

REDUCTION FORMULAE

4.1 Reduction formulae for $\sin^n x$ and $\cos^n x$:

$$\text{Let } I_n = \int \sin^n x dx = \int \sin^{n-1} x \sin x dx$$

Integrating by parts by taking $\sin^{n-1} x$ as first function and $\sin x$ as second function.

$$\begin{aligned} I_n &= \sin^{n-1} x (-\cos x) - \int (n-1) \sin^{n-2} x \cdot \cos x (-\cos x) dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x \cos^2 x dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x (1 - \sin^2 x) dx \\ &= -\sin^{n-1} x \cos x + (n-1) \int \sin^{n-2} x dx - (n-1) \int \sin^n x dx \end{aligned}$$

$$\begin{aligned} I_n &= -\sin^{n-1} x \cos x + (n-1) I_{n-2} - (n-1) I_n \\ \Rightarrow I_n (1 + (n-1)) &= -\sin^{n-1} x \cos x + (n-1) I_{n-2} \\ \Rightarrow I_n &= \frac{-\sin^{n-1} x \cos x}{n} + \frac{(n-1)}{n} I_{n-2} \end{aligned}$$

is the required reduction formula for $\int \sin^n x dx$

$$\text{Similarly } \int \cos^n x dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{(n-1)}{n} I_{n-2}$$

Derivation of formula for $\int_0^{\pi/2} \sin^n x dx$

$$\int \sin^n x dx = -\frac{1}{n} (\sin^{n-1} x \cos x) + \left(\frac{n-1}{n}\right) I_{n-2} \text{ (By reduction formula for } \int \sin^n x dx)$$

$$\begin{aligned} \therefore \int_0^{\pi/2} \sin^n x dx &= -\frac{1}{n} \left[\sin^{n-1} x \cos x \right]_0^{\pi/2} + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} \sin^{n-2} x dx \\ &= 0 + \left(\frac{n-1}{n}\right) \int_0^{\pi/2} \sin^{n-2} x dx \end{aligned}$$

$$\therefore I_n = \left(\frac{n-1}{n}\right) I_{n-2} \text{ (where } I_n = \int_0^{\pi/2} \sin^n x dx)$$

Changing n to $n-2, n-4, n-6, \dots$ in successive steps, we get

$$I_{n-2} = \left(\frac{n-3}{n-2}\right) I_{n-4}$$

$$I_{n-4} = \left(\frac{n-5}{n-4}\right) I_{n-6} \text{ and so on.}$$

$$\therefore I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} I_{n-6}$$

Case (i) If n is an even positive integer, then

$$I_n = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \dots \cdot \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \int_0^{\pi/2} 1 dx$$

$$\int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{5}{6} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if } n \text{ is even}$$

Case (ii) If n is an odd positive integer, then

$$\begin{aligned} I_n &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \int_0^{\pi/2} \sin x \, dx \\ &= \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} [-\cos x]_0^{\pi/2} \end{aligned}$$

$$\therefore \int_0^{\pi/2} \sin^n x \, dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1, \text{ if } n \text{ is odd}$$

Example 1 Find $I_n = \int_0^{\pi/2} \cos^n x \, dx$

Solution: $I_n = \int_0^{\pi/2} \cos^n \left(\frac{\pi}{2} - x\right) \, dx$ ($\because \int_0^a f(x) \, dx = \int_0^a f(a-x) \, dx$, if f is continuous function on $[0, a]$)

$$= \int_0^{\pi/2} \sin^n x \, dx = \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdot \frac{n-5}{n-4} \cdots \frac{4}{5} \cdot \frac{2}{3} \cdot 1, \text{ if } n \text{ is odd} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{5}{8} \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, \text{ if } n \text{ is even} \end{cases}$$

Example 2 Evaluate $\int_0^{\pi/2} \sin^4 x \, dx$

Solution: $\int_0^{\pi/2} \sin^4 x \, dx = \frac{(4-1)(4-3)}{4(4-2)} \frac{\pi}{2}$ ($\because n = 4$ is even) $= \frac{3\pi}{16}$

Example 3 Evaluate $\int_0^{\infty} \frac{dx}{(1+x^2)^4}$

Solution: Put $x = \tan \theta \Rightarrow dx = \sec^2 \theta \, d\theta$

When $x \rightarrow 0$, $\theta \rightarrow 0$ and when $x \rightarrow \infty$, $\theta \rightarrow \frac{\pi}{2}$

