## INFINITE SERIES

The sum of an infinite sequence of numbers is known as infinite series. Let $\langle u_n \rangle$ given by $u_1, u_2, u_3, \ldots, u_n, \ldots$ be a sequence, then the sum $\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is known as the infinite series

**Sequence:** A sequence in a set $S$ is a rule which assigns to each natural number a unique element of $S$. A sequence can be expressed as $\langle u_1, u_2, u_3, \ldots, u_n, \ldots \rangle$ or $\langle u_n \rangle$.

**Bounded Sequence:** A sequence $\langle u_n \rangle$ is said to be bounded, if it is bounded both above and below. Hence if a sequence $\langle u_n \rangle$ is bounded, then there exist two real numbers $a$ and $b$, such that $a \leq u_n \leq b \; \forall \, n \in N$ .

- The sequence $\langle u_n \rangle$ defined by $u_n = \frac{1}{n^2}$ is a bounded sequence.

  Here $\langle u_n \rangle = \langle 1, \frac{1}{2^2}, \frac{1}{3^2}, \ldots \rangle$, as $0 < u_n \leq 1 \; \forall \, n \in N$ , $\therefore u_n$ is bounded.
- The sequence $\langle u_n \rangle$ defined by $u_n = 2^{n-1}$ is bounded below.
  Here $\langle u_n \rangle = \langle 1, 2, 2^2, 2^3, \ldots \rangle$, as $u_n \geq 1 \; \forall \, n \in N$ , $\therefore u_n$ is bounded below sequence.
- The sequence $\langle u_n \rangle = \langle \ldots, -4, -3, -2, -1, 0 \rangle$ is bounded above sequence.

**Convergent sequence**: A sequence $\langle u_n \rangle$ converges to a number $l$, if for any small positive number $\varepsilon$, there exists $m \in N$ , such that $|u_n - l| < \varepsilon \, , \forall \, n \geq m$.

Here $l$ is called the limit of the given sequence $\therefore \lim_{n \to \infty} u_n = l$ or $u_n \to l$.

**Divergent Sequence**: A sequence $\langle u_n \rangle$ is said to diverge to $+\infty$, if for each positive number $k$ (however large), there exists a positive integer $m$ such that $u_n > k \;\; \forall \, n \geq m$.

A sequence $\langle u_n \rangle$ is said to diverge to $-\infty$, if for each negative number $p$ (however small), there exists a positive integer $m$ such that

$u_n < p \;\; \forall \, n \leq m$.

A sequence which diverges to either $+\infty$ or $-\infty$ is called a divergent sequence.

**Remark**: Every bounded monotonic sequence converges and every unbounded sequence diverges.

**Infinite Series**

If $\langle u_n \rangle$ is a sequence of real numbers, then expression $u_1 + u_2 + u_3 + \cdots + u_n + \cdots$ is known as the infinite series.

It is denoted by $\sum_{n=1}^{\infty} u_n$ or $\sum u_n$.

**Positive term series:** An infinite series in which all the terms after a certain term are positive, then the series is called a positive term series.

For example, $-4-3-2-1+0+1+\ 2+3+4+\cdots$ is a positive term series.

**Alternating Series:** A series in which all the terms are alternatively positive or negative is called an alternating series.

For example, $1-2+3-4+5-6+\cdots$ is an alternating series.

**Basic Tests for Checking Convergence of Infinite Series**

**I Partial Sum Test for Convergence of Infinite Series**

Consider an infinite series $\sum_{n=1}^{\infty} u_n = u_1+u_2+u_3+\cdots$

Let us define $S_n = u_1+u_2+u_3+\cdots+u_n$ , here $\langle S_n\rangle$ is known as the sequence of partial sums.

An infinite series $\sum u_n$ converges, diverges or oscillates (finitely or infinitely) according as the sequence $\langle S_n\rangle$ of partial sums converges, diverges or oscillates.

Thus $i.$ $\sum u_n$ is convergent if $\lim_{n\to\infty} S_n$ is finite

$ii.$ $\sum u_n$ is divergent if $\lim_{n\to\infty} S_n$ is $+\infty$ or $-\infty$

$iii.$ $\sum u_n$ oscillates finitely if the sequence $\langle S_n\rangle$ oscillates finitely

$iv.$ $\sum u_n$ oscillates infinitely if $\langle S_n\rangle$ oscillates infinitely.

