

Chapter 3

DIFFERENCE EQUATIONS

3.1 Introduction

Differential equations are applicable for continuous systems and cannot be used for discrete variables. Difference equations are the discrete equivalent of differential equations and arise whenever an independent variable can have only discrete values. They are of growing importance in engineering in view of their applications in discrete time- systems used in association with microprocessors.

Some Useful Results

➤ ‘ Δ ’ is a Forward difference operator such that $\Delta f(x) = f(x+h) - f(x)$

$$\therefore \Delta y_x = y_{x+1} - y_x \quad \text{Taking } h \text{ as one unit}$$

$$\Delta y_0 = y_1 - y_0$$

$$\Delta y_1 = y_2 - y_1$$

⋮

$$\Delta y_n = y_{n+1} - y_n$$

$$\text{Also } \Delta^2 y_0 = \Delta y_1 - \Delta y_0 = (y_2 - y_1) - (y_1 - y_0) = y_2 - 2y_1 + y_0$$

⋮

$$\Delta^n y_0 = y_n - {}^n C_1 y_{n-1} + {}^n C_2 y_{n-2} - \dots + (-1)^{n-1} {}^n C_{n-1} y_1 + (-1)^n y_0$$

$$\text{Generalizing } \Delta^n y_r = y_{n+r} - {}^n C_1 y_{n+r-1} + {}^n C_2 y_{n+r-2} - \dots + (-1)^r y_r$$

Properties of operator ‘ Δ ’

$$\Delta C = 0, C \text{ being a constant}$$

$$\Delta C f(x) = C f(x)$$

$$\Delta [af(x) \pm bg(x)] = a\Delta f(x) \pm b\Delta g(x)$$

Example 1 Evaluate the following:

$$\text{i. } \Delta e^x \quad \text{ii. } \Delta^2 e^x \quad \text{iii. } \Delta \tan^{-1} x \quad \text{iv. } \Delta \left(\frac{x+1}{x^2-3x+2} \right)$$

Solution: i. $\Delta e^x = e^{x+h} - e^x = e^x(e^h - 1)$

$$\Delta e^x = e^x(e - 1), \text{ if } h = 1$$

$$\begin{aligned} \text{ii. } \Delta^2 e^x &= \Delta(\Delta e^x) \\ &= \Delta[e^x(e^h - 1)] \\ &= (e^h - 1) \Delta e^x \\ &= (e^h - 1) [e^{x+h} - e^x] \\ &= (e^h - 1) e^x(e^h - 1) \end{aligned}$$

$$= e^x (e^h - 1)^2$$

$$\text{iii. } \Delta \tan^{-1} x = \tan^{-1}(x+h) - \tan^{-1} x$$

$$= \tan^{-1} \left(\frac{x+h-x}{1+(x+h)x} \right)$$

$$= \tan^{-1} \frac{h}{1+(x+h)x}$$

$$\text{iv. } \Delta \left(\frac{x+1}{x^2-3x+2} \right) = \Delta \left(\frac{x+1}{(x-1)(x-2)} \right)$$

$$= \Delta \left(\frac{-2}{x-1} + \frac{3}{x-2} \right) = \Delta \left(\frac{-2}{x-1} \right) + \Delta \left(\frac{3}{x-2} \right)$$

$$= -2 \left(\frac{1}{x+1-1} - \frac{1}{x-1} \right) + 3 \left(\frac{1}{x+1-2} - \frac{1}{x-2} \right)$$

$$= -2 \left(\frac{1}{x} - \frac{1}{x-1} \right) + 3 \left(\frac{1}{x-1} - \frac{1}{x-2} \right)$$

$$= -\frac{(x+4)}{x(x-1)(x-2)}$$

➤ The shift operator 'E' is defined as $Ef(x) = f(x+h)$

$$\therefore Ey_x = y_{x+h}$$

Clearly effect of the shift operator E is to shift the function value to the next higher value $f(x+h)$ or y_{x+h}

$$\text{Also } E^2 f(x) = E(Ef(x)) = Ef(x+h) = f(x+2h)$$

$$\therefore E^n f(x) = f(x+nh)$$

Moreover $E^{-1}f(x) = f(x-h)$, where E^{-1} is the inverse operator.

➤ Relation between Δ and E is given by $E \equiv 1 + \Delta$

Proof: we know that $\Delta y_n = y_{n+1} - y_n$

$$= Ey_n - y_n$$

$$\Rightarrow \Delta y_n = (E - 1) y_n$$

$$\Rightarrow \Delta \equiv E - 1 \text{ or } E \equiv 1 + \Delta$$

➤ **Factorial Notation of a Polynomial**

A product of the form $x(x-1)(x-2) \dots (x-r+1)$ is called a factorial and is denoted by $[x]^r$

$$\therefore [x] = x$$

$$[x]^2 = x(x-1)$$

$$[x]^3 = x(x-1)(x-2)$$

⋮

$$[x]^n = x(x-1)(x-2) \dots (x-n+1)$$

In case, the interval of differencing is h , then

$$[x]^n = x(x-h)(x-h) \dots \left(\overline{x-n-1} \ h \right)$$

The results of differencing $[x]^r$ are analogous to that differentiating x^r

$$\begin{aligned} \therefore \Delta[x]^n &= n[x]^{n-1} \\ \Delta^2[x]^n &= n(n-1)[x]^{n-2} \\ \Delta^3[x]^n &= n(n-1)(n-2)[x]^{n-3} \\ &\vdots \\ \Delta^n[x]^n &= n(n-1)(n-2) \dots 3.2.1 = n! \\ \Delta^{n+1}[x]^n &= 0 \end{aligned}$$

Also $\frac{1}{\Delta}[x] = \frac{[x]^2}{2}$, $\frac{1}{\Delta}[x]^2 = \frac{[x]^3}{3}$ and so on

Remark:

- i. Every polynomial of degree n can be expressed as a factorial polynomial of the same degree and vice-versa.
- ii. The coefficient of highest power of x and also the constant term remains unchanged while transforming a polynomial to factorial notation.

Example2 Express the polynomial $2x^2 - 3x + 1$ in factorial notation.

Solution: $2x^2 - 3x + 1 = 2x^2 - 2x - x + 1$
 $= 2x(x-1) - x + 1$
 $= [x]^2 - [x] + 1$

Example3 Express the polynomial $3x^3 - x + 2$ in factorial notation.

Solution: $3x^3 - x + 2 = 3[x]^3 + A[x]^2 + B[x] + 2$ Using remarks i. and ii.
 $= 3x(x-1)(x-2) + Ax(x-1) + Bx + 2$
 $= 3x^3 + (A-9)x^2 + (-A+B+6)x + 2$

Comparing the coefficients on both sides

$$\begin{aligned} A - 9 &= 0, & -A + B + 6 &= -1 \\ \Rightarrow A &= 9, & B &= 2 \end{aligned}$$

$$\therefore 3x^3 - x + 2 = 3[x]^3 + 9[x]^2 + 2[x] + 2$$

or

We can also find factorial polynomial using synthetic division as shown below

$$\text{Let } 3x^3 - x + 2 = 3[x]^3 + A[x]^2 + B[x] + 2$$

Now coefficients A and B can be found as remainders under x^2 and x columns

1	x^3	x^2	x	2
	3	0	-1	
	-	3	3	
2	3	3	2 = B	
	-	6		
	3	9 = A		

3.2 Order of Difference Equation

Order of a difference equation is the difference between the largest and the smallest argument occurring in the difference equation.

Order of the difference equation $x_{n+1} - x_n = 2$ is one $\because n + 1 - n = 1$

Similarly order of the difference equation $x_{n+2} - x_{n+1} + 2x_n = n$ is 2 $\because n + 2 - n = 2$

The equation $x_{n+1} \cdot x_{n-1} = 2x_{n-2}$ is of order 3 as $(n + 1) - (n - 2) = 3$

Also the equation $y_{n+2} + y_n = 1$ is having order 2.

In general, the higher the order of an equation, the more difficult it is to solve.

3.2 Formation of Difference Equations

A difference equation is formed by eliminating the arbitrary constants from a given relation. The order of the difference equation is equal to the number of arbitrary constants in the given relation. Following examples illustrate the formation of difference equations.

