}$ is given by $\boldsymbol{y}$ **= C.F**

3.3.1 Rules for Finding Complimentary Function (C.F.)

Consider the difference equation $f(E) y_n = F(n) \ldots$ ①

$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r) y_n = F(n)$

Step 1: Find the Auxiliary Equation (A.E) given by $f(E) = 0$

$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r) = 0 \ldots$②

Step 2: Solve the auxiliary equation given by ②

i. If the *n* roots of A.E. are real and distinct say $m_1, m_2, \ldots, m_n$

 C.F. = $c_1 {m_1}^n + c_2 {m_2}^n + \cdots + c_n {m_n}^n$

ii. If two or more roots are equal i.e. $m_1 = m_2 = \cdots = m_k, k \leq n$

 C.F. = $(c_1 + c_2 n + c_3 n^2 + \cdots + c_k n^{k-1}) {m_1}^n + \cdots + c_n {m_n}^n$

iii. If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

 C.F. = $r^n (c_1 \cos n\theta + c_2 \sin n\theta) + c_3 {m_3}^n + \cdots + c_n {m_n}^n$

 where $r = \sqrt{\alpha^2 + \beta^2}$ and $\theta = \tan^{-1} \frac{\beta}{\alpha}$

Example 11 Solve the difference equation $(\Delta^2 - 3\Delta + 2) y_n = 0$

Solution: Putting $\Delta \equiv E - 1$

$\Rightarrow ((E-1)^2 - 3(E-1) + 2)y_n = 0$

$\Rightarrow (E^2 - 5E + 6)y_n = 0$

Auxiliary Equation (A.E.) is

$E^2 - 5E + 6 = 0$

$\Rightarrow (E-2)(E-3) = 0$

$\Rightarrow E = 2,3$

$\therefore C.F. = c_1 2^n + c_2 3^n$

The solution is y_n = C.F.

$\Rightarrow y_n = c_1 2^n + c_2 3^n$

Example 12 Solve the difference equation $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

Solution: Given difference equation is $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

$\Rightarrow (E^3 - 2E^2 - 5E + 6)U_n = 0$

$\Rightarrow E^3 - E^2 - E^2 + E - 6E + 6 = 0$

$\Rightarrow E^2(E-1) - E(E-1) - 6(E-1) = 0$

$\Rightarrow (E-1)(E^2 - E - 6) = 0$

$\Rightarrow (E-1)(E-3)(E+2) = 0$

$\Rightarrow E = 1, 3, -2$

$\therefore C.F. = c_1 + c_2 3^n + c_2 3^n$

The solution is y_n = C.F.

$\Rightarrow y_n = c_1 + c_2 3^n + c_2 3^n$

Example 13 Solve the difference equation: $y_{n+4} - 2y_{n+2} + y_n = 0$

Solution: Given difference equation is $y_{n+4} - 2y_{n+2} + y_n = 0$

$\Rightarrow (E^4 - 2E^2 + 1)y_n = 0$

Auxiliary equation is: $E^4 - 2E^2 + 1 = 0$

$\Rightarrow (E^2 - 1)^2 = 0$

$\Rightarrow (E+1)^2(E-1)^2 = 0$

$\Rightarrow E = -1, -1, 1, 1$

C.F. = $(c_1 + c_2 n)(-1)^n + (c_3 + c_4 n)(1)^n$

Since $F(n) = 0$, solution is given by y_n = C.F

$$\Rightarrow y_n = (c_1 + c_2 n)(-1)^n + (c_3 + c_4 n)$$

Example 14 Solve the difference equation: $y_{n+3} - 2y_{n+1} + 4y_n = 0$

Solution: Given difference equation is $y_{n+3} - 2y_{n+1} + 4y_n = 0$

$$\Rightarrow (E^3 - 2E + 4)y_n = 0$$

Auxiliary equation is: $E^3 - 2E + 4 = 0$ ...①

By hit and trial $(E + 2)$ is a factor of ①

∴① May be rewritten as

$$E^3 + 2E^2 - 2E^2 - 4E + 2E + 4 = 0$$

$$\Rightarrow E^2(E + 2) - 2E(E + 2) + 2(E + 2) = 0$$

$$\Rightarrow (E + 2)(E^2 - 2E + 2) = 0$$

$$\Rightarrow E = -2, 1 \pm i$$

C.F. = $c_1(-2)^n + r^n(c_2 \cos n\theta + c_3 \sin n\theta)$

$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{1+1} = \sqrt{2}$, where $1 \pm i \equiv \alpha \pm i\beta$

$\theta = \tan^{-1}\frac{\beta}{\alpha} = \tan^{-1} 1 = \frac{\pi}{4}$

∴ C.F. = $c_1(-2)^n + r^n(c_2 \cos n\theta + c_3 \sin n\theta)$

Since $F(n) = 0$, solution is given by $y_n =$ C.F

$$\Rightarrow y_n = c_1 e^{-2x} + \left(\sqrt{2}\right)^n \left(c_2 \cos\frac{n\pi}{4} + c_3 \sin\frac{n\pi}{4}\right)$$