Also, if $\lim_{n\to\infty} S_n = l$ then $l$ is called as the limit of the given series.

**Example 1** Discuss the convergence of the series $\sum \frac{1}{n(n+1)}$

**Solution**: Here $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$

$$u_1 = \frac{1}{1} - \frac{1}{2}$$

$$u_2 = \frac{1}{2} - \frac{1}{3}$$

$$u_3 = \frac{1}{3} - \frac{1}{4}$$

$$\vdots$$

$$u_n = \frac{1}{n} - \frac{1}{n+1}$$

$$\therefore S_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{n-1} - \frac{1}{n} + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$$

$\lim_{n\to\infty} S_n = 1 - 0 = 1$ , $\therefore$ $\langle S_n \rangle$ converges to 1 $\Rightarrow \sum u_n$ converges to 1

**Example 2** Show that the series $\sum \frac{1}{n(n+2)}$ converges to $\frac{3}{4}$.

**Solution**: Here $u_n = \frac{1}{n(n+2)} = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$

$$\therefore\ u_1 = \frac{1}{2}\left[\frac{1}{1} - \frac{1}{3}\right]$$

$$u_2 = \frac{1}{2}\left[\frac{1}{2} - \frac{1}{4}\right]$$

$$u_3 = \frac{1}{2}\left[\frac{1}{3} - \frac{1}{5}\right]$$

$$\vdots$$

$$u_{n-1} = \frac{1}{2}\left[\frac{1}{n-1} - \frac{1}{n+1}\right]$$

$$u_n = \frac{1}{2}\left[\frac{1}{n} - \frac{1}{n+2}\right]$$

$$\therefore S_n = \frac{1}{2}\left[1 + \frac{1}{2} - \frac{1}{n+1} - \frac{1}{n+2}\right]$$

$\lim_{n\to\infty} S_n = \frac{3}{4}$ , $\therefore$ $\langle S_n \rangle$ converges to $\frac{3}{4} \Rightarrow \sum u_n$ converges to $\frac{3}{4}$

**Example 3** Show that the series $1^2 + 2^2 + 3^2 + \cdots + n^2 + \cdots$ diverges

**Solution**: Here $S_n = 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

$\lim_{n\to\infty} S_n = +\infty$ , $\therefore$ $\langle S_n \rangle$ diverges to $+\infty$

$\Rightarrow \sum u_n$ diverges to $+\infty$

**Example 4** Show that the series $-1-2-3-\cdots-n-\cdots$ diverges

**Solution**: Here $S_n = -1-2-3-\cdots-n = -(1+2+3+\cdots+n)$

$= -\frac{n(n+1)}{2}$

$\lim_{n\to\infty} S_n = -\infty$ , $\therefore$ $\langle S_n \rangle$ diverges to$-\infty$

$\Rightarrow \sum u_n$ diverges to$-\infty$

**Example 5** Discuss the convergence of following series:

$i.\ \sum(-1)^{n-1}$ $ii.\ \sum n(-1)^n$ $iii.\ 3-2-1+3-2-1+\cdots$

**Solution:** $i.$ Here $u_n = (-1)^{n-1}$

$\therefore$ $S_n = 1-1+1-1+\cdots$ to $n$ terms

$=$ 1or 0 according as $n$ is odd or even

$\therefore$ $\langle S_{2n-1} \rangle$ converges to 1 and $\langle S_{2n} \rangle$converges to 0

Now since $\langle S_n \rangle$ is bounded and oscillates finitely, $\therefore \sum u_n$ oscillates finitely.

$ii.$ Here $u_n = n(-1)^n$

$\therefore$ $S_n = -1+2-3+4-5+6-\cdots$ to $n$ terms

$\therefore$ $\langle S_{2n-1} \rangle$ diverges to $-\infty$ and $\langle S_{2n} \rangle$ diverges to $+\infty$

Hence $\langle S_n \rangle$ oscillates infinitely, $\therefore \sum u_n$ oscillates infinitely.

$iii.$ Here $S_n = 3-2-1+3-2-1+\cdots$ to $n$ terms

$\therefore$ $\langle S_{3n} \rangle$ converges to 0, $\langle S_{3n+1} \rangle$ converges to 3 , $\langle S_{3n+2} \rangle$ converges to 1

Hence $\langle S_n \rangle$ oscillates finitely, $\therefore \sum u_n$ oscillates finitely.