Example 4 Write the given difference equation in the subscript notation

$$\Delta^3 y_x + \Delta^2 y_x + \Delta y_x + y_x = 0$$

Solution: Using the relation $\Delta^n y_r = y_{n+r} - {}^n C_1 y_{n+r-1} + {}^n C_2 y_{n+r-2} - \dots + (-1)^r y_r$
 $\Rightarrow (y_{x+3} - 3y_{x+2} + 3y_{x+1} - y_x) + (y_{x+2} - 2y_{x+1} + y_x) + (y_{x+1} - y_x) - y_x = 0$
 $\Rightarrow y_{x+3} - 2y_{x+2} + 2y_{x+1} = 0$

Example 5 Find a difference equation satisfied by the relation $y = A2^n + n3^{n-1}$

Solution: Given that $y_n = A2^n + n3^{n-1} \dots$ ①

Since there is only one arbitrary constant A , we need only first difference

$$\Rightarrow y_{n+1} = A2^{n+1} + (n+1)3^n$$

$$\Rightarrow y_{n+1} = 2A2^n + 3(n+1)3^{n-1} \dots$$
 ②

Subtracting 2 times ① from ②, we get the required difference equation

$$y_{n+1} - 2y_n = (n+3)3^{n-1}$$

We can also form the difference equation by the method given below:

Given that $y_n = A2^n + n3^{n-1} \dots$ ①

Since there is only one arbitrary constant A , taking the first difference

$$\therefore \Delta y_n = A\Delta 2^n + \Delta n3^{n-1}$$

$$\Rightarrow y_{n+1} - y_n = A(2^{n+1} - 2^n) + (n+1)3^n - n3^{n-1}$$

$$\Rightarrow y_{n+1} - y_n = A2^n(2 - 1) + 3(n+1)3^{n-1} - n3^{n-1}$$

$$\Rightarrow y_{n+1} - y_n = A2^n + (2n+3)3^{n-1} \dots$$
 ②

Subtracting ① from ②, we get the required difference equation

$$\Rightarrow y_{n+1} - 2y_n = (n+3)3^{n-1}$$

Example 6 Find a difference equation satisfied by the relation $y = ax^2 - bx$

Solution: Since there are 2 arbitrary constants a and b , taking 1st and 2nd differences

$$\begin{aligned}y_x &= ax^2 - bx \\ \Rightarrow y_{x+1} &= a(x+1)^2 - b(x+1) \\ \text{and } y_{x+2} &= a(x+2)^2 - b(x+2)\end{aligned}$$

Eliminating arbitrary constants a and b from the given set of equations

$$\Rightarrow \begin{vmatrix} y_x & x^2 & x \\ y_{x+1} & (x+1)^2 & (x+1) \\ y_{x+2} & (x+2)^2 & (x+2) \end{vmatrix} = 0$$

$$\begin{aligned}\Rightarrow y_x[(x+1)^2(x+2) - (x+2)^2(x+1)] - y_{x+1}[x^2(x+2) - (x+2)^2x] \\ + y_{x+2}[x^2(x+1) - (x+1)^2x] &= 0 \\ \Rightarrow y_x[(x+1)(x+2)(x+1-x-2)] - y_{x+1}[x(x+2)(x-x-2)] \\ + y_{x+2}[x(x+1)(x-x-1)] &= 0\end{aligned}$$

\therefore The required difference equation is given by:

$$y_{x+2}(x^2 + x) - 2y_{x+1}(x^2 + 2x) + y_x(x^2 + 3x + 2) = 0$$

We can also form the difference equation by the method given below:

$$\text{Given that } y = ax^2 - bx \dots \textcircled{1}$$

Taking the first difference

$$\begin{aligned}\Delta y &= a\Delta x^2 - b\Delta x \\ &= a[(x+1)^2 - x^2] - b[x+1-x] \\ \Rightarrow \Delta y &= a(2x+1) - b \dots \textcircled{2}\end{aligned}$$

Again taking the second difference

$$\begin{aligned}\Delta^2 y &= 2a\Delta x + 0 \\ &= 2a(x+1-x) = 2a \\ \Rightarrow a &= \frac{1}{2}\Delta^2 y \dots \textcircled{3}\end{aligned}$$

$$\text{Also from } \textcircled{2}, b = a(2x+1) - \Delta y = \frac{1}{2}\Delta^2 y(2x+1) - \Delta y \text{ using } \textcircled{3}$$

$$\Rightarrow b = \frac{1}{2}\Delta^2 y(2x+1) - \Delta y \dots \textcircled{4}$$

Using $\textcircled{3}$ and $\textcircled{4}$ in $\textcircled{1}$, we get

$$\begin{aligned}y &= \frac{1}{2}\Delta^2 y(x^2) - \left(\frac{1}{2}\Delta^2 y(2x+1) - \Delta y\right)x \\ \Rightarrow 2y &= \Delta^2 y(x^2 - 2x^2 - x) + 2x\Delta y \\ \Rightarrow (x^2 + x)\Delta^2 y - 2x\Delta y + 2y &= 0\end{aligned}$$

Writing in subscript notation, we get

$$\begin{aligned}(x^2 + x)(y_{x+2} - 2y_{x+1} + y_x) - 2x(y_{x+1} - y_x) + 2y_x &= 0 \\ \Rightarrow y_{x+2}(x^2 + x) - 2y_{x+1}(x^2 + 2x) + y_x(x^2 + 3x + 2) &= 0\end{aligned}$$

Example 7 Find a difference equation satisfied by the relation $y = \frac{a}{x} + b$

Solution: Since there are 2 arbitrary constants a and b , taking 1st and 2nd differences

$$y_x = \frac{a}{x} + b$$

$$\Rightarrow y_{x+1} = \frac{a}{x+1} + b$$

and $y_{x+2} = \frac{a}{x+2} + b$

Eliminating arbitrary constants a and b from the given set of equations

$$\Rightarrow \begin{vmatrix} y_x & \frac{1}{x} & 1 \\ y_{x+1} & \frac{1}{x+1} & 1 \\ y_{x+2} & \frac{1}{x+2} & 1 \end{vmatrix} = 0$$

$$\Rightarrow y_x \left[\frac{1}{x+1} - \frac{1}{x+2} \right] - y_{x+1} \left[\frac{1}{x} - \frac{1}{x+2} \right] + y_{x+2} \left[\frac{1}{x} - \frac{1}{x+1} \right] = 0$$

$$\Rightarrow y_x \left[\frac{x+2-x-1}{(x+1)(x+2)} \right] - y_{x+1} \left[\frac{x+2-x}{x(x+2)} \right] + y_{x+2} \left[\frac{x+1-x}{x(x+1)} \right] = 0$$

\therefore The required difference equation is given by:

$$(x+2)y_{x+2} - 2(x+1)y_{x+1} - xy_x = 0$$

Example 8 Find a difference equation satisfied by the relation $y = A2^n + B3^n + \frac{1}{2}$

Solution: Since there are 2 arbitrary constants A and B , taking 1st and 2nd differences

$$y_n = A2^n + B3^n + \frac{1}{2}$$

$$\Rightarrow y_{n+1} = A2^{n+1} + B3^{n+1} + \frac{1}{2}$$

and $y_{n+2} = A2^{n+2} + B3^{n+2} + \frac{1}{2}$

Rewriting the given equations as

$$y_n - \frac{1}{2} = A2^n + B3^n$$

$$\Rightarrow y_{n+1} - \frac{1}{2} = 2A2^n + 3B3^n$$

and $y_{n+2} - \frac{1}{2} = 4A2^n + 9B3^n$

Eliminating arbitrary constants A and B from the given set of equations

$$\Rightarrow \begin{vmatrix} y_n - \frac{1}{2} & 1 & 1 \\ y_{n+1} - \frac{1}{2} & 2 & 3 \\ y_{n+2} - \frac{1}{2} & 4 & 9 \end{vmatrix} = 0$$

$$\Rightarrow \left(y_n - \frac{1}{2}\right)(18 - 12) - \left(y_{n+1} - \frac{1}{2}\right)(9 - 4) + \left(y_{n+2} - \frac{1}{2}\right)(3 - 2) = 0$$