\therefore Given integral becomes

$$\begin{aligned} \int_0^{\pi/2} \frac{\sec^2 \theta \, d\theta}{(1+\tan^2 \theta)^4} &= \int_0^{\pi/2} \frac{\sec^2 \theta}{(\sec^2 \theta)^4} \, d\theta = \int_0^{\pi/2} \frac{\sec^2 \theta}{\sec^6 \theta} \, d\theta \\ &= \int_0^{\pi/2} \frac{1}{\sec^4 \theta} \, d\theta = \int_0^{\pi/2} \cos^4 \theta \, d\theta \\ &= \frac{(6-1)(6-3)(6-5)}{6(6-2)(6-4)} \frac{\pi}{2} = \frac{15\pi}{32} \end{aligned}$$

Example 4 Obtain the reduction formula for $\int \sin^m x \cos^n x \, dx$

Solution: Let $I_{m,n} = \int \sin^m x \cos^n x \, dx$

$$= \int \sin^m x \cos^{n-1} x \cos x \, dx$$

$$= \int \cos^{n-1} x (\sin^m x \cos x) \, dx$$

$$= \cos^{n-1} x \frac{\sin^{m+1} x}{m+1} - \int (n-1) \cos^{n-2} x (-\sin x) \cdot \frac{\sin^{m+1} x}{m+1} \, dx$$

$$\begin{aligned}
& \text{(Integrating by parts)} \left(\because \int \sin^m x \cos x \, dx = \frac{\sin^{m+1} x}{m+1} \right) \\
&= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^{m+2} x \, dx \\
&= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \sin^2 x \, dx \\
&= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x (1 - \cos^2 x) \, dx \\
&= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \sin^m x \, dx - \frac{(n-1)}{m+1} \int \cos^n x \sin^m x \, dx \\
I_{m,n} &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2} - \frac{(n-1)}{m+1} I_{m,n} \\
\left(1 + \frac{n-1}{m+1}\right) I_{m,n} &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+1} + \frac{n-1}{m+n} I_{m,n-2} \\
\Rightarrow I_{m,n} (m+n) &= \sin^{m+1} x \cos^{n-1} x + (n-1) I_{m,n-2} \\
\Rightarrow I_{m,n} &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx \\
\Rightarrow \int \sin^m x \cos^n x \, dx &= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cos^{n-2} x \, dx
\end{aligned}$$

Example 5 If $U_n = \int_0^{\pi/2} x^n \sin x \, dx$ and $n > 1$ prove that

$$U_n + n(n-1)U_{n-2} = n \left(\frac{\pi}{2}\right)^{n-1}$$

Solution: $U_n = \int_0^{\pi/2} x^n \sin x \, dx$

$$\begin{aligned}
&= x^n \int \sin x \, dx - \int_0^{\pi/2} \left\{ \frac{d}{dx} (x^n) [\int \sin x \, dx] \right\} dx \\
&= [x^n (-\cos x)]_0^{\pi/2} - \int_0^{\pi/2} n x^{n-1} (-\cos x) \, dx \\
&= -\left[\left(\frac{\pi}{2}\right)^n \cos \frac{\pi}{2} - 0\right] + \int_0^{\pi/2} n x^{n-1} \cos x \, dx \\
&= n \int_0^{\pi/2} x^{n-1} \cos x \, dx \\
&= n \left\{ [x^{n-1} \sin x]_0^{\pi/2} - \int_0^{\pi/2} (n-1) x^{n-2} \sin x \, dx \right\} \\
&= n \left[x^{n-1} \sin x \right]_0^{\pi/2} - n(n-1) \int_0^{\pi/2} x^{n-2} \sin x \, dx \\
\Rightarrow U_n &= n \left[\left(\frac{\pi}{2}\right)^{n-1} \sin \frac{\pi}{2} - 0 \right] - n(n-1)U_{n-2} \\
\Rightarrow U_n + n(n-1)U_{n-2} &= n \left(\frac{\pi}{2}\right)^{n-1}
\end{aligned}$$