## II Geometric Series

The Geometric series $\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + ar^3 + \cdots$

1. Converges if $|r| < 1$ i.e. $-1 < r < 1$
2. Diverges if $r \geq 1$
3. Oscillates finitely if $r = -1$
4. Oscillates infinitely if $r < -1$

**Example 6** Discuss the convergence of following series:

$i.\ 1 + \frac{2}{3} + \frac{4}{9} + \frac{8}{27} + \cdots$ $\quad ii.\ 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \cdots$ $\quad iii.\ \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$

$iv.\ 2 - 2 + 2 - 2 + \cdots$ $\quad v.\ 1 - \frac{3}{2} + \frac{9}{4} - \frac{27}{8} + \cdots$

**Solution**: $i$. Comparing with geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

$a = 1$, $r = \frac{2}{3} < 1$, $\therefore$ the given series converges

$ii$. Comparing with geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

$a = 1$, $r = \frac{3}{2} > 1$, $\therefore$ the given series diverges

$iii$. Comparing with geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

$a = \frac{1}{2}$, $r = 1$, $\therefore$ the given series diverges

$iv$. Comparing with geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

$a = 2$, $r = -1$, $\therefore$ the given series oscillates finitely

$v$ Comparing with geometric series $\sum_{n=1}^{\infty} ar^{n-1}$

$a = 1$, $r = -\frac{3}{2} < -1$, $\therefore$ the given series oscillates infinitely

## III p-Series

A series of the form $\sum \frac{1}{n^p}$, $p > 0$ is called p-series.

It converges if $p > 1$ and diverges if $p \leq 1$.

- $\sum \frac{1}{n^3} = \frac{1}{1^3} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots$ converges $\because p = 3 > 1$
- $\sum \frac{1}{n^{5/2}} = \frac{1}{1^{5/2}} + \frac{1}{2^{5/2}} + \frac{1}{3^{5/2}} + \cdots$ converges $\because p = \frac{5}{2} > 1$
- $\sum \frac{1}{n} = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$ diverges $\because p = 1$
- $\sum \frac{1}{n^{1/2}} = \frac{1}{1^{1/2}} + \frac{1}{2^{1/2}} + \frac{1}{3^{1/2}} + \cdots$ diverges $\because p = \frac{1}{2} < 1$

## IV Test for Divergence of an Infinite Series

**Necessary condition for convergence:**

If an infinite series $\sum_{n=1}^{\infty} u_n$ is convergent then $\lim_{n\to\infty} u_n = 0$. However, converse need not be true.

**Proof:** Let an infinite series $\sum_{n=1}^{\infty} u_n$ be convergent

Consider the sequence $\langle S_n \rangle$ of partial sums of the series $\sum u_n$

i.e. $S_n = u_1 + u_2 + u_3 + \cdots\cdots\cdots\cdots + u_{n-1} + u_n$

$= S_{n-1} + u_n$

$\Rightarrow S_n - S_{n-1} = u_n$

$\Rightarrow \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} u_n$

$\Rightarrow \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} u_n$ ...①

Now as $\sum_{n=1}^{\infty} u_n$ is convergent $\therefore$ sequence $\langle S_n \rangle$ of its partial sums is also convergent.

Let $\lim_{n\to\infty} S_n = l$, then $\lim_{n\to\infty} S_{n-1} = l$

Substituting these values in ①, we get $\lim_{n\to\infty} u_n = 0$

Hence proved.

Now to show that converse may not be true, consider the series $\sum_{n=1}^{\infty} \frac{1}{n}$

Here $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{n} = 0$

But $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series by $p$-series test

$\therefore \lim_{n\to\infty} u_n = 0$ does not imply that the series is convergent

**Result:** If $\lim_{n\to\infty} u_n \neq 0$, then the series $\sum_{n=1}^{\infty} u_n$ diverges

**Example 7** Test the convergence of the series $\sum cos\frac{1}{n}$

**Solution**: Here $u_n = cos\frac{1}{n} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} cos\frac{1}{n} = 1 \neq 0$

Hence the given series is divergent.

**Example 8** Test the convergence of the series $\sum \sqrt{\frac{n}{n+1}}$

**Solution**: Here $u_n = \sqrt{\frac{n}{n+1}} \Rightarrow \lim_{n\to\infty} u_n = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} = \lim_{n\to\infty} \sqrt{\frac{1}{1+\frac{1}{n}}} = 1 \neq 0$

Hence the given series is divergent.