\therefore The required difference equation is given by:

$$y_{n+2} - 5y_{n+1} + 6y_n = 1$$

Example 9 Find a difference equation satisfied by the relation

$$y = A \cos n\theta + B \sin n\theta$$

Solution: Since there are 2 arbitrary constants A and B , taking 1st and 2nd differences

$$y_n = A \cos n\theta + B \sin n\theta$$

$$\Rightarrow y_{n+1} = A \cos(n+1)\theta + B \sin(n+1)\theta$$

$$\text{and } y_{n+2} = A \cos(n+2)\theta + B \sin(n+2)\theta$$

Eliminating arbitrary constants A and B from the given set of equations

$$\Rightarrow \begin{vmatrix} y_n & \cos n\theta & \sin n\theta \\ y_{n+1} & \cos(n+1)\theta & \sin(n+1)\theta \\ y_{n+2} & \cos(n+2)\theta & \sin(n+2)\theta \end{vmatrix} = 0$$

$$\Rightarrow y_n [\sin(n+2)\theta \cdot \cos(n+1)\theta - \cos(n+2)\theta \cdot \sin(n+1)\theta] -$$

$$y_{n+1} [\sin(n+2)\theta \cdot \cos n\theta - \cos(n+2)\theta \cdot \sin n\theta]$$

$$+ y_{n+2} [\sin(n+1)\theta \cdot \cos n\theta - \cos(n+1)\theta \cdot \sin n\theta] = 0$$

$$\Rightarrow y_n \sin(n+2-n-1)\theta - y_{n+1} \sin(n+2-n)\theta + y_{n+2} \sin(n+1-n)\theta = 0$$

\therefore The required difference equation is given by:

$$y_{n+2} \sin \theta - y_{n+1} \sin 2\theta + y_n \sin \theta = 0$$

$$\Rightarrow y_{n+2} - 2 \cos \theta y_{n+1} + y_n = 0$$

Example 10 Form the difference equation from the given relation

$$\frac{\log(1+z)}{(1-z)} = y_0 + y_1 z + y_2 z^2 + \dots + y_n z^n + \dots$$

Solution: Rewriting given relation as

$$\log(1+z) = (1-z)(y_0 + y_1 z + y_2 z^2 + \dots + y_n z^n + \dots)$$

$$\Rightarrow z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots = y_0 + (y_1 - y_0)z + (y_2 - y_1)z^2 + (y_3 - y_2)z^3 + \dots +$$

Equating the powers of z on both sides, we get

$$(y_1 - y_0) = 1$$

$$(y_2 - y_1) = -\frac{1}{2}$$

$$(y_3 - y_2) = \frac{1}{3}$$

$$\begin{array}{c} \vdots \\ (y_{n+1} - y_n) = \frac{(-1)^n}{n+1} \\ \vdots \end{array}$$

\therefore The required difference equation is given by $(y_{n+1} - y_n) = \frac{(-1)^n}{n+1}$

$$\text{Or } \Delta y_n = \frac{(-1)^n}{n+1}$$

3.3 Solution of Difference Equations

A general linear difference equation of order 'r' with constant coefficients is given by:

$$(a_0 E^r + a_1 E^{r-1} + \dots + a_{r-1} E + a_r) y_n = F(n)$$

where $a_0, a_1, a_2, \dots, a_r$ are constants and $F(n)$ is a function of 'n' alone or constant.

Note:

- If the difference equation involves 'Δ' instead of 'E' use $\Delta \equiv E - 1$
- If it involves ' y_{n+r} ' instead of ' y_n ' use $y_{n+r} = E^r y_n$

Hence any linear difference equation may be written in symbolic form as:

$$f(E)y_n = F(n)$$

Complete solution of equation $f(E)y_n = F(n)$ is given by $y = \mathbf{C.F} + \mathbf{P.I}$.

where C.F. denotes complimentary function and P.I. is particular integral.

When $F(n) = \mathbf{0}$, then solution of equation $f(E)y_n = \mathbf{0}$ is given by $y = \mathbf{C.F}$

3.3.1 Rules for Finding Complimentary Function (C.F.)

Consider the difference equation $f(E)y_n = F(n) \dots \textcircled{1}$

$$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \dots + a_{r-1} E + a_r) y_n = F(n)$$

Step 1: Find the Auxiliary Equation (A.E) given by $f(E) = 0$

$$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \dots + a_{r-1} E + a_r) = 0 \dots \textcircled{2}$$

Step 2: Solve the auxiliary equation given by $\textcircled{2}$

- If the n roots of A.E. are real and distinct say m_1, m_2, \dots, m_n
C.F. = $c_1 m_1^n + c_2 m_2^n + \dots + c_n m_n^n$
- If two or more roots are equal i.e. $m_1 = m_2 = \dots = m_k, k \leq n$
C.F. = $(c_1 + c_2 n + c_3 n^2 + \dots + c_k n^{k-1}) m_1^n + \dots + c_n m_n^n$
- If A.E. has a pair of imaginary roots i.e. $m_1 = \alpha + i \beta, m_2 = \alpha - i \beta$

$$\text{C.F.} = r^n (c_1 \cos n\theta + c_2 \sin n\theta) + c_3 m_3^n + \dots + c_n m_n^n$$

$$\text{where } r = \sqrt{\alpha^2 + \beta^2} \text{ and } \theta = \tan^{-1} \frac{\beta}{\alpha}$$

Example 11 Solve the difference equation $(\Delta^2 - 3\Delta + 2)y_n = 0$

Solution: Putting $\Delta \equiv E - 1$

$$\Rightarrow ((E - 1)^2 - 3(E - 1) + 2)y_n = 0$$

$$\Rightarrow (E^2 - 5E + 6)y_n = 0$$

Auxiliary Equation (A.E.) is

$$E^2 - 5E + 6 = 0$$

$$\Rightarrow (E - 2)(E - 3) = 0$$

$$\Rightarrow E = 2, 3$$

$$\therefore C.F. = c_1 2^n + c_2 3^n$$

The solution is $y_n = C.F.$

$$\Rightarrow y_n = c_1 2^n + c_2 3^n$$

Example 12 Solve the difference equation $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

Solution: Given difference equation is $U_{n+3} - 2U_{n+2} - 5U_{n+1} + 6U_n = 0$

$$\Rightarrow (E^3 - 2E^2 - 5E + 6)U_n = 0$$

$$\Rightarrow E^3 - E^2 - E^2 + E - 6E + 6 = 0$$

$$\Rightarrow E^2(E - 1) - E(E - 1) - 6(E - 1) = 0$$

$$\Rightarrow (E - 1)(E^2 - E - 6) = 0$$

$$\Rightarrow (E - 1)(E - 3)(E + 2) = 0$$

$$\Rightarrow E = 1, 3, -2$$

$$\therefore C.F. = c_1 + c_2 3^n + c_3 (-2)^n$$

The solution is $y_n = C.F.$

$$\Rightarrow y_n = c_1 + c_2 3^n + c_3 (-2)^n$$

Example 13 Solve the difference equation: $y_{n+4} - 2y_{n+2} + y_n = 0$

Solution: Given difference equation is $y_{n+4} - 2y_{n+2} + y_n = 0$

$$\Rightarrow (E^4 - 2E^2 + 1)y_n = 0$$

Auxiliary equation is: $E^4 - 2E^2 + 1 = 0$

$$\Rightarrow (E^2 - 1)^2 = 0$$

$$\Rightarrow (E + 1)^2 (E - 1)^2 = 0$$

$$\Rightarrow E = -1, -1, 1, 1$$

$$C.F. = (c_1 + c_2 n)(-1)^n + (c_3 + c_4 n)(1)^n$$

Since $F(n) = 0$, solution is given by $y_n = C.F.$

$$\Rightarrow y_n = (c_1 + c_2 n)(-1)^n + (c_3 + c_4 n)$$

Example 14 Solve the difference equation: $y_{n+3} - 2y_{n+1} + 4y_n = 0$

Solution: Given difference equation is $y_{n+3} - 2y_{n+1} + 4y_n = 0$

$$\Rightarrow (E^3 - 2E + 4)y_n = 0$$

$$\text{Auxiliary equation is: } E^3 - 2E + 4 = 0 \quad \dots \textcircled{1}$$

By hit and trial $(E + 2)$ is a factor of $\textcircled{1}$

$\therefore \textcircled{1}$ May be rewritten as

$$E^3 + 2E^2 - 2E^2 - 4E + 2E + 4 = 0$$

$$\Rightarrow E^2(E + 2) - 2E(E + 2) + 2(E + 2) = 0$$

$$\Rightarrow (E + 2)(E^2 - 2E + 2) = 0$$

$$\Rightarrow E = -2, 1 \pm i$$

$$\text{C.F.} = c_1(-2)^n + r^n(c_2 \cos n\theta + c_3 \sin n\theta)$$

$$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{1 + 1} = \sqrt{2}, \text{ where } 1 \pm i \equiv \alpha \pm i\beta$$

$$\theta = \tan^{-1} \frac{\beta}{\alpha} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$\therefore \text{C.F.} = c_1(-2)^n + r^n(c_2 \cos n\theta + c_3 \sin n\theta)$$