3.3.2 Rules for Finding Particular Integral (P.I.)

Consider the difference equation $f(E)y_n = F(n)$ ... ①

$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \cdots + a_{r-1} E + a_r)y_n = F(n)$

Then P.I = $\frac{1}{f(E)} F(n)$, Clearly P.I. = 0 if $F(n) = 0$

Case I: When $F(n) = a^n$

Use the rule P.I $= \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$, $f(a) \neq 0$

In case of failure, i.e. if $f(a) = 0$

Then $\frac{1}{E-a} a^n = n a^{n-1}$

$$\frac{1}{(E-a)^2} a^n = \frac{n(n-1)!}{2!} a^{n-2}$$

$$\frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)!}{3!} a^{n-3}$$

$$\vdots$$

Case II: When $F(n) = \sin kn$ or $\cos kn$

Use the exponential values of $\sin kn$ or $\cos kn$

i.e. $\sin kn = \frac{e^{ikn} - e^{-ikn}}{2i}$, $\cos kn = \frac{e^{ikn} + e^{-ikn}}{2}$ and find P.I. as in case I.

Case III: When $F(n) = n^p$, P.I $= \frac{1}{f(E)} n^p$

1. Replace E by Δ by substituting $E \equiv 1 + \Delta$, so that P.I. $= \frac{1}{f(1+\Delta)} n^p$
2. Take the lowest degree term common from $f(1 + \Delta)$ to get an expression of the form $[1 \pm \phi(\Delta)]$ and take it to the numerator to become $[1 \pm \phi(\Delta)]^{-1}$
3. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to p^{th} degree as $(p+1)^{\text{th}}$ difference of n^p is zero. Following expansions will be useful to expand $[1 \pm \phi(\Delta)]^{-1}$ in ascending powers of Δ
   - $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
   - $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \cdots$
4. Express $F(n) = n^p$ into factorial notation and operate on it term by term.

Case IV: When $F(n) = a^n g(n)$, where $g(n)$ is any function of n

Take P.I $= \frac{1}{f(E)} a^n g(n)$ and use the rule: $\frac{1}{f(E)} a^n g(n) = a^n \left(\frac{1}{f(aE)} g(n) \right)$ and proceed as case III.

Examples on Case I

Example 15 Solve the difference equation: $y_{n+2} - 3y_{n+1} + 2y_n = 5^n$

Solution: Given difference equation is $y_{n+2} - 3y_{n+1} + 2y_n = 5^n$

$$\Rightarrow (E^2 - 3E + 2)y_n = 5^n$$

Auxiliary equation is: $E^2 - 3E + 2 = 0$

$$\Rightarrow (E - 1)(E - 2) = 0$$

$$\Rightarrow E = 1, 2$$

C.F. $= c_1 + c_2 2^n$

P.I. $= \frac{1}{f(E)} F(n) = \frac{1}{(E-1)(E-2)} 5^n$

$= \frac{5^n}{(5-1)(5-2)}$, by putting $E = 5$ $\because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$

$= \frac{1}{4.3} 5^n = \frac{5^n}{12}$

Complete solution is: $y_n =$ C.F. + P.I

$\Rightarrow y_n = c_1 + c_2 2^n + \frac{5^n}{12}$

Example 16 Solve the difference equation: $U_{n+2} - 7U_{n+1} + 10U_n = 3e^{2n} - 3^n$

Solution: Given difference equation is $U_{n+2} - 7U_{n+1} + 10U_n = 3e^{2n} - 3^n$

$\Rightarrow (E^2 - 7E + 10)U_n = 3e^{2n} - 3^n$

Auxiliary equation is: $E^2 - 7E + 10 = 0$

$\Rightarrow (E-2)(E-5) = 0$

$\Rightarrow E = 2, 5$

C.F. = $c_1 2^n + c_2 5^n$

P.I. = $\frac{1}{f(E)} F(n) = \frac{1}{E^2-7E+10}(3e^{2n} - 3^n)$

$= \frac{3}{E^2-7E+10}(e^2)^n - \frac{1}{E^2-7E+10} 3^n$

$= \frac{3(e^2)^n}{(e^2)^2-7(e^2)+10} - \frac{3^n}{3^2-7(3)+10}$ $\because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$

$= \frac{3e^{2n}}{e^4-7e^2+10} + \frac{3^n}{2}$

Complete solution is: $y_n =$ C.F. + P.I

$\Rightarrow y_n = c_1 2^n + c_2 5^n + \frac{3e^{2n}}{e^4-7e^2+10} + \frac{3^n}{2}$

Example 17 Solve the difference equation: $y_{n+2} - 2y_{n+1} + y_n - 1 = 0$

Solution: Given difference equation is $y_{n+2} - 2y_{n+1} + y_n = 1$

$\Rightarrow (E^2 - 2E + 1)y_n = 5^n$

Auxiliary equation is: $E^2 - 2E + 1 = 0$

$\Rightarrow (E-1)^2 = 0$

$\Rightarrow E = 1, 1$

C.F. = $(c_1 + c_2 n)1^n = (c_1 + c_2 n)$

P.I. = $\frac{1}{f(E)} F(n) = \frac{1}{(E-1)^2} 1^n$

$= \frac{n(n-1)}{2!} 1^{n-2} \qquad \because \frac{1}{(E-a)^2} a^n = \frac{n(n-1)!}{2!} a^{n-2}$ if $f(a) = 0$