Example 6 Evaluate $\int_0^{\pi/2} x^4 \sin x \, dx$

Solution: $U_n = \int_0^{\pi/2} x^4 \sin x \, dx$

$$\text{Now } U_n + n(n-1)U_{n-2} - 2 = n \left(\frac{\pi}{2}\right)^{n-1} \dots\dots\dots(1)$$

Putting $n = 4$ in (1), we get

$$U_4 + 4(4-1)U_{4-2} = 4 \left(\frac{\pi}{2}\right)^{4-1}$$

$$\Rightarrow U_4 + 12U_2 = \frac{\pi^3}{2} \dots\dots\dots(2)$$

Putting $n = 2$ in (1), we get

$$U_2 + 2(2-1)U_{2-2} = 2 \left(\frac{\pi}{2}\right)^{2-1}$$

$$U_2 + 2U_0 = \pi \dots\dots\dots(3)$$

$$\begin{aligned} \text{Now } U_0 &= \int_0^{\pi/2} x^0 \sin x \, dx = \int_0^{\pi/2} \sin x \, dx \\ &= [-\cos x]_0^{\pi/2} = -\cos \frac{\pi}{2} + \cos 0 = 1 \end{aligned}$$

Hence equation (3) becomes

$$\begin{aligned} U_2 + 2(1) &= \pi \\ \Rightarrow U_2 &= \pi - 2 \end{aligned}$$

$$\therefore (2) \text{ becomes } U_4 + 12(\pi - 2) = \frac{\pi^3}{2}$$

$$U_4 = \frac{\pi^3}{2} - 12\pi + 24$$

$$\Rightarrow \int_0^{\pi/2} x^4 \sin x \, dx = \frac{\pi^3}{2} - 12\pi + 24$$

Example 7 If $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$ then prove that

$$I_{m,n} = \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \cdot \frac{m-5}{m+n-4} \dots\dots\dots \frac{2}{3+n} \cdot \frac{1}{1+n}$$

where m is an odd positive integer and n is a positive integer, even or odd.

Solution: $\int \sin^m x \cos^n x \, dx = \int \sin^{m-1} x (\sin x \cos^n) \, dx$

$$= \frac{-\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \cos^{n+1} x \sin^{m-2} x \cos x \, dx$$

(Integrating using by parts)

$$= \frac{-\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \cos^{m-2} x \cos^n x (1 - \sin^2 x) \, dx$$

$$\Rightarrow \left(1 + \frac{m-1}{n+1}\right) \int \sin^m x \cos^n x \, dx = -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \, dx$$

$$\Rightarrow \int \sin^m x \cos^n x \, dx = -\frac{\cos^{n+1} x \sin^{m-1} x}{n+1} + \frac{m-1}{n+1} \int \sin^{m-2} x \cos^n x \, dx$$

$$\text{Now } I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \left[\frac{-\cos^{n+1}x \sin^{m-1}x}{m+n} \right]_0^{\pi/2} + \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2}x \cos^n x dx$$

$$\Rightarrow \int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{m-1}{m+n} \int_0^{\pi/2} \sin^{m-2}x \cos^n x dx$$

$$\text{Hence, } I_{m,n} = \frac{m-1}{m+n} I_{m-2,n}$$

Replacing m by m - 2, m - 4,,3, 2, we obtain

$$I_{m-2,n} = \frac{m-3}{m+n-2} I_{m-4,n}$$

$$I_{m-4,n} = \frac{m-5}{m+n-4} I_{m-6,n}$$

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$$I_{3,n} = \frac{2}{3+n} I_{1,n}$$

$$I_{2,n} = \frac{1}{2+n} I_{0,n}$$

From these relations, we obtain

$$I_{m,n} = \begin{cases} \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots \frac{2}{3+n} I_{1,n}, & \text{if } m \text{ is odd} \\ \frac{m-1}{m+n} \frac{m-3}{m+n-2} \frac{m-5}{m+n-4} \dots \frac{1}{2+n} I_{0,n}, & \text{if } m \text{ is even} \end{cases}$$