Since $F(n) = 0$, solution is given by $y_n = \text{C.F.}$

$$\Rightarrow y_n = c_1 e^{-2x} + (\sqrt{2})^n \left(c_2 \cos \frac{n\pi}{4} + c_3 \sin \frac{n\pi}{4} \right)$$

3.3.2 Rules for Finding Particular Integral (P.I.)

Consider the difference equation $f(E)y_n = F(n) \dots \textcircled{1}$

$$\Rightarrow (a_0 E^r + a_1 E^{r-1} + \dots + a_{r-1} E + a_r)y_n = F(n)$$

Then P.I. = $\frac{1}{f(E)} F(n)$, Clearly P.I. = 0 if $F(n) = 0$

Case I: When $F(n) = a^n$

Use the rule P.I. = $\frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$, $f(a) \neq 0$

In case of failure, i.e. if $f(a) = 0$

Then $\frac{1}{E-a} a^n = n a^{n-1}$

$$\frac{1}{(E-a)^2} a^n = \frac{n(n-1)!}{2!} a^{n-2}$$

$$\frac{1}{(E-a)^3} a^n = \frac{n(n-1)(n-2)!}{3!} a^{n-3}$$

⋮

Case II: When $F(n) = \sin kn$ or $\cos kn$

Use the exponential values of $\sin kn$ or $\cos kn$

i.e. $\sin kn = \frac{e^{ikn} - e^{-ikn}}{2i}$, $\cos kn = \frac{e^{ikn} + e^{-ikn}}{2}$ and find P.I. as in case I.

Case III: When $F(n) = n^p$, P.I. = $\frac{1}{f(E)} n^p$

1. Replace E by Δ by substituting $E \equiv 1 + \Delta$, so that P.I. = $\frac{1}{f(1+\Delta)} n^p$
2. Take the lowest degree term common from $f(1 + \Delta)$ to get an expression of the form $[1 \pm \phi(\Delta)]$ and take it to the numerator to become $[1 \pm \phi(\Delta)]^{-1}$
3. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to p^{th} degree as $(p+1)^{\text{th}}$ difference of n^p is zero. Following expansions will be useful to expand $[1 \pm \phi(\Delta)]^{-1}$ in ascending powers of Δ
 - $(1 + x)^{-1} = 1 - x + x^2 - x^3 + \dots$
 - $(1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$
4. Express $F(n) = n^p$ into factorial notation and operate on it term by term.

Case IV: When $F(n) = a^n g(n)$, where $g(n)$ is any function of n

Take P.I. = $\frac{1}{f(E)} a^n g(n)$ and use the rule: $\frac{1}{f(E)} a^n g(n) = a^n \left(\frac{1}{f(aE)} g(n) \right)$

and proceed as case III.

Examples on Case I

Example 15 Solve the difference equation: $y_{n+2} - 3y_{n+1} + 2y_n = 5^n$

Solution: Given difference equation is $y_{n+2} - 3y_{n+1} + 2y_n = 5^n$

$$\Rightarrow (E^2 - 3E + 2)y_n = 5^n$$

Auxiliary equation is: $E^2 - 3E + 2 = 0$

$$\Rightarrow (E - 1)(E - 2) = 0$$

$$\Rightarrow E = 1, 2$$

$$\text{C.F.} = c_1 + c_2 2^n$$

$$\text{P.I.} = \frac{1}{f(E)} F(n) = \frac{1}{(E-1)(E-2)} 5^n$$

$$= \frac{5^n}{(5-1)(5-2)}, \text{ by putting } E = 5 \quad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$$

$$= \frac{1}{4 \cdot 3} 5^n = \frac{5^n}{12}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c_1 + c_2 2^n + \frac{5^n}{12}$$

Example 16 Solve the difference equation: $U_{n+2} - 7U_{n+1} + 10U_n = 3e^{2n} - 3^n$

Solution: Given difference equation is $U_{n+2} - 7U_{n+1} + 10U_n = 3e^{2n} - 3^n$

$$\Rightarrow (E^2 - 7E + 10)U_n = 3e^{2n} - 3^n$$

Auxiliary equation is: $E^2 - 7E + 10 = 0$

$$\Rightarrow (E - 2)(E - 5) = 0$$

$$\Rightarrow E = 2, 5$$

$$\text{C.F.} = c_1 2^n + c_2 5^n$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{E^2 - 7E + 10} (3e^{2n} - 3^n) \\ &= \frac{3}{E^2 - 7E + 10} (e^2)^n - \frac{1}{E^2 - 7E + 10} 3^n \\ &= \frac{3(e^2)^n}{(e^2)^2 - 7(e^2) + 10} - \frac{3^n}{3^2 - 7(3) + 10} \quad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)} \\ &= \frac{3e^{2n}}{e^4 - 7e^2 + 10} + \frac{3^n}{2} \end{aligned}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c_1 2^n + c_2 5^n + \frac{3e^{2n}}{e^4 - 7e^2 + 10} + \frac{3^n}{2}$$

Example 17 Solve the difference equation: $y_{n+2} - 2y_{n+1} + y_n - 1 = 0$

Solution: Given difference equation is $y_{n+2} - 2y_{n+1} + y_n = 1$

$$\Rightarrow (E^2 - 2E + 1)y_n = 5^n$$

Auxiliary equation is: $E^2 - 2E + 1 = 0$

$$\Rightarrow (E - 1)^2 = 0$$

$$\Rightarrow E = 1, 1$$

$$\text{C.F.} = (c_1 + c_2 n)1^n = (c_1 + c_2 n)$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{(E-1)^2} 1^n \\ &= \frac{n(n-1)}{2!} 1^{n-2} \quad \because \frac{1}{(E-a)^2} a^n = \frac{n(n-1)!}{2!} a^{n-2} \text{ if } f(a) = 0 \\ &= \frac{n(n-1)}{2} \end{aligned}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = (c_1 + c_2 n) + \frac{n(n-1)}{2}$$

Example 18 Solve the difference equation: $y_{n+2} - 3y_{n+1} - 4y_n + 12 = 0$

Solution: Given difference equation is $y_{n+2} - 3y_{n+1} - 4y_n = -12$

$$\Rightarrow (E^2 - 3E - 4)y_n = -12$$

Auxiliary equation is: $E^2 - 3E - 4 = 0$

$$\Rightarrow (E - 4)(E + 1) = 0$$

$$\Rightarrow E = 4, -1$$

$$\text{C.F.} = c_1 4^n + c_2 (-1)^n$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(E)} F(n) = (-12) \frac{1}{(E-4)(E+1)} 1^n \\ &= (-12) \frac{1}{(1-4)(1+1)}, \text{ by putting } E = 1 \quad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)} \\ &= (-12) \frac{1}{-6} = 2 \end{aligned}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c_1 4^n + c_2 (-1)^n + 2$$

Example 19 Solve the difference equation: $y_{n+2} - 4y_n = 2^n$

Solution: Given difference equation is $y_{n+2} - 4y_n = 2^n$

$$\Rightarrow (E^2 - 4)y_n = 2^n$$

Auxiliary equation is: $E^2 - 4 = 0$

$$\Rightarrow E = \pm 2$$

$$\text{C.F.} = c_1 2^n + c_2 (-2)^n$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{(E+2)(E-2)} 2^n \\
&= \frac{1}{(E-2)} \left(\frac{1}{(E+2)} 2^n \right) \\
&= \frac{1}{(E-2)} \left(\frac{2^n}{(2+2)} \right) & \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)} \\
&= \frac{1}{4(E-2)} 2^n = \frac{n2^{n-1}}{4} & \because \frac{1}{E-a} a^n = na^{n-1} \text{ if } f(a) = 0
\end{aligned}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c_1 2^n + c_2 (-2)^n + \frac{n2^{n-1}}{4}$$