$= \frac{n(n-1)}{2}$

Complete solution is: y_n = C.F. + P.I

$\Rightarrow y_n = (c_1 + c_{2n}) + \frac{n(n-1)}{2}$

Example 18 Solve the difference equation: $y_{n+2} - 3y_{n+1} - 4y_n + 12 = 0$

Solution: Given difference equation is $y_{n+2} - 3y_{n+1} - 4y_n = -12$

$\Rightarrow (E^2 - 3E - 4)y_n = -12$

Auxiliary equation is: $E^2 - 3E - 4 = 0$

$\Rightarrow (E-4)(E+1) = 0$

$\Rightarrow E = 4, -1$

C.F. = $c_1 4^n + c_2(-1)^n$

P.I. = $\frac{1}{f(E)} F(n) = (-12)\frac{1}{(E-4)(E+1)} 1^n$

$= (-12)\frac{1}{(1-4)(1+1)}$, by putting $E = 1 \qquad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$

$= (-12)\frac{1}{-6} = 2$

Complete solution is: y_n = C.F. + P.I

$\Rightarrow y_n = c_1 4^n + c_2(-1)^n + 2$

Example 19 Solve the difference equation: $y_{n+2} - 4y_n = 2^n$

Solution: Given difference equation is $y_{n+2} - 4y_n = 2^n$

$\Rightarrow (E^2 - 4)y_n = 2^n$

Auxiliary equation is: $E^2 - 4 = 0$

$\Rightarrow E = \pm 2$

C.F. = $c_1 2^n + c_2(-2)^n$

P.I. = $\frac{1}{f(E)}\ F(n)\ = \frac{1}{(E+2)(E-2)}2^n$

$= \frac{1}{(E-2)}\left(\frac{1}{(E+2)}2^n\right)$

$= \frac{1}{(E-2)}\left(\frac{2^n}{(2+2)}\right)$ $\quad\quad \because \frac{1}{f(E)}a^n = \frac{a^n}{f(a)}$

$= \frac{1}{4}\frac{1}{(E-2)}2^n = \frac{n2^{n-1}}{4}$ $\quad\quad \because \frac{1}{E-a}a^n = na^{n-1}$ if $f(a) = 0$

Complete solution is: $y_n\ =\ $ C.F. + P.I

$\Rightarrow y_n\ =\ \ c_1 2^n + c_2(-2)^n + \frac{n2^{n-1}}{4}$

Examples on Case II

Example 20 Solve the difference equation: $y_{n+1} - 2y_n = \cos 2n$

Solution: Given difference equation is $y_{n+1} - 2y_n = \cos 2n$

$\Rightarrow (E-2)y_n = \cos 2n$

Auxiliary equation is: $E - 2 = 0$

$\Rightarrow\ E = 2$

C.F. = $c2^n$

P.I. = $\frac{1}{f(E)}\ F(n)\ = \frac{1}{E-2}\cos 2n$

$= \frac{1}{E-2}\frac{e^{2ni} + e^{-2ni}}{2}$

$= \frac{1}{2}\frac{1}{E-2}(e^{2i})^n + \frac{1}{2}\frac{1}{E-2}(e^{-2i})^n$

$= \frac{1}{2}\frac{e^{2ni}}{e^{2i}-2} + \frac{1}{2}\frac{e^{-2ni}}{e^{-2i}-2}$ $\quad\quad \because \frac{1}{f(E)}a^n = \frac{a^n}{f(a)}$

$= \frac{1}{2}\left(\frac{e^{2ni}}{e^{2i}-2} + \frac{e^{-2ni}}{e^{-2i}-2}\right)$

$= \frac{1}{2}\left(\frac{e^{2ni}(e^{-2i}-2)+e^{-2ni}(e^{2i}-2)}{(e^{2i}-2)(e^{-2i}-2)}\right)$

$= \frac{1}{2}\left(\frac{(e^{2(n-1)i}+e^{-2(n-1)i})-2(e^{2ni}+e^{-2ni})}{1-2(e^{2i}+e^{-2i})+4}\right)$

$= \frac{1}{2}\left(\frac{2\cos 2(n-1)-4\cos 2n}{5-4\cos 2}\right)$ $\quad\quad \because \cos ax = \frac{e^{iax}+e^{-iax}}{2}$