**V Tests for the convergence of infinite series by Comparison (Limit Test)**

In this test we compare a given positive term series $\sum u_n$ with a series $\sum v_n$ , whose nature is already known.

- If $\sum v_n$ is convergent and $u_n \leq v_n \; \forall \, n$, then $u_n$ is also convergent
- If $\sum v_n$ is divergent and $u_n \geq v_n \; \forall \, n$, then $u_n$ is also divergent
- If the two positive term series $\sum u_n$ and $\sum v_n$ are such that $\lim_{n\to\infty} \frac{u_n}{v_n}$ is non-zero and finite, then $\sum u_n$ and $\sum v_n$ behave alike, i.e. either both converge or both diverge.

**Example 9** Discuss the convergence of the following series

$i.\ \sum_{n=1}^{\infty}\frac{1}{n^n}$ $\qquad ii.\ \sum_{n=2}^{\infty}\frac{1}{log\, n}$ $\qquad iii. \sum_{n=1}^{\infty}\frac{1}{2^n+x},\ \forall x>0$

**Solution**: $i.$ Here $u_n=\frac{1}{n^n}$

Now $n^n>2^n$ for $n>2, \therefore \frac{1}{n^n}<\frac{1}{2^n}$ for $n>2$

Now $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series with common ratio $\frac{1}{2}<1$ , ∴ it is convergent

Thus, by comparison test $\sum_{n=1}^{\infty}\frac{1}{n^n}$ is also convergent.

$ii.$ Here $u_n=\frac{1}{\log n}$

Now $\log n<n$ for $n\geq 2$ , $\therefore \frac{1}{\log n}>\frac{1}{n}$ for $n\geq 2$

Now $\sum\frac{1}{n}$ is a divergent series (by $p$ series test)

Thus, by comparison test $\sum_{n=2}^{\infty}\frac{1}{\log n}$ is also divergent.

$iii.$ Here $u_n=\frac{1}{2^n+x}$

Now $2^n+x>2^n$ (as $x>0$) , $\therefore \frac{1}{2^n+x}<\frac{1}{2^n}$

Now $\sum_{n=1}^{\infty}\frac{1}{2^n}$ is a geometric series with common ratio $\frac{1}{2}<1$ , ∴ it is convergent

Thus by comparison test $\sum_{n=1}^{\infty}\frac{1}{2^n+x}$ is also convergent.

**Example 10** Discuss the convergence of the series $\sum_{n=1}^{\infty}\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

**Solution:** Here $u_n=\left[\frac{1}{n^2}+\frac{1}{(n+1)^2}\right]$

Clearly $\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (by $p$ series test).

Thus, by comparison test $\sum_{n=1}^{\infty} \left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$ is also convergent.

**Example 11** Discuss the convergence of the following series

$i.\ \frac{1}{1.3} + \frac{1}{3.5} + \frac{1}{5.7} + \cdots \qquad ii.\ \frac{1}{3.7} + \frac{1}{4.9} + \frac{1}{5.11} + \cdots$

**Solution:** $i.$ Here $u_n = \frac{1}{(2n-1)(2n+1)} = \frac{1}{n^2\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)}$

Let $v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2\left(2-\frac{1}{n}\right)\left(2+\frac{1}{n}\right)} = \frac{1}{4}$ which is non zero and finite

$\therefore \sum u_n$ and $\sum v_n$ behave alike, i.e. either both converge or both diverge.

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (by $p$ series test).

Thus by comparison test $\sum u_n$ is also convergent

$ii.$ Here $u_n = \frac{1}{(n+2)(2n+5)} = \frac{1}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)}$

Let $v_n = \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n^2\left(1+\frac{2}{n}\right)\left(2+\frac{5}{n}\right)} = \frac{1}{2}$ which is non zero and finite

$\therefore \sum u_n$ and $\sum v_n$ behave alike, i.e. either both converge or both diverge.

Now $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (by $p$ series test).

Thus by comparison test $\sum u_n = \sum_{n=1}^{\infty} \frac{1}{(n+2)(2n+5)}$ is also convergent

**Example 12** Discuss the convergence of the series $\frac{1.2}{3.4.5} + \frac{2.3}{4.5.6} + \frac{3.4}{5.6.7} + \cdots$

**Solution:** Here $u_n = \frac{n(n+1)}{(n+2)(n+3)(n+4)} = \frac{n^2\left(1+\frac{1}{n}\right)}{n^3\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)} = \frac{\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)}$

Let $v_n = \frac{1}{n}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n\left(1+\frac{1}{n}\right)}{n\left(1+\frac{2}{n}\right)\left(1+\frac{3}{n}\right)\left(1+\frac{4}{n}\right)} = 1$ which is non zero and finite

$\therefore \sum u_n$ and $\sum v_n$ behave alike, i.e. either both converge or both diverge.