Now, we have

$$I_{1,n} = \int_0^{\pi/2} \sin x \cos^n x dx = - \left[\frac{\cos^{n+1}x}{n+1} \right]_0^{\pi/2} = \frac{1}{n+1}$$

$$\text{And } I_{0,n} = \int_0^{\pi/2} \cos^n x dx = \begin{cases} \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{2}{3} \cdot 1, & \text{if } n \text{ is odd} \\ \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2} & \text{if } n \text{ is even} \end{cases}$$

$$\therefore I_{m,n} = \begin{cases} \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \frac{2}{3+n} \cdot \frac{1}{1+n} \\ \text{if } m \text{ is odd and } n \text{ may be even or odd} \\ \frac{m-1}{m+n} \cdot \frac{m-3}{m+n-2} \dots \frac{1}{2+n} \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{2}{3} \\ \text{if } m \text{ is even and } n \text{ is odd} \\ \frac{m-1}{m+n} \frac{m-3}{m+n-2} \dots \frac{1}{2+n} \frac{n-1}{n} \frac{n-3}{n-2} \dots \frac{1}{2} \cdot \frac{\pi}{2} \\ \text{if } m \text{ is even \& } n \text{ is even} \end{cases}$$

These formulae can be expressed as a single formula

$$\int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3)\dots\dots\dots(n-1)(n-3)\dots\dots\dots}{(m+n)(m+n-2)(m+n-4)\dots\dots\dots}$$

to be multiplied by $\frac{\pi}{2}$ when m & n both are even integers.

Example 8 Find $\int_0^{\pi/2} \sin^6 x \cos^5 x dx$

Solution: Here m = 6 and n = 5

$$\int_0^{\pi/2} \sin^6 x \cos^5 x dx = \frac{(6-1)(6-3)(6-5)(5-1)(5-3)}{(6+5)(6+5-2)(6+5-4)(6+5-6)(6+5-8)(6+5-10)} = \frac{8}{693}$$

Example 9 Evaluate $\int_0^{\pi} x \sin^7 x \cos^4 x dx$

Solution: Let $I = \int_0^{\pi} x \sin^7 x \cos^4 x dx$

$$= \int_0^{\pi} (\pi-x) \sin^7 (\pi-x) \cos^4 (\pi-x) dx \quad (\because \int_0^a f(x)dx = \int_0^a f(a-x)dx)$$

$$= \int_0^{\pi} (\pi-x) \sin^7 x \cos^4 x dx$$

$$= \pi \int_0^{\pi} \sin^7 x \cos^4 x dx - \int_0^{\pi} x \sin^7 x \cos^4 x dx$$

$$\Rightarrow 2I = \pi \int_0^{\pi} \sin^7 x \cos^4 x dx$$

$$= 2 \int_0^{\pi/2} \sin^7 x \cos^4 x dx \quad \because \int_0^{2a} f(x)dx = \begin{cases} 2 \int_0^a f(x)dx, & \text{if } f(2a-x)=f(x) \\ 0, & \text{if } f(2a-x)=-f(x) \end{cases}$$

$$\Rightarrow I = \pi \int_0^{\pi/2} \sin^7 x \cos^4 x dx$$

$$= \frac{\pi(7-1)(7-3)(7-5)(4-1)(4-3)}{(7+4)(7+4-2)(7+4-4)(7+4-6)(7+4-8)} = \frac{16\pi}{385}$$

$$\left(\text{using } \int_0^{\pi/2} \sin^m x \cos^n x dx = \frac{(m-1)(m-3)\dots\dots(n-1)(n-3)\dots\dots}{(m+n)(m+n-2)(m+n-4)\dots\dots\dots} \right)$$

Example 10 Evaluate $\int_0^4 x^3 \sqrt{4x-x^2} dx$

$$\begin{aligned} \text{Solution: Let } I &= \int_0^4 x^3 \sqrt{4x-x^2} dx = \int_0^4 x^3 \sqrt{x(4-x)} dx \\ &= \int_0^4 x^3 \sqrt{x} \sqrt{(4-x)} dx \\ &= \int_0^4 x^{7/2} (4-x)^{1/2} dx \end{aligned}$$

$$\text{Putting } x = 4 \sin^2 \theta \Rightarrow dx = 8 \sin \theta \cos \theta d\theta$$

$$\text{Hence } I = \int_0^{\pi/2} 4^{7/2} \sin^7 \theta (4-4 \sin^2 \theta)^{1/2} 8 \sin \theta \cos \theta d\theta$$

$$= \int_0^{\pi/2} 4^{7/2} 4^{1/2} 8 \sin^7 \theta (1-\sin^2 \theta)^{1/2} \sin \theta \cos \theta d\theta$$