$$= \frac{\cos 2(n-1) - 2\cos 2n}{5 - 4\cos 2}$$

Complete solution is: $y_n = $ C.F. + P.I

$$\Rightarrow y_n = c2^n + \frac{\cos 2(n-1) - 2\cos 2n}{5 - 4\cos 2}$$

Example 21 Solve the difference equation: $y_{n+2} + y_{n+1} + 2y_n = \sin 2n$

Solution: Given difference equation is $y_{n+2} + y_{n+1} + 2y_n = \sin 2n$

$$\Rightarrow (E^2 + E + 2)y_n = \sin 2n$$

Auxiliary equation is: $E^2 + E + 2 = 0$

$$\Rightarrow (E-1)(E+2) = 0$$

$$\Rightarrow E = 1, -2$$

C.F. = $c_1 + c_2(-2)^n$

P.I. = $\frac{1}{f(E)} F(n) = \frac{1}{E^2+E+2} \sin 2n$

$$= \frac{1}{E^2+E+2} \frac{e^{2ni} - e^{-2ni}}{2i}$$

$$= \frac{1}{2i} \cdot \frac{1}{E^2+E+2} e^{2ni} - \frac{1}{2i} \cdot \frac{1}{E^2+E+2} e^{-2ni}$$

$$= \frac{1}{2i} \cdot \frac{1}{E^2+E+2} (e^{2i})^n - \frac{1}{2i} \cdot \frac{1}{E^2+E+2} (e^{-2i})^n$$

$$= \frac{1}{2i} \cdot \frac{e^{2ni}}{e^{4i}+e^{2i}+2} - \frac{1}{2i} \cdot \frac{e^{-2ni}}{e^{-4i}+e^{-2i}+2} \qquad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$$

$$= \frac{1}{2i}\left(\frac{e^{2ni}}{e^{4i}+e^{2i}+2} - \frac{e^{-2ni}}{e^{-4i}+e^{-2i}+2}\right)$$

Complete solution is: $y_n = $ C.F. + P.I

$$\Rightarrow y_n = c_1 + c_2(-2)^n + \frac{1}{2i}\left(\frac{e^{2ni}}{e^{4i}+e^{2i}+2} - \frac{e^{-2ni}}{e^{-4i}+e^{-2i}+2}\right)$$

Example 22 Solve the difference equation $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos \alpha n$

Solution: Given difference equation is $y_{n+2} - 2\cos\alpha\, y_{n+1} + y_n = \cos \alpha n$

$$\Rightarrow (E^2 - 2\cos\alpha\, E + 1)y_n = \cos \alpha n$$

Auxiliary equation is: $E^2 - 2\cos\alpha\, E + 1 = 0$ ...①

$$\Rightarrow E = \cos\alpha \pm i\sin\alpha$$

C.F. = $r^n(c_1 \cos n\theta + c_2 \sin n\theta)$

$r = \sqrt{\cos^2\alpha + sin^2\alpha} = 1$, $\theta = \tan^{-1}\frac{\sin\alpha}{\cos\alpha} = \tan^{-1}\tan\alpha = \alpha$

$\therefore$ C.F. = $(c_1 \cos n\alpha + c_2 \sin n\alpha)$

P.I. = $\frac{1}{f(E)}\ F(n) = \frac{1}{E^2 - 2\cos\alpha E + 1}\cos\alpha n$

$= \frac{1}{E^2 - (e^{i\alpha} + e^{-i\alpha})E + 1}\left(\frac{e^{i\alpha n} + e^{-i\alpha n}}{2}\right)$

$= \frac{1}{2}\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})}(e^{i\alpha n} + e^{-i\alpha n})$

$\because [x^2 - (a+b)x + ab = (x-a)(x-b)]$

$= \frac{1}{2}\left[\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})}(e^{i\alpha})^n + \frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})}(e^{-i\alpha})^n\right]$

$= \frac{1}{2(E - e^{i\alpha})}\left[\frac{e^{i\alpha n}}{e^{i\alpha} - e^{-i\alpha}}\right] + \frac{1}{2(E - e^{-i\alpha})}\left[\frac{e^{-i\alpha n}}{e^{-i\alpha} - e^{i\alpha}}\right] \quad \because \frac{1}{f(E)}a^n = \frac{a^n}{f(a)}$

$= \left[\frac{1}{2(E - e^{i\alpha})(2iSin\alpha)}(e^{i\alpha})^n + \frac{1}{2\,(E - e^{-i\alpha})\,(-2i\,Sin\alpha)}(e^{-i\alpha})^n\right]$

$= \frac{1}{4i\ Sin\ \alpha}\left[\frac{1}{E - e^{i\alpha}}(e^{i\alpha})^n - \frac{1}{E - e^{-i\alpha}}(e^{-i\alpha})^n\right]$

$= \frac{1}{4iSin\alpha}[n(e^{i\alpha})^{n-1} - n(e^{-i\alpha})^{n-1}] \quad \because \frac{1}{E-a}a^n = na^{n-1}$ if $f(a) = 0$

Complete solution is: y_n = C.F. + P.I

$\Rightarrow y_n = (c_1 \cos n\alpha + c_2 \sin n\alpha) + \frac{1}{4iSin\alpha}[n(e^{i\alpha})^{n-1} - n(e^{-i\alpha})^{n-1}]$

Examples on Case III

Example 23 Solve the difference equation: $y_{n+2} + y_{n+1} + y_n = n^2$

Solution: Given difference equation is $y_{n+2} + y_{n+1} + y_n = n^2$

$\Rightarrow (E^2 + E + 1)y_n = n^2$

Auxiliary equation is: $E^2 + E + 1 = 0$

$\Rightarrow E = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

C.F. = $r^n(c_1 \cos n\theta + c_2 \sin n\theta)$

$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1$, where $-\frac{1}{2} \pm i\frac{\sqrt{3}}{2} \equiv \alpha \pm i\beta$