Examples on Case II

Example 20 Solve the difference equation: $y_{n+1} - 2y_n = \cos 2n$

Solution: Given difference equation is $y_{n+1} - 2y_n = \cos 2n$

$$\Rightarrow (E - 2)y_n = \cos 2n$$

Auxiliary equation is: $E - 2 = 0$

$$\Rightarrow E = 2$$

$$\text{C.F.} = c2^n$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{E-2} \cos 2n \\
&= \frac{1}{E-2} \frac{e^{2ni} + e^{-2ni}}{2} \\
&= \frac{1}{2} \frac{1}{E-2} (e^{2i})^n + \frac{1}{2} \frac{1}{E-2} (e^{-2i})^n \\
&= \frac{1}{2} \frac{e^{2ni}}{e^{2i}-2} + \frac{1}{2} \frac{e^{-2ni}}{e^{-2i}-2} & \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)} \\
&= \frac{1}{2} \left(\frac{e^{2ni}}{e^{2i}-2} + \frac{e^{-2ni}}{e^{-2i}-2} \right) \\
&= \frac{1}{2} \left(\frac{e^{2ni}(e^{-2i}-2) + e^{-2ni}(e^{2i}-2)}{(e^{2i}-2)(e^{-2i}-2)} \right) \\
&= \frac{1}{2} \left(\frac{(e^{2(n-1)i} + e^{-2(n-1)i}) - 2(e^{2ni} + e^{-2ni})}{1 - 2(e^{2i} + e^{-2i}) + 4} \right) \\
&= \frac{1}{2} \left(\frac{2 \cos 2(n-1) - 4 \cos 2n}{5 - 4 \cos 2} \right) & \because \cos ax = \frac{e^{iax} + e^{-iax}}{2}
\end{aligned}$$

$$= \frac{\cos 2(n-1) - 2 \cos 2n}{5 - 4 \cos 2}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c2^n + \frac{\cos 2(n-1) - 2 \cos 2n}{5 - 4 \cos 2}$$

Example 21 Solve the difference equation: $y_{n+2} + y_{n+1} + 2y_n = \sin 2n$

Solution: Given difference equation is $y_{n+2} + y_{n+1} + 2y_n = \sin 2n$

$$\Rightarrow (E^2 + E + 2)y_n = \sin 2n$$

Auxiliary equation is: $E^2 + E + 2 = 0$

$$\Rightarrow (E - 1)(E + 2) = 0$$

$$\Rightarrow E = 1, -2$$

$$\text{C.F.} = c_1 + c_2(-2)^n$$

$$\text{P.I.} = \frac{1}{f(E)} F(n) = \frac{1}{E^2 + E + 2} \sin 2n$$

$$= \frac{1}{E^2 + E + 2} \frac{e^{2ni} - e^{-2ni}}{2i}$$

$$= \frac{1}{2i} \cdot \frac{1}{E^2 + E + 2} e^{2ni} - \frac{1}{2i} \cdot \frac{1}{E^2 + E + 2} e^{-2ni}$$

$$= \frac{1}{2i} \cdot \frac{1}{E^2 + E + 2} (e^{2i})^n - \frac{1}{2i} \cdot \frac{1}{E^2 + E + 2} (e^{-2i})^n$$

$$= \frac{1}{2i} \cdot \frac{e^{2ni}}{e^{4i} + e^{2i} + 2} - \frac{1}{2i} \cdot \frac{e^{-2ni}}{e^{-4i} + e^{-2i} + 2} \quad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$$

$$= \frac{1}{2i} \left(\frac{e^{2ni}}{e^{4i} + e^{2i} + 2} - \frac{e^{-2ni}}{e^{-4i} + e^{-2i} + 2} \right)$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = c_1 + c_2(-2)^n + \frac{1}{2i} \left(\frac{e^{2ni}}{e^{4i} + e^{2i} + 2} - \frac{e^{-2ni}}{e^{-4i} + e^{-2i} + 2} \right)$$

Example 22 Solve the difference equation $y_{n+2} - 2 \cos \alpha y_{n+1} + y_n = \cos \alpha n$

Solution: Given difference equation is $y_{n+2} - 2 \cos \alpha y_{n+1} + y_n = \cos \alpha n$

$$\Rightarrow (E^2 - 2 \cos \alpha E + 1)y_n = \cos \alpha n$$

Auxiliary equation is: $E^2 - 2 \cos \alpha E + 1 = 0 \quad \dots \textcircled{1}$

$$\Rightarrow E = \cos \alpha \pm i \sin \alpha$$

$$\text{C.F.} = r^n(c_1 \cos n\theta + c_2 \sin n\theta)$$

$$r = \sqrt{\cos^2 \alpha + \sin^2 \alpha} = 1, \quad \theta = \tan^{-1} \frac{\sin \alpha}{\cos \alpha} = \tan^{-1} \tan \alpha = \alpha$$

$$\therefore \text{C.F.} = (c_1 \cos n\alpha + c_2 \sin n\alpha)$$

$$\text{P.I.} = \frac{1}{f(E)} F(n) = \frac{1}{E^2 - 2 \cos \alpha E + 1} \cos \alpha n$$

$$= \frac{1}{E^2 - (e^{i\alpha} + e^{-i\alpha})E + 1} \left(\frac{e^{ian} + e^{-ian}}{2} \right)$$

$$= \frac{1}{2} \frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} (e^{ian} + e^{-ian})$$

$$\therefore [x^2 - (a+b)x + ab = (x-a)(x-b)]$$

$$= \frac{1}{2} \left[\frac{1}{(E - e^{i\alpha})(E - e^{-i\alpha})} (e^{i\alpha})^n + \frac{1}{(E - e^{-i\alpha})(E - e^{i\alpha})} (e^{-i\alpha})^n \right]$$

$$= \frac{1}{2(E - e^{i\alpha})} \left[\frac{e^{ian}}{e^{i\alpha} - e^{-i\alpha}} \right] + \frac{1}{2(E - e^{-i\alpha})} \left[\frac{e^{-ian}}{e^{-i\alpha} - e^{i\alpha}} \right] \quad \therefore \frac{1}{f(E)} a^n = \frac{a^n}{f(a)}$$

$$= \left[\frac{1}{2(E - e^{i\alpha})(2i \sin \alpha)} (e^{i\alpha})^n + \frac{1}{2(E - e^{-i\alpha})(-2i \sin \alpha)} (e^{-i\alpha})^n \right]$$

$$= \frac{1}{4i \sin \alpha} \left[\frac{1}{E - e^{i\alpha}} (e^{i\alpha})^n - \frac{1}{E - e^{-i\alpha}} (e^{-i\alpha})^n \right]$$

$$= \frac{1}{4i \sin \alpha} [n(e^{i\alpha})^{n-1} - n(e^{-i\alpha})^{n-1}] \quad \therefore \frac{1}{E-a} a^n = n a^{n-1} \text{ if } f(a) = 0$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = (c_1 \cos n\alpha + c_2 \sin n\alpha) + \frac{1}{4i \sin \alpha} [n(e^{i\alpha})^{n-1} - n(e^{-i\alpha})^{n-1}]$$