$$= 8 \cdot 4^4 \int_0^{\pi/2} \sin^8 x \cos^2 dx$$

$$= 8 \cdot 4^4 \frac{(8-1)(8-3)(8-5)(8-7)}{(8+2)(8+2-2)(8+2-4)(8+2-6)(8+2-8)} \frac{\pi}{2} = \frac{8 \cdot 4^4 \cdot 7 \cdot 5 \cdot 3}{10 \cdot 8 \cdot 6 \cdot 4 \cdot 2} \frac{\pi}{2} = 28 \pi$$

Example 11 Evaluate $\int_0^\infty \frac{x^6 - x^3}{(1+x^3)^5} x^2 dx$

Solution: Let $I = \int_0^\infty \frac{(x^6 - x^3)}{(1+x^3)^5} x^2 dx$

Put $x^3 = \tan^2 \theta \Rightarrow 3x^2 dx = 2 \tan \theta \sec^2 \theta d\theta$

$$\begin{aligned} \text{Then } I &= \int_0^{\pi/2} \frac{(\tan^4 \theta - \tan^2 \theta)}{(1+\tan^2 \theta)^5} \frac{2}{3} \tan \theta \sec^2 \theta d\theta \\ &= \frac{2}{3} \int_0^{\pi/2} \frac{\tan^5 \theta}{(\sec^2 \theta)^5} \sec^2 \theta d\theta - \frac{2}{3} \int_0^{\pi/2} \frac{\tan^3 \theta}{(\sec^2 \theta)^5} \sec^2 \theta d\theta \\ &= \frac{2}{3} \int_0^{\pi/2} \frac{\tan^5 \theta}{\sec^8 \theta} d\theta - \frac{2}{3} \int_0^{\pi/2} \frac{\tan^3 \theta}{\sec^8 \theta} d\theta \\ &= \frac{2}{3} \int_0^{\pi/2} \sin^5 \theta \cos^3 \theta d\theta - \frac{2}{3} \int_0^{\pi/2} \sin^3 \theta \cos^5 \theta d\theta \\ &= \frac{2}{3} \int_0^{\pi/2} \sin^5 \theta \cos^3 \theta d\theta - \frac{2}{3} \int_0^{\pi/2} \sin^3 \left(\frac{\pi}{2} - \theta\right) \cos^5 \left(\frac{\pi}{2} - \theta\right) d\theta \\ &= \frac{2}{3} \int_0^{\pi/2} \sin^5 \theta \cos^3 \theta d\theta - \frac{2}{3} \int_0^{\pi/2} \cos^3 \theta \sin^5 \theta d\theta \\ &\quad [\because \int_0^a f(x) dx = \int_0^a f(a-x) dx] \\ &= 0 \end{aligned}$$

Example 12 Evaluate $\int_0^{\pi/2} \sin^5 x dx$

Solution: We know $\int \sin^n x dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x dx$

$$\begin{aligned} \therefore \int_0^{\pi/4} \sin^5 x dx &= \left[\frac{-\sin^{5-1} x \cos x}{5} \right]_0^{\pi/4} + \frac{5-1}{5} \int_0^{\pi/4} \sin^{5-2} x dx \\ &= \frac{-1}{5} [\sin^4 x \cos x]_0^{\pi/4} + \frac{4}{5} \int_0^{\pi/4} \sin^3 x dx \\ &= \frac{-1}{5} \left[\left(\frac{1}{\sqrt{2}}\right)^4 \left(\frac{1}{\sqrt{2}}\right) \right] + \frac{4}{5} \int_0^{\pi/4} \sin^3 x dx \dots\dots\dots(1) \end{aligned}$$

$$\begin{aligned} \text{Now } \int_0^{\pi/4} \sin^3 x dx &= \left[-\frac{\sin^{3-1} x \cos x}{3} \right]_0^{\pi/4} + \frac{3-1}{3} \int_0^{\pi/4} \sin^{3-2} x dx \\ &= -\frac{1}{3} [\sin^2 x \cos x]_0^{\pi/4} + \frac{2}{3} \int_0^{\pi/4} \sin x dx \\ &= -\frac{1}{3} \left[\left(\frac{1}{\sqrt{2}}\right)^2 \frac{1}{\sqrt{2}} \right] + \frac{2}{3} (-\cos x) \Big|_0^{\pi/4} \\ &= \frac{-1}{3 \cdot 2\sqrt{2}} - \frac{2}{3} \left(\frac{1}{\sqrt{2}} - 1 \right) \end{aligned}$$