$\theta = \tan^{-1}\frac{\beta}{\alpha} = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$

$\therefore$ C.F. = $\left(c_1 \cos\frac{n\pi}{3} - c_2 \sin\frac{n\pi}{3}\right)$

P.I. = $\frac{1}{f(E)}\ F(n) = \frac{1}{E^2+E+1}(n^2)$

$$= \frac{1}{E^2+E+1}(n^2 - n + n)$$

$$= \frac{1}{(1+\Delta)^2 + (1+\Delta)+1}(n^2 - n + n)$$

$$= \frac{1}{3+3\Delta+\Delta^2}(n(n-1) + n)$$

$$= \frac{1}{3\left(1+\Delta+\frac{1}{3}\Delta^2\right)}([n]^2 + [n])$$

$$= \frac{1}{3}\left[1 + \left(\Delta + \frac{1}{3}\Delta^2\right)\right]^{-1}([n]^2 + [n])$$

$$= \frac{1}{3}\left[1 - \left(\Delta + \frac{1}{3}\Delta^2\right) + \left(\Delta + \frac{1}{3}\Delta^2\right)^2 + \cdots\right]([n]^2 + [n])$$

$$\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$= \frac{1}{3}\left[1 - \Delta - \frac{1}{3}\Delta^2 + \Delta^2 + \cdots\right]([n]^2 + [n])$$

$$= \frac{1}{3}\left[1 - \Delta + \frac{2}{3}\Delta^2 + \cdots\right]([n]^2 + [n])$$

$$= \frac{1}{3}\left[[n]^2 + [n] - \Delta([n]^2 + [n]) + \frac{2}{3}\Delta^2([n]^2 + [n])\right]$$

$$= \frac{1}{3}\left[[n]^2 + [n] - (2[n] + 1) + \frac{2}{3}(2 + 0)\right]$$

$$\because \Delta[n]^2 = 2[n],\ \Delta[n] = 1,\ \Delta^2[n]^2 = 2!,\ \Delta^2[n] = 0$$

$$= \frac{1}{3}\left[[n]^2 - [n] + \frac{1}{3}\right]$$

$$= \frac{1}{3}\left[n(n-1) - n + \frac{1}{3}\right]$$

$$= \frac{1}{3}\left[n^2 - 2n + \frac{1}{3}\right]$$

Complete solution is: y_n = C.F. + P.I

$\Rightarrow y_n = c_1 \cos\frac{n\pi}{3} - c_2 \sin\frac{n\pi}{3} + \frac{1}{3}\left[n^2 - 2n + \frac{1}{3}\right]$

Example 24 Solve the difference equation: $y_{n+2} - 5y_{n+1} + 6y_n = n^2 + 3n + 2$

Solution: Given difference equation is $y_{n+2} - 5y_{n+1} + 6y_n = n^2 + 3n + 2$

$\Rightarrow (E^2 - 5E + 6)y_n = n^2 + 3n + 2$

Auxiliary equation is: $E^2 - 5E + 6 = 0$

$\Rightarrow (E-2)(E-3) = 0$

$\Rightarrow E = 2, 3$

C.F. = $c_1 2^n + c_2 3^n$

P.I. = $\frac{1}{f(E)} F(n) = \frac{1}{E^2-5E+6}(n^2+3n+2)$

$$= \frac{1}{E^2-5E+6}(n^2+3n+2)$$

$$= \frac{1}{(1+\Delta)^2 - 5(1+\Delta)+6}(n^2 - n + 4n + 2)$$

$$= \frac{1}{2-3\Delta+\Delta^2}(n(n-1)+4n+2)$$

$$= \frac{1}{2-3\Delta+\Delta^2}([n]^2+4[n]+2)$$

$$= \frac{1}{2\left(1-\frac{3}{2}\Delta+\frac{1}{2}\Delta^2\right)}([n]^2+4[n]+2)$$

$$= \frac{1}{2}\left[1-\left(\frac{3}{2}\Delta-\frac{1}{2}\Delta^2\right)\right]^{-1}([n]^2+4[n]+2)$$

$$= \frac{1}{2}\left[1+\left(\frac{3}{2}\Delta-\frac{1}{2}\Delta^2\right)+\left(\frac{3}{2}\Delta-\frac{1}{2}\Delta^2\right)^2+\cdots\right]([n]^2+4[n]+2)$$

$$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{1}{2}\left[1+\frac{3}{2}\Delta-\frac{1}{2}\Delta^2+\frac{9}{4}\Delta^2+\cdots\right]([n]^2+4[n]+2)$$

$$= \frac{1}{2}\left[1+\frac{3}{2}\Delta+\frac{7}{4}\Delta^2+\cdots\right]([n]^2+4[n]+2)$$

$$= \frac{1}{2}\left[[n]^2+4[n]+2+\frac{3}{2}\Delta([n]^2+4[n]+2)+\frac{7}{4}\Delta^2([n]^2+4[n]+2)\right]$$