Now $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (by $p$ series test).

Thus by comparison test $\sum u_n$ is also divergent.

**Example 13** Test the convergence of the series

$i. \frac{1}{\sqrt{2}+\sqrt{3}} + \frac{1}{\sqrt{3}+\sqrt{4}} + \frac{1}{\sqrt{4}+\sqrt{5}} + \cdots$ $\qquad ii. \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$

**Solution**: $i.$ Here $u_n = \frac{1}{\sqrt{n+1}+\sqrt{n+2}} = \frac{1}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}\right)}$

Let $v_n = \frac{1}{\sqrt{n}}$.

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1+\frac{2}{n}}\right)} = \frac{1}{2}$ which is non zero and finite

$\therefore \sum u_n$ and $\sum v_n$ behave alike, i.e. either both converge or both diverge.

Now $\sum v_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent series (by $p$ series test).

Thus by comparison test $\sum u_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n+1}+\sqrt{n+2}}$ is also divergent.

$ii.$ Here $u_n = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} = \frac{\sqrt{n+1}-\sqrt{n-1}}{n} \cdot \frac{\sqrt{n+1}+\sqrt{n-1}}{\sqrt{n+1}+\sqrt{n-1}} = \frac{(n+1)-(n-1)}{n(\sqrt{n+1}+\sqrt{n-1})}$

$$= \frac{2}{n(\sqrt{n+1}+\sqrt{n-1})} = \frac{2}{n\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)}$$

Let $v_n = \frac{1}{n\sqrt{n}}$

$\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{2n\sqrt{n}}{n\sqrt{n}\left(\sqrt{1+\frac{1}{n}}+\sqrt{1-\frac{1}{n}}\right)} = 1$ which is non zero and finite

$\therefore \sum u_n$ and $\sum v_n$ behave alike, i.e. either both converge or both diverge.

Now $\sum v_n = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}$ is a convergent series (by $p$ series test).

Thus by comparison test $\sum u_n = \sum_{n=1}^{\infty} \frac{\sqrt{n+1}-\sqrt{n-1}}{n}$ is also convergent.

**Example 14** Test the convergence of the following series

$i.\ \sum_{n=1}^{\infty} \left[(n^3+1)^{1/3} - n\right]$ $\qquad$ $ii.\ \sum_{n=1}^{\infty} sin\frac{1}{n}$

**Solution**: $i.$ Here $u_n = (n^3+1)^{1/3} - n = n\left(1+\frac{1}{n^3}\right)^{1/3} - n$

$$= n\left[1 + \frac{1}{3n^3} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}.\frac{1}{n^6} + \frac{\frac{1}{3}\left(\frac{1}{3}-1\right)\left(\frac{1}{3}-2\right)}{3!}.\frac{1}{n^9} + \cdots\right] - n$$

$$= \frac{1}{3n^2} - \frac{1}{9n^5} + \cdots$$

Let $v_n = \frac{1}{n^2}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{3n^2} - \frac{n^2}{9n^5} + \cdots = \frac{1}{3}$ which is non zero and finite

Now $\sum v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (by $p$ series test).

Thus, by comparison test $\sum u_n = \sum_{n=1}^{\infty} \left[(n^3+1)^{1/3} - n\right]$ is also convergent.

$ii.$ Here $u_n = sin\frac{1}{n}$

Let $v_n = \frac{1}{n}$.

Then $\lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{sin\frac{1}{n}}{\frac{1}{n}} = 1$ which is non zero and finite

Now $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent series (by $p$ series test).

Thus, by comparison test $\sum u_n = \sum_{n=1}^{\infty} sin\frac{1}{n}$ is also divergent.