Putting this value in (1), we get

$$\begin{aligned} \int_0^{\pi/4} \sin^5 x dx &= -\frac{1}{5} \left[\left(\frac{1}{\sqrt{2}}\right)^4 \frac{1}{\sqrt{2}} \right] + \frac{4}{5} \left[\frac{-1}{6\sqrt{2}} - \frac{1}{2} \left(\frac{1}{\sqrt{2}} - 1\right) \right] \\ &= \frac{-1}{5 \cdot 4\sqrt{2}} - \frac{4}{5} \left[\frac{1}{6\sqrt{2}} + \frac{1}{2\sqrt{2}} - \frac{1}{2} \right] \end{aligned}$$

Example 13 Evaluate $\int_0^1 \frac{x^5}{2\sqrt{1-x^2}} dx$

Solution: Put $x = \sin \theta \Rightarrow dx = \cos \theta d\theta$

Then the given integral becomes

$$\begin{aligned} \frac{1}{2} \int_0^{\pi/2} \frac{\sin^5 \theta}{\sqrt{1-\sin^2 \theta}} \cos \theta d\theta &= \frac{1}{2} \int_0^{\pi/2} \sin^4 \theta d\theta \\ &= \frac{1}{2} \frac{(5-1)(5-3)}{5(5-2)(5-4)} = \frac{4}{15} \end{aligned}$$

Example 14 Evaluate $\int_{-\pi/2}^{\pi/2} \cos^3 \theta (1 + \sin \theta)^2 d\theta$

Solution: $\int_{-\pi/2}^{\pi/2} \cos^3 \theta (1 - \sin \theta)^2 d\theta = \int_{-\pi/2}^{\pi/2} \cos^3 \theta (1 + \sin^2 \theta + 2\sin \theta) d\theta$

$$\begin{aligned} &= \int_{-\pi/2}^{\pi/2} \cos^3 \theta d\theta + \int_{-\pi/2}^{\pi/2} \cos^3 \theta \sin^2 \theta + 2 \int_{-\pi/2}^{\pi/2} \cos^3 \theta \sin \theta d\theta \\ &= 2 \int_0^{\pi/2} \cos^3 \theta d\theta + 2 \int_0^{\pi/2} \cos^3 \theta \sin^2 \theta + 0 \\ &= \frac{2(3-1)}{3(3-2)} + 2 \frac{(2-1)(3-1)}{(3+2)(3+2-2)(3+2-4)} \\ &= \frac{4}{3} + \frac{4}{15} = \frac{8}{5} \end{aligned}$$

Exercise 7A

1. Evaluate $\int_0^{2a} x^3 (2ax - x^2)^{3/2} dx$ (Ans. $\frac{9\pi a^7}{16}$)
2. Evaluate $\int_0^{\infty} \frac{x^3}{(1+x^2)^{9/2}} dx$ (Ans. $\frac{2}{35}$)
3. Evaluate $\int_0^{\pi/2} (\cos 2\theta)^{3/2} \cos \theta d\theta$ (Ans. $\frac{3\pi}{16\sqrt{2}}$)
4. Evaluate $\int_0^{\pi/2} \sin^4 x \cos 3x dx$ (Ans. $\frac{-13}{35}$)
5. Evaluate $\int_0^a x^2 \sqrt{ax - x^2} dx$ (Ans. $\frac{5\pi a^4}{128}$)
6. Evaluate $\int_0^{\pi/2} \frac{\cos^2 \theta}{\cos^2 \theta + 4 \sin^2 \theta} d\theta$ (Ans. $\frac{\pi}{6}$)
7. Evaluate $\int_0^{\pi} \frac{\sin^4 \theta \sqrt{1-\cos \theta}}{(1+\cos \theta)^2} d\theta$ (Ans. $\frac{64\sqrt{2}}{15}$)
8. Evaluate $\int_0^1 x^{3/2} (1-x)^{3/2} dx$ (Ans. $\frac{3\pi}{128}$)