$$= \frac{1}{2}\left[[n]^2+4[n]+2+\frac{3}{2}(2[n]+4+0)+\frac{7}{4}(2+0+0)\right]$$

$$\because \Delta[n]^2 = 2[n], \Delta[n] = 1, \ \Delta^2[n]^2 = 2!$$

$$= \frac{1}{2}\left[[n]^2+4[n]+2+3[n]+6+\frac{7}{2}\right]$$

$$= \frac{1}{2}\left[[n]^2+7[n]+\frac{23}{2}\right] = \frac{1}{2}\left[n(n-1)+7n+\frac{23}{2}\right]$$

$$= \frac{1}{2}\left[n^2+6n+\frac{23}{2}\right]$$

Complete solution is: y_n = C.F. + P.I

$$\Rightarrow y_n = c_1 2^n + c_2 3^n + \frac{1}{2}\left[n^2+6n+\frac{23}{2}\right]$$

Examples on Case IV

Example 25 Solve the difference equation: $U_{n+1} - 2U_n = 3^n n^2$

Solution: Given difference equation is $U_{n+1} - 2U_n = 3^n n^2$

$\Rightarrow (E-2)U_n = 3^n n^2$

Auxiliary equation is: $E - 2 = 0$

$\Rightarrow E = 2$

C.F. = $C2^n$

P.I. = $\frac{1}{f(E)} F(n) = \frac{1}{E-2}(3^n n^2)$

$$= 3^n \frac{1}{3E-2}(n^2) \qquad \because \frac{1}{f(E)} a^n g(n) = a^n \frac{1}{f(aE)} g(n)$$

$$= 3^n \frac{1}{3(1+\Delta)-2}(n^2 - n + n)$$

$$= 3^n \frac{1}{1+3\Delta}(n(n-1) + n)$$

$$= 3^n[1+3\Delta]^{-1}([n]^2 + [n])$$

$$= 3^n[1 - 3\Delta + 9\Delta^2 - \cdots]([n]^2 + [n])$$

$$\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$$

$$= 3^n[[n]^2 + [n] - 3\Delta([n]^2 + [n]) + 9\Delta^2([n]^2 + [n])]$$

$$= 3^n[[n]^2 + [n] - 3(2[n] + 1) + 9(2 + 0)]$$

$$\because \Delta[n]^2 = 2[n], \Delta[n] = 1, \ \Delta^2[n]^2 = 2!, \Delta^2[n] = 0$$

$$= 3^n[[n]^2 - 5[n] + 15]$$

$$= 3^n[n(n-1) - 5n + 15]$$

$$= 3^n[n^2 - 6n + 15]$$

Complete solution is: y_n = C.F. + P.I

$\Rightarrow y_n = C2^n + 3^n[n^2 - 6n + 15]$

Example 26 Solve the difference equation

$$y_{n+2} - 2y_{n+1} + 4y_n = 2^n\left(\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right)$$

Solution: Given difference equation may be written as:

$\Rightarrow (E^2 - 2E + 4)y_n = 2^n\left(\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right)$

Auxiliary equation is: $E^2 - 2E + 4 = 0$ ...①

$\Rightarrow E = 1 \pm i\sqrt{3}$

$$\text{C.F.} = r^n(c_1 \cos n\theta + c_2 \sin n\theta)$$

$$r = \sqrt{1+3} = 2\,, \quad \theta = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$$

$$\therefore \text{C.F.} = 2^n\left(c_1 \cos\frac{n\pi}{3} + c_2 \sin\frac{n\pi}{3}\right)$$

$$\text{P.I.} = \frac{1}{f(E)}\,F(n) = \frac{1}{E^2-2E+4}2^n\left(\cos\frac{n\pi}{3} + \sqrt{3}\sin\frac{n\pi}{3}\right)$$

$$= 2^n\frac{1}{4E^2-4E+4}2\left(\frac{1}{2}\cos\frac{n\pi}{3} + \frac{\sqrt{3}}{2}\sin\frac{n\pi}{3}\right) \quad \because \frac{1}{f(E)}\,a^n g(n) = a^n\frac{1}{f(aE)}\,g(n)$$

$$= 2^{n+1}\frac{1}{4(E^2-E+1)}\left(\cos\frac{\pi}{3}\cos\frac{n\pi}{3} + \sin\frac{\pi}{3}\sin\frac{n\pi}{3}\right)$$

$$= \frac{2^{n+1}}{4}\frac{1}{(E^2-E+1)}\cos(n-1)\frac{\pi}{3}$$

$$= 2^{n-1}\frac{1}{(E^2-2E\cos\frac{\pi}{3}+1)}\cos(n-1)\frac{\pi}{3}$$

$$= 2^{n-1}\frac{1}{E^2-E\left(e^{\frac{i\pi}{3}}+e^{\frac{-i\pi}{3}}\right)+1}\left(\frac{e^{i(n-1)\frac{\pi}{3}}+e^{-i(n-1)\frac{\pi}{3}}}{2}\right) \quad \because \cos\theta = \frac{(e^{i\theta}+e^{-i\theta})}{2}$$