## Exercise 2A

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} e^{-n^2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{1}{n^2 \log n}$ Ans. Convergent
3. $\sum_{n=1}^{\infty} \left(\sqrt{n^3+1} - \sqrt{n^3}\right)$ Ans. Convergent
4. $\frac{1}{5} + \frac{\sqrt{2}}{7} + \frac{\sqrt{3}}{9} + \frac{\sqrt{4}}{11}$….. Ans. Divergent
5. $\frac{1}{1.2^2} + \frac{1}{2.3^2} + \frac{1}{3.4^2} +$….. Ans. Convergent
6. $\sum_{n=1}^{\infty} \left((n^3+1)^{1/3} - n\right)$ Ans. Divergent

7. $\sum_{n=1}^{\infty}\frac{n^2}{n^3+1}$ Ans. Divergent

8. $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}tan\frac{1}{n}$ Ans. Convergent

9. $\frac{1}{1+x}+\frac{1}{2+x}+\frac{1}{3+x}+\ldots..$ Ans. Divergent

10. $\sum_{n=1}^{\infty}\frac{1}{n-1}$ Ans. Divergent

## 2.4 Advanced Tests for Testing Convergence of Infinite Series

At times behavior of infinite series cannot be analyzed using basic comparison tests or they may be complicated to apply. We need to apply advanced tests to study the nature of the series.

### 2.4.1 D' Alembert's Ratio Test or Ratio Test

Let $\sum_{n=1}^{\infty}u_n$ be a positive term series such that $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=l$

Then (*i*) $\sum_{n=1}^{\infty}u_n$ converges if $l>1$

(*ii*) $\sum_{n=1}^{\infty}u_n$ diverges if $l<1$

(*iii*) Test fails if $l=1$

**Example 15** Test the convergence of the following series:

*i.* $\frac{1}{3}+\frac{1}{2.3^2}+\frac{1}{3.3^3}+\frac{1}{4.3^4}+\cdots$ *ii.* $\frac{1^2 2^2}{1}+\frac{2^2 3^2}{2!}+\frac{3^2 4^2}{3!}+\frac{4^2 5^2}{4!}+\cdots$ *iii.* $\sum_{n=1}^{\infty}\frac{n!}{n^n}$

**Solution**: *i.* Here $u_n=\frac{1}{n3^n}$ $\Rightarrow u_{n+1}=\frac{1}{(n+1)3^{n+1}}$

Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{(n+1)3^{n+1}}{n3^n}=\lim_{n\to\infty}\frac{3(n+1)}{n}=\lim_{n\to\infty}3\left(1+\frac{1}{n}\right)=3>1$

Hence by Ratio test, the given series converges.

*ii.* Here $u_n=\frac{n^2(n+1)^2}{n!}\Rightarrow u_{n+1}=\frac{(n+1)^2(n+2)^2}{(n+1)!}$

Then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n^2(n+1)^2}{n!} \cdot \frac{(n+1)!}{(n+1)^2(n+2)^2}$

$$= \lim_{n\to\infty} \frac{n^2(n+1)}{(n+2)^2} = \lim_{n\to\infty} \frac{n^3\left(1+\frac{1}{n}\right)}{n^2\left(1+\frac{2}{n}\right)^2} = \infty > 1$$

Hence by Ratio test, the given series converges.

$iii.$ Here $u_n = \frac{n!}{n^n} \Rightarrow u_{n+1} = \frac{(n+1)!}{(n+1)^{(n+1)}}$

Then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{n^n} \cdot \frac{(n+1)^{(n+1)}}{(n+1)!} = \lim_{n\to\infty} \frac{(n+1)^n}{n^n}$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e = 2.7183 > 1$$

Hence by Ratio test, the given series converges.

**Example 16** Test the convergence of the following series:

$i.$ $\frac{1}{7} + \frac{2!}{7^2} + \frac{3!}{7^3} + \frac{4!}{7^4} + \cdots$ $ii.$ $\left(\frac{1}{3}\right)^2 + \left(\frac{1.2}{3.5}\right)^2 + \left(\frac{1.2.3}{3.5.7}\right)^2 + \left(\frac{1.2.3.4}{3.5.7.9}\right)^2 + \cdots$

**Solution**: $i.$ Here $u_n = \frac{n!}{7^n} \Rightarrow u_{n+1} = \frac{(n+1)!}{7^{n+1}}$

Then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{n!}{7^n} \cdot \frac{7^{n+1}}{(n+1)!}$

$$= \lim_{n\to\infty} \frac{7}{n+1} = 0 < 1$$

Hence by Ratio test, the given series diverges.