Examples on Case III

Example 23 Solve the difference equation: $y_{n+2} + y_{n+1} + y_n = n^2$

Solution: Given difference equation is $y_{n+2} + y_{n+1} + y_n = n^2$

$$\Rightarrow (E^2 + E + 1)y_n = n^2$$

Auxiliary equation is: $E^2 + E + 1 = 0$

$$\Rightarrow E = -\frac{1}{2} \pm i \frac{\sqrt{3}}{2}$$

$$\text{C.F.} = r^n(c_1 \cos n\theta + c_2 \sin n\theta)$$

$$r = \sqrt{\alpha^2 + \beta^2} = \sqrt{\frac{1}{4} + \frac{3}{4}} = 1, \text{ where } -\frac{1}{2} \pm i \frac{\sqrt{3}}{2} \equiv \alpha \pm i\beta$$

$$\theta = \tan^{-1} \frac{\beta}{\alpha} = \tan^{-1}(-\sqrt{3}) = -\frac{\pi}{3}$$

$$\therefore \text{C.F.} = \left(c_1 \cos \frac{n\pi}{3} - c_2 \sin \frac{n\pi}{3} \right)$$

$$\text{P.I.} = \frac{1}{f(E)} F(n) = \frac{1}{E^2 + E + 1} (n^2)$$

$$= \frac{1}{E^2 + E + 1} (n^2 - n + n)$$

$$= \frac{1}{(1+\Delta)^2 + (1+\Delta) + 1} (n^2 - n + n)$$

$$= \frac{1}{3 + 3\Delta + \Delta^2} (n(n-1) + n)$$

$$= \frac{1}{3(1 + \Delta + \frac{1}{3}\Delta^2)} ([n]^2 + [n])$$

$$= \frac{1}{3} \left[1 + \left(\Delta + \frac{1}{3}\Delta^2 \right) \right]^{-1} ([n]^2 + [n])$$

$$= \frac{1}{3} \left[1 - \left(\Delta + \frac{1}{3}\Delta^2 \right) + \left(\Delta + \frac{1}{3}\Delta^2 \right)^2 + \dots \right] ([n]^2 + [n])$$

$$\because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$= \frac{1}{3} \left[1 - \Delta - \frac{1}{3}\Delta^2 + \Delta^2 + \dots \right] ([n]^2 + [n])$$

$$= \frac{1}{3} \left[1 - \Delta + \frac{2}{3}\Delta^2 + \dots \right] ([n]^2 + [n])$$

$$= \frac{1}{3} \left[[n]^2 + [n] - \Delta([n]^2 + [n]) + \frac{2}{3}\Delta^2([n]^2 + [n]) \right]$$

$$= \frac{1}{3} \left[[n]^2 + [n] - (2[n] + 1) + \frac{2}{3}(2 + 0) \right]$$

$$\because \Delta[n]^2 = 2[n], \Delta[n] = 1, \Delta^2[n]^2 = 2!, \Delta^2[n] = 0$$

$$= \frac{1}{3} \left[[n]^2 - [n] + \frac{1}{3} \right]$$

$$= \frac{1}{3} \left[n(n-1) - n + \frac{1}{3} \right]$$

$$= \frac{1}{3} \left[n^2 - 2n + \frac{1}{3} \right]$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y_n = c_1 \cos \frac{n\pi}{3} - c_2 \sin \frac{n\pi}{3} + \frac{1}{3} \left[n^2 - 2n + \frac{1}{3} \right]$$

Example 24 Solve the difference equation: $y_{n+2} - 5y_{n+1} + 6y_n = n^2 + 3n + 2$

Solution: Given difference equation is $y_{n+2} - 5y_{n+1} + 6y_n = n^2 + 3n + 2$

$$\Rightarrow (E^2 - 5E + 6)y_n = n^2 + 3n + 2$$

Auxiliary equation is: $E^2 - 5E + 6 = 0$

$$\Rightarrow (E - 2)(E - 3) = 0$$

$$\Rightarrow E = 2, 3$$

$$\text{C.F.} = c_1 2^n + c_2 3^n$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{E^2 - 5E + 6} (n^2 + 3n + 2) \\ &= \frac{1}{E^2 - 5E + 6} (n^2 + 3n + 2) \\ &= \frac{1}{(1+\Delta)^2 - 5(1+\Delta) + 6} (n^2 - n + 4n + 2) \\ &= \frac{1}{2 - 3\Delta + \Delta^2} (n(n-1) + 4n + 2) \\ &= \frac{1}{2 - 3\Delta + \Delta^2} ([n]^2 + 4[n] + 2) \\ &= \frac{1}{2(1 - \frac{3}{2}\Delta + \frac{1}{2}\Delta^2)} ([n]^2 + 4[n] + 2) \\ &= \frac{1}{2} \left[1 - \left(\frac{3}{2}\Delta - \frac{1}{2}\Delta^2 \right) \right]^{-1} ([n]^2 + 4[n] + 2) \\ &= \frac{1}{2} \left[1 + \left(\frac{3}{2}\Delta - \frac{1}{2}\Delta^2 \right) + \left(\frac{3}{2}\Delta - \frac{1}{2}\Delta^2 \right)^2 + \dots \right] ([n]^2 + 4[n] + 2) \\ &\quad \because (1-x)^{-1} = 1 + x + x^2 + x^3 + \dots \\ &= \frac{1}{2} \left[1 + \frac{3}{2}\Delta - \frac{1}{2}\Delta^2 + \frac{9}{4}\Delta^2 + \dots \right] ([n]^2 + 4[n] + 2) \\ &= \frac{1}{2} \left[1 + \frac{3}{2}\Delta + \frac{7}{4}\Delta^2 + \dots \right] ([n]^2 + 4[n] + 2) \\ &= \frac{1}{2} \left[[n]^2 + 4[n] + 2 + \frac{3}{2}\Delta([n]^2 + 4[n] + 2) + \frac{7}{4}\Delta^2([n]^2 + 4[n] + 2) \right] \\ &= \frac{1}{2} \left[[n]^2 + 4[n] + 2 + \frac{3}{2}(2[n] + 4 + 0) + \frac{7}{4}(2 + 0 + 0) \right] \\ &\quad \because \Delta[n]^2 = 2[n], \Delta[n] = 1, \Delta^2[n]^2 = 2! \\ &= \frac{1}{2} \left[[n]^2 + 4[n] + 2 + 3[n] + 6 + \frac{7}{2} \right] \\ &= \frac{1}{2} \left[[n]^2 + 7[n] + \frac{23}{2} \right] = \frac{1}{2} \left[n(n-1) + 7n + \frac{23}{2} \right] \\ &= \frac{1}{2} \left[n^2 + 6n + \frac{23}{2} \right] \end{aligned}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y_n = c_1 2^n + c_2 3^n + \frac{1}{2} \left[n^2 + 6n + \frac{23}{2} \right]$$

Example 25 Solve the difference equation: $y_{x+2} - 2y_{x+1} + y_x = 3x + 4$, given that

$$y_0 = 0 \text{ and } y_1 = 1$$

Solution: Given difference equation is $y_{x+2} - 2y_{x+1} + y_x = 3x + 4$ with the conditions

$$y = 0 \text{ at } x = 0 \text{ and } y = 1 \text{ at } x = 1$$

$$\Rightarrow (E^2 - 2E + 1)y_x = 3x + 4$$

$$\text{Auxiliary equation is: } E^2 - 2E + 1 = 0$$

$$\Rightarrow (E - 1)^2 = 0$$

$$\Rightarrow E = 1, 1$$

$$\text{C.F.} = c_1 + c_2x$$

$$\text{P.I.} = \frac{1}{f(E)} F(x) = \frac{1}{(E-1)^2} (3x + 4)$$

$$= \frac{1}{\Delta^2} (3[x] + 4)$$

$$= \frac{1}{\Delta} \left(\frac{3[x]^2}{2} + 4[x] \right) \quad \because \frac{1}{\Delta} [x] = \frac{[x]^2}{2}$$

$$= \frac{1}{2} [x]^3 + 2[x]^2 \quad \because \frac{1}{\Delta} [x] = \frac{[x]^2}{2} \text{ And } \frac{1}{\Delta} [x]^2 = \frac{[x]^3}{3}$$

$$= \frac{1}{2} x(x-1)(x-2) + 2x(x-1)$$

$$= \frac{1}{2} x(x-1)(x+2)$$

Complete solution is: $y_x = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_x = c_1 + c_2x + \frac{1}{2}x(x-1)(x+2) \quad \dots \textcircled{1}$$