$$= 2^{n-1}\frac{1}{\left(E-e^{\frac{i\pi}{3}}\right)\left(E-e^{\frac{-i\pi}{3}}\right)}\left(\frac{e^{i(n-1)\frac{\pi}{3}}+e^{-i(n-1)\frac{\pi}{3}}}{2}\right)$$

$$\because [x^2-(a+b)x+ab = (x-a)(x-b)]$$

$$= 2^{n-2}\left[\frac{1}{\left(E-e^{\frac{i\pi}{3}}\right)\left(E-e^{\frac{-i\pi}{3}}\right)}\left(e^{\frac{i\pi}{3}}\right)^{n-1} + \frac{1}{\left(E-e^{\frac{i\pi}{3}}\right)\left(E-e^{\frac{-i\pi}{3}}\right)}\left(e^{-\frac{i\pi}{3}}\right)^{n-1}\right]$$

$$= \frac{2^{n-2}}{\left(E-e^{\frac{i\pi}{3}}\right)}\left[\frac{e^{\frac{i(n-1)\pi}{3}}}{\left(e^{\frac{i\pi}{3}}-e^{\frac{-i\pi}{3}}\right)}\right] + \frac{2^{n-2}}{\left(E-e^{\frac{-i\pi}{3}}\right)}\left[\frac{e^{\frac{-i(n-1)\pi}{3}}}{\left(e^{\frac{-i\pi}{3}}-e^{\frac{i\pi}{3}}\right)}\right] \quad \because \frac{1}{f(E)}a^n = \frac{a^n}{f(a)}$$

$$= \frac{2^{n-2}}{\left(E-e^{\frac{i\pi}{3}}\right)}\left[\frac{e^{\frac{i(n-1)\pi}{3}}}{2i\sin\frac{\pi}{3}}\right] + \frac{2^{n-2}}{\left(E-e^{\frac{-i\pi}{3}}\right)}\left[\frac{e^{\frac{-i(n-1)\pi}{3}}}{-2i\sin\frac{\pi}{3}}\right]$$

$$= \frac{2^{n-2}}{i\sqrt{3}\left(E-e^{\frac{i\pi}{3}}\right)}e^{\frac{i(n-1)\pi}{3}} - \frac{2^{n-2}}{i\sqrt{3}\left(E-e^{\frac{-i\pi}{3}}\right)}e^{\frac{-i(n-1)\pi}{3}}$$

$$= \frac{2^{n-2}}{i\sqrt{3}}\left[\frac{1}{\left(E-e^{\frac{i\pi}{3}}\right)}\left(e^{\frac{i\pi}{3}}\right)^{n-1} - \frac{1}{\left(E-e^{\frac{-i\pi}{3}}\right)}\left(e^{\frac{-i\pi}{3}}\right)^{n-1}\right]$$

$$= \frac{2^{n-2}}{i\sqrt{3}}\left[(n-1)\left(e^{\frac{i\pi}{3}}\right)^{n-2} - (n-1)\left(e^{\frac{-i\pi}{3}}\right)^{n-2}\right] \because \frac{1}{E-a}a^n = na^{n-1} \text{ if } f(a) = 0$$

$$= \frac{2^{n-2}}{i\sqrt{3}}\left[(n-1)\left(e^{\frac{i(n-2)\pi}{3}} - e^{\frac{-i(n-2)\pi}{3}}\right)\right]$$

$$= \frac{2^{n-2}}{i\sqrt{3}}\left[(n-1)2i\sin\frac{(n-2)\pi}{3}\right] = \frac{2^{n-1}}{\sqrt{3}}(n-1)\sin\frac{(n-2)\pi}{3}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = 2^n\left(c_1\cos\frac{n\pi}{3} + c_2\sin\frac{n\pi}{3}\right) + \frac{2^{n-1}}{\sqrt{3}}(n-1)\sin\frac{(n-2)\pi}{3}$$

3.4 Solution of Simultaneous Difference Equations with Constant Coefficients

Method of solving simultaneous difference equations is analogous to that of solving differential equations as demonstrated in following examples:

Example 27 Solve the simultaneous difference equations

$$u_{n+1} - 7u_n - 10v_n = 0 \,, \; v_{n+1} - u_n - 4v_n = 0$$

Given that $u_0 = 3, \quad v_0 = 2$

Solution: Given equations can be written as:

$(E-7)u_n - 10v_n = 0$ ... ①

$(E-4)v_n - u_n = 0$ ... ②

Multiplying ① by $(E-4)$, ② by 10 and adding

$(E-4)(E-7)u_n - 10u_n = 0$

$\Rightarrow (E^2 - 11E + 18)u_n = 0$ ... ③

For solving ③, Auxiliary equation is given by:

$E^2 - 11E + 18 = 0$

$\Rightarrow (E-2)(E-9) = 0$

$\Rightarrow E = 2, 9$

C.F. = $c_1 2^n + c_2 9^n$

$\Rightarrow u_n = c_1 2^n + c_2 9^n$ ... ④

Now given that $u_0 = 3$ i.e. $u = 3$ at $n = 0$ ... ⑤

Using ⑤in④, we get $c_1 + c_2 = 3$ ...⑥

Now using ④in ①, we get

$(E-7)(c_1 2^n + c_2 9^n) - 10v_n = 0$

$\Rightarrow E(c_1 2^n + c_2 9^n) - 7(c_1 2^n + c_2 9^n) - 10v_n = 0$

$\Rightarrow (c_1 2^{n+1} + c_2 9^{n+1}) - 7(c_1 2^n + c_2 9^n) - 10v_n = 0$

$\Rightarrow (2c_1 2^n + 9c_2 9^n) - 7(c_1 2^n + c_2 9^n) - 10v_n = 0$

$\Rightarrow v_n = \frac{1}{10}(-5c_1 2^n + 2c_2 9^n)$ ...⑦

Also given that $v_0 = 2$ i.e. $v = 2$ at $n = 0$ ... ⑧

Using ⑧in⑦, we get $-5c_1 + 2c_2 = 20$ ...⑨

Solving ⑥and ⑨, we get $c_1 = -2$, $c_2 = 5$ ... ⑩

Using ⑩ in ④and ⑦, we get

$u_n = -2.2^n + 5.9^n$, $v_n = 2^n + 9^n$

Example 28 Solve the simultaneous difference equations

$$x_{n+1} - 3x_n - 2y_n = -n \,,\; y_{n+1} - x_n - 2y_n = n$$

Given that $x_0 = 1$, $y_0 = 0$

Solution: Given equations can be written as:

$(E-3)x_n - 2y_n = -n$ ... ①

$(E-2)y_n - x_n = n$ ... ②

Multiplying ② by $(E-3)$ and adding to ①, we get

$(E-2)(E-3)y_n - 2y_n = -n + (E-3)n$

$\Rightarrow (E^2 - 5E + 4)y_n = -3n + 1$ $\quad \because En = n + 1$ ... ③

For solving ③, Auxiliary equation is given by:

$E^2 - 5E + 4 = 0$

$\Rightarrow (E-1)(E-4) = 0$

$\Rightarrow E = 1, 4$

C.F. = $c_1 + c_2 4^n$

P.I. $= \frac{1}{E^2-5E+4}(-3n+1)$

$$= \frac{1}{E^2-5E+4}(-3[n]+1)$$

$$= \frac{1}{(1+\Delta)^2 - 5\,(1+\Delta)+4}(-3[n]+1)$$

$$= \frac{1}{-3\Delta+\Delta^2}(-3[n]+1)$$

$$= \frac{1}{-3\Delta\left(1-\frac{1}{3}\Delta\right)}(-3[n]+1)$$

$$= \frac{-1}{3\Delta}\left[\left(1-\frac{1}{3}\Delta\right)\right]^{-1}(-3[n]+1)$$

$$= \frac{-1}{3\Delta}\left[1 + \frac{1}{3}\Delta + \cdots\right](-3[n]+1)$$

$$\because (1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$$

$$= \frac{-1}{3\Delta}\left[-3[n] + 1 + \frac{1}{3}\Delta(-3[n]+1)\right]$$

$$= \frac{-1}{3\Delta}\left[-3[n] + 1 + \frac{1}{3}(-3)\right]$$

$$= \frac{1}{\Delta}[n] = \frac{[n]^2}{2} = \frac{n(n-1)}{2}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c_1 + c_2 4^n + \frac{n(n-1)}{2} \quad \text{... ④}$$

Now given that $y_0 = 0$ i.e. $y = 0$ at $n = 0$ ... ⑤

Using ⑤in④, we get $c_1 + c_2 = 0$ ...⑥

Now using ④in ②, we get

$$(E-2)\left(c_1 + c_2 4^n + \frac{n(n-1)}{2}\right) - x_n = n$$

$$\Rightarrow E\left(c_1 + c_2 4^n + \frac{n(n-1)}{2}\right) - 2\left(c_1 + c_2 4^n + \frac{n(n-1)}{2}\right) - x_n = n$$

$$\Rightarrow \left(c_1 + c_2 4^{n+1} + \frac{(n+1)n}{2}\right) - 2c_1 - 2c_2 4^n - n(n-1) - x_n = n$$

$$\Rightarrow (1-2)c_1 + (4-2)c_2 4^n + \frac{(n+1)n}{2} - n(n-1) - x_n = n$$

$$\Rightarrow x_n = -c_1 + 2c_2 4^n - \frac{n(n-1)}{2} \quad \text{...⑦}$$

Also given that $x_0 = 1$ i.e. $x = 1$ at $n = 0$ ... ⑧

Using ⑧in⑦, we get $-c_1 + 2c_2 = 1$ ...⑨

Solving ⑥and ⑨, we get $c_1 = -\frac{1}{3}, c_2 = \frac{1}{3}$ ... ⑩

Using ⑩ in ④and ⑦, we get

$$x_n = \frac{1}{3} + \frac{2}{3}4^n - \frac{n(n-1)}{2}, \; y_n = -\frac{1}{3} + \frac{1}{3}4^n + \frac{n(n-1)}{2}$$