Now given that $y = 0$ at $x = 0$ and $y = 1$ at $x = 1$

$$\Rightarrow 0 = c_1 + 0 \text{ And } 1 = c_1 + c_2 + 0$$

$$\Rightarrow c_1 = 0 \text{ And } c_2 = 1 \quad \dots \textcircled{2}$$

Using $\textcircled{2}$ in $\textcircled{1}$, we get

$$y_x = x + \frac{1}{2}x(x-1)(x+2)$$

Examples on Case IV

Example 26 Solve the difference equation: $U_{n+1} - 2U_n = 3^n n^2$

Solution: Given difference equation is $U_{n+1} - 2U_n = 3^n n^2$

$$\Rightarrow (E - 2)U_n = 3^n n^2$$

$$\text{Auxiliary equation is: } E - 2 = 0$$

$$\Rightarrow E = 2$$

$$\text{C.F.} = C2^n$$

$$\begin{aligned} \text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{E-2} (3^n n^2) \\ &= 3^n \frac{1}{3E-2} (n^2) \quad \because \frac{1}{f(E)} a^n g(n) = a^n \frac{1}{f(aE)} g(n) \\ &= 3^n \frac{1}{3(1+\Delta)-2} (n^2 - n + n) \\ &= 3^n \frac{1}{1+3\Delta} (n(n-1) + n) \\ &= 3^n [1 + 3\Delta]^{-1} ([n]^2 + [n]) \\ &= 3^n [1 - 3\Delta + 9\Delta^2 - \dots] ([n]^2 + [n]) \\ &\quad \because (1+x)^{-1} = 1 - x + x^2 - x^3 + \dots \\ &= 3^n [[n]^2 + [n] - 3\Delta([n]^2 + [n]) + 9\Delta^2([n]^2 + [n])] \\ &= 3^n [[n]^2 + [n] - 3(2[n] + 1) + 9(2 + 0)] \\ &\quad \because \Delta[n]^2 = 2[n], \Delta[n] = 1, \Delta^2[n]^2 = 2!, \Delta^2[n] = 0 \\ &= 3^n [[n]^2 - 5[n] + 15] \\ &= 3^n [n(n-1) - 5n + 15] \\ &= 3^n [n^2 - 6n + 15] \end{aligned}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y_n = C2^n + 3^n [n^2 - 6n + 15]$$

Example 27 Solve the difference equation

$$y_{n+2} - 2y_{n+1} + 4y_n = 2^n \left(\cos \frac{n\pi}{3} + \sqrt{3} \sin \frac{n\pi}{3} \right)$$

Solution: Given difference equation may be written as:

$$\Rightarrow (E^2 - 2E + 4)y_n = 2^n \left(\cos \frac{n\pi}{3} + \sqrt{3} \sin \frac{n\pi}{3} \right)$$

$$\text{Auxiliary equation is: } E^2 - 2E + 4 = 0 \quad \dots \textcircled{1}$$

$$\Rightarrow E = 1 \pm i\sqrt{3}$$

$$\text{C.F.} = r^n (c_1 \cos n\theta + c_2 \sin n\theta)$$

$$r = \sqrt{1+3} = 2, \quad \theta = \tan^{-1} \sqrt{3} = \frac{\pi}{3}$$

$$\therefore \text{C.F.} = 2^n \left(c_1 \cos \frac{n\pi}{3} + c_2 \sin \frac{n\pi}{3} \right)$$

$$\begin{aligned}
\text{P.I.} &= \frac{1}{f(E)} F(n) = \frac{1}{E^2-2E+4} 2^n \left(\cos \frac{n\pi}{3} + \sqrt{3} \sin \frac{n\pi}{3} \right) \\
&= 2^n \frac{1}{4E^2-4E+4} 2 \left(\frac{1}{2} \cos \frac{n\pi}{3} + \frac{\sqrt{3}}{2} \sin \frac{n\pi}{3} \right) \quad \because \frac{1}{f(E)} a^n g(n) = a^n \frac{1}{f(aE)} g(n) \\
&= 2^{n+1} \frac{1}{4(E^2-E+1)} \left(\cos \frac{\pi}{3} \cos \frac{n\pi}{3} + \sin \frac{\pi}{3} \sin \frac{n\pi}{3} \right) \\
&= \frac{2^{n+1}}{4} \frac{1}{(E^2-E+1)} \cos(n-1) \frac{\pi}{3} \\
&= 2^{n-1} \frac{1}{(E^2-2E \cos \frac{\pi}{3} + 1)} \cos(n-1) \frac{\pi}{3} \\
&= 2^{n-1} \frac{1}{E^2-E \left(e^{\frac{i\pi}{3}} + e^{-\frac{i\pi}{3}} \right) + 1} \left(\frac{e^{i(n-1)\frac{\pi}{3}} + e^{-i(n-1)\frac{\pi}{3}}}{2} \right) \quad \because \cos \theta = \frac{(e^{i\theta} + e^{-i\theta})}{2} \\
&= 2^{n-1} \frac{1}{\left(E - e^{\frac{i\pi}{3}} \right) \left(E - e^{-\frac{i\pi}{3}} \right)} \left(\frac{e^{i(n-1)\frac{\pi}{3}} + e^{-i(n-1)\frac{\pi}{3}}}{2} \right) \\
&\quad \because [x^2 - (a+b)x + ab = (x-a)(x-b)] \\
&= 2^{n-2} \left[\frac{1}{\left(E - e^{\frac{i\pi}{3}} \right) \left(E - e^{-\frac{i\pi}{3}} \right)} \left(e^{\frac{i\pi}{3}} \right)^{n-1} + \frac{1}{\left(E - e^{\frac{i\pi}{3}} \right) \left(E - e^{-\frac{i\pi}{3}} \right)} \left(e^{-\frac{i\pi}{3}} \right)^{n-1} \right] \\
&= \frac{2^{n-2}}{\left(E - e^{\frac{i\pi}{3}} \right)} \left[\frac{e^{\frac{i(n-1)\pi}{3}}}{\left(e^{\frac{i\pi}{3}} - e^{-\frac{i\pi}{3}} \right)} \right] + \frac{2^{n-2}}{\left(E - e^{-\frac{i\pi}{3}} \right)} \left[\frac{e^{-\frac{i(n-1)\pi}{3}}}{\left(e^{-\frac{i\pi}{3}} - e^{\frac{i\pi}{3}} \right)} \right] \quad \because \frac{1}{f(E)} a^n = \frac{a^n}{f(a)} \\
&= \frac{2^{n-2}}{\left(E - e^{\frac{i\pi}{3}} \right)} \left[\frac{e^{\frac{i(n-1)\pi}{3}}}{2i \sin \frac{\pi}{3}} \right] + \frac{2^{n-2}}{\left(E - e^{-\frac{i\pi}{3}} \right)} \left[\frac{e^{-\frac{i(n-1)\pi}{3}}}{-2i \sin \frac{\pi}{3}} \right] \\
&= \frac{2^{n-2}}{i\sqrt{3} \left(E - e^{\frac{i\pi}{3}} \right)} e^{\frac{i(n-1)\pi}{3}} - \frac{2^{n-2}}{i\sqrt{3} \left(E - e^{-\frac{i\pi}{3}} \right)} e^{-\frac{i(n-1)\pi}{3}} \\
&= \frac{2^{n-2}}{i\sqrt{3}} \left[\frac{1}{\left(E - e^{\frac{i\pi}{3}} \right)} \left(e^{\frac{i\pi}{3}} \right)^{n-1} - \frac{1}{\left(E - e^{-\frac{i\pi}{3}} \right)} \left(e^{-\frac{i\pi}{3}} \right)^{n-1} \right] \\
&= \frac{2^{n-2}}{i\sqrt{3}} \left[(n-1) \left(e^{\frac{i\pi}{3}} \right)^{n-2} - (n-1) \left(e^{-\frac{i\pi}{3}} \right)^{n-2} \right] \quad \because \frac{1}{E-a} a^n = na^{n-1} \text{ if } f(a) = 0 \\
&= \frac{2^{n-2}}{i\sqrt{3}} \left[(n-1) \left(e^{\frac{i(n-2)\pi}{3}} - e^{-\frac{i(n-2)\pi}{3}} \right) \right]
\end{aligned}$$

$$= \frac{2^{n-2}}{i\sqrt{3}} \left[(n-1)2i \sin \frac{(n-2)\pi}{3} \right] = \frac{2^{n-1}}{\sqrt{3}} (n-1) \sin \frac{(n-2)\pi}{3}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I}$

$$\Rightarrow y_n = 2^n \left(c_1 \cos \frac{n\pi}{3} + c_2 \sin \frac{n\pi}{3} \right) + \frac{2^{n-1}}{\sqrt{3}} (n-1) \sin \frac{(n-2)\pi}{3}$$

3.4 Solution of Simultaneous Difference Equations with Constant Coefficients

Method of solving simultaneous difference equations is analogous to that of solving differential equations as demonstrated in following examples:

Example 28 Solve the simultaneous difference equations

$$u_{n+1} - 7u_n - 10v_n = 0, v_{n+1} - u_n - 4v_n = 0$$

Given that $u_0 = 3, v_0 = 2$

Solution: Given equations can be written as:

$$(E - 7)u_n - 10v_n = 0 \quad \dots \textcircled{1}$$

$$(E - 4)v_n - u_n = 0 \quad \dots \textcircled{2}$$

Multiplying $\textcircled{1}$ by $(E - 4)$, $\textcircled{2}$ by 10 and adding

$$(E - 4)(E - 7)u_n - 10u_n = 0$$

$$\Rightarrow (E^2 - 11E + 18)u_n = 0 \quad \dots \textcircled{3}$$

For solving $\textcircled{3}$, Auxiliary equation is given by:

$$E^2 - 11E + 18 = 0$$

$$\Rightarrow (E - 2)(E - 9) = 0$$

$$\Rightarrow E = 2, 9$$

$$\text{C.F.} = c_1 2^n + c_2 9^n$$

$$\Rightarrow u_n = c_1 2^n + c_2 9^n \quad \dots \textcircled{4}$$

Now given that $u_0 = 3$ i.e. $u = 3$ at $n = 0$... $\textcircled{5}$

Using $\textcircled{5}$ in $\textcircled{4}$, we get $c_1 + c_2 = 3$... $\textcircled{6}$

Now using $\textcircled{4}$ in $\textcircled{1}$, we get

$$(E - 7)(c_1 2^n + c_2 9^n) - 10v_n = 0$$

$$\Rightarrow E(c_1 2^n + c_2 9^n) - 7(c_1 2^n + c_2 9^n) - 10v_n = 0$$

$$\Rightarrow (c_1 2^{n+1} + c_2 9^{n+1}) - 7(c_1 2^n + c_2 9^n) - 10v_n = 0$$

$$\Rightarrow (2c_1 2^n + 9c_2 9^n) - 7(c_1 2^n + c_2 9^n) - 10v_n = 0$$

$$\Rightarrow v_n = \frac{1}{10} (-5c_1 2^n + 2c_2 9^n) \quad \dots \textcircled{7}$$

Also given that $v_0 = 2$ i.e. $v = 2$ at $n = 0$... $\textcircled{8}$

Using $\textcircled{8}$ in $\textcircled{7}$, we get $-5c_1 + 2c_2 = 20$... $\textcircled{9}$

Solving $\textcircled{6}$ and $\textcircled{9}$, we get $c_1 = -2, c_2 = 5$... $\textcircled{10}$

Using ⑩ in ④ and ⑦, we get

$$u_n = -2 \cdot 2^n + 5 \cdot 9^n, v_n = 2^n + 9^n$$

Example 29 Solve the simultaneous difference equations

$$x_{n+1} - 3x_n - 2y_n = -n, y_{n+1} - x_n - 2y_n = n$$

$$\text{Given that } x_0 = 1, y_0 = 0$$

Solution: Given equations can be written as:

$$(E - 3)x_n - 2y_n = -n \quad \dots \text{①}$$

$$(E - 2)y_n - x_n = n \quad \dots \text{②}$$

Multiplying ② by $(E - 3)$ and adding to ①, we get

$$(E - 2)(E - 3)y_n - 2y_n = -n + (E - 3)n$$

$$\Rightarrow (E^2 - 5E + 4)y_n = -3n + 1 \quad \because En = n + 1 \quad \dots \text{③}$$

For solving ③, Auxiliary equation is given by:

$$E^2 - 5E + 4 = 0$$

$$\Rightarrow (E - 1)(E - 4) = 0$$

$$\Rightarrow E = 1, 4$$

$$\text{C.F.} = c_1 + c_2 4^n$$

$$\text{P.I.} = \frac{1}{E^2 - 5E + 4} (-3n + 1)$$

$$= \frac{1}{E^2 - 5E + 4} (-3[n] + 1)$$

$$= \frac{1}{(1+\Delta)^2 - 5(1+\Delta) + 4} (-3[n] + 1)$$

$$= \frac{1}{-3\Delta + \Delta^2} (-3[n] + 1)$$

$$= \frac{1}{-3\Delta(1 - \frac{1}{3}\Delta)} (-3[n] + 1)$$

$$= \frac{-1}{3\Delta} \left[\left(1 - \frac{1}{3}\Delta \right) \right]^{-1} (-3[n] + 1)$$

$$= \frac{-1}{3\Delta} \left[1 + \frac{1}{3}\Delta + \dots \right] (-3[n] + 1)$$

$$\because (1 - x)^{-1} = 1 + x + x^2 + x^3 + \dots$$

$$= \frac{-1}{3\Delta} \left[-3[n] + 1 + \frac{1}{3}\Delta(-3[n] + 1) \right]$$

$$= \frac{-1}{3\Delta} \left[-3[n] + 1 + \frac{1}{3}(-3) \right]$$

$$= \frac{1}{\Delta} [n] = \frac{[n]^2}{2} = \frac{n(n-1)}{2}$$

Complete solution is: $y_n = \text{C.F.} + \text{P.I.}$

$$\Rightarrow y_n = c_1 + c_2 4^n + \frac{n(n-1)}{2} \quad \dots \textcircled{4}$$

Now given that $y_0 = 0$ i.e. $y = 0$ at $n = 0$... $\textcircled{5}$

Using $\textcircled{5}$ in $\textcircled{4}$, we get $c_1 + c_2 = 0$... $\textcircled{6}$

Now using $\textcircled{4}$ in $\textcircled{2}$, we get

$$\begin{aligned} & (E - 2) \left(c_1 + c_2 4^n + \frac{n(n-1)}{2} \right) - x_n = n \\ \Rightarrow & E \left(c_1 + c_2 4^n + \frac{n(n-1)}{2} \right) - 2 \left(c_1 + c_2 4^n + \frac{n(n-1)}{2} \right) - x_n = n \\ \Rightarrow & \left(c_1 + c_2 4^{n+1} + \frac{(n+1)n}{2} \right) - 2c_1 - 2c_2 4^n - n(n-1) - x_n = n \\ \Rightarrow & (1-2)c_1 + (4-2)c_2 4^n + \frac{(n+1)n}{2} - n(n-1) - x_n = n \\ \Rightarrow & x_n = -c_1 + 2c_2 4^n - \frac{n(n-1)}{2} \quad \dots \textcircled{7} \end{aligned}$$

Also given that $x_0 = 1$ i.e. $x = 1$ at $n = 0$... $\textcircled{8}$

Using $\textcircled{8}$ in $\textcircled{7}$, we get $-c_1 + 2c_2 = 1$... $\textcircled{9}$

Solving $\textcircled{6}$ and $\textcircled{9}$, we get $c_1 = -\frac{1}{3}$, $c_2 = \frac{1}{3}$... $\textcircled{10}$

Using $\textcircled{10}$ in $\textcircled{4}$ and $\textcircled{7}$, we get

$$x_n = \frac{1}{3} + \frac{2}{3} 4^n - \frac{n(n-1)}{2}, \quad y_n = -\frac{1}{3} + \frac{1}{3} 4^n + \frac{n(n-1)}{2}$$

Exercise 3A

1. Form a difference equation satisfied by the relation $y = An + B2^n$
2. Solve the Fibonacci difference equation $y_n = y_{n-1} + y_{n-2}$, $n > 2$
3. Solve the difference equation $y_{n+2} - 3y_{n+1} + 2y_n = 4^n$
4. Solve the difference equation $y_{n+2} - 2y_{n+1} + y_n = 3n + 4$
5. Solve the difference equation $y_{n+2} - 6y_{n+1} + 8y_n = 2^n + 6n$

Answers

1. $y_{n+2}(1-x) + y_{n+2}(3x-2) - y_{n+2}(2x) = 0$

2. $y_n = \frac{5-\sqrt{5}}{10} \left(\frac{1+\sqrt{5}}{2} \right)^n + \frac{5+\sqrt{5}}{10} \left(\frac{1-\sqrt{5}}{2} \right)^n$

Hint: Fibonacci series is given by 0, 1, 1, 2, 3, 5, 8, 13, 21, ...

$$\therefore y_1 = 0, y_2 = 1$$

3. $y_n = c_1 + c_2 2^n + \frac{1}{6} 4^n$

4. $y_n = c_1 + c_2 n + \frac{n}{2} (n-1)(n+2)$

5. $y_n = \left(c_1 - \frac{x}{2} \right) 2^n + c_2 4^n + 2n + \frac{8}{3}$