$ii.$ Here $u_n = \left[\frac{1.2.3.4\ldots.n}{3.5.7.9\ldots.(2n+1)}\right]^2 \Rightarrow u_{n+1} = \left[\frac{1.2.3.4\ldots..n(n+1)}{3.5.7.9\ldots.(2n+1)(2n+3)}\right]^2$

Then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\frac{2n+3}{n+1}\right)^2 = 4 > 1$

Hence by Ratio test, the given series converges.

**Example 17** Test the convergence of the following series:

$i.\ \frac{x}{\sqrt{5}}+\frac{x^3}{\sqrt{7}}+\frac{x^5}{\sqrt{9}}+\frac{x^7}{\sqrt{11}}+\cdots$ $\qquad ii.\ \frac{x}{1.3}+\frac{x^2}{2.4}+\frac{x^3}{3.5}+\frac{x^4}{4.6}+\cdots(x>0)$

**Solution**: $i.$ Here $u_n=\frac{x^{2n-1}}{\sqrt{2n+3}}\ \Rightarrow u_{n+1}=\frac{x^{2n+1}}{\sqrt{2n+5}}$

Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{x^{2n-1}}{\sqrt{2n+3}}\frac{\sqrt{2n+5}}{x^{2n+1}}=\frac{\mathbf{1}}{x^2}$

Hence by Ratio test, the given series converges if $x^2<1$, diverges if $x^2>1$ and the test fails if $x^2=1$,

When $x^2=1,\ u_n=\frac{1}{\sqrt{2n+3}}$, let $v_n=\frac{1}{\sqrt{n}}$

Then $\lim_{n\to\infty}\frac{u_n}{v_n}=\lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{2n+3}}=\frac{1}{\sqrt{2}}$ which is non zero and finite

Now $\sum_{n=1}^{\infty}v_n=\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ is a divergent series (by $p$ series test).

Thus by comparison test $\sum_{n=1}^{\infty}\frac{1}{\sqrt{2n+3}}$ is also divergent.

$\therefore$ The series $\sum\frac{x^{2n-1}}{\sqrt{2n+3}}$ converges for $x^2<1$ and diverges for $x^2\geq 1$ .

$ii.$ Here $u_n=\frac{x^n}{n(n+2)}\Rightarrow u_{n+1}=\frac{x^{n+1}}{(n+1)(n+3)}$

Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{x^n}{n(n+2)}.\frac{(n+1)(n+3)}{x^{n+1}}=\frac{1}{x}$

Hence by Ratio test, the given series converges if $x<1$ and diverges if $x>1$ and the test fails if $x=1$

When $x=1,\ u_n=\frac{1}{n(n+2)}$ , let $v_n=\frac{1}{n^2}$

$\Rightarrow \lim_{n\to\infty} \frac{u_n}{v_n} = \lim_{n\to\infty} \frac{n^2}{n(n+2)} = 1$ which is non zero and finite

Now $\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a convergent series (by $p$ series test).

Thus by comparison test $\sum_{n=1}^{\infty} \frac{1}{n(n+2)}$ is also convergent.

$\therefore$ The series converges for $x \leq 1$ and diverges for $x > 1$.

### 2.4.2 Raabe's Test

Raabe's test is usually used when D' Alembert's Ratio test fails and the term $\frac{u_n}{u_{n+1}}$ does not involve the exponential function ' $e$'. When $\frac{u_n}{u_{n+1}}$ involves the exponential function ' $e$', we apply logarithmic test after Ratio test and not Raabe's test.

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that $\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}} - 1\right) = l$

Then $(i)$ $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$

$(ii)$ $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$ $\qquad$ $(iii)$ Test fails if $l = 1$

**Example 18** Test the convergence of the following series:

$i.$ $\frac{2}{3} + \frac{2.4}{3.5} + \frac{2.4.6}{3.5.7} + \frac{2.4.6.8}{3.5.7.9} + \cdots$ $\qquad$ $ii.$ $1 + \frac{3x}{7} + \frac{3.6\,x^2}{7.10} + \frac{3.6.9x^3}{7.10.13} + \cdots (x > 0)$

$iii.$ $x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + \frac{1.3.5}{2.4.6}\frac{x^7}{7} + \cdots \ (x > 0)$

**Solution**: $i.$ Here $u_n = \frac{2.4.6....2n}{1.3.5....(2n+1)} \Rightarrow u_{n+1} = \frac{2.4.6....2n(2n+2)}{1.3.5....(2n+1)(2n+3)}$

Then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{2n+3}{2n+2} = 1$

As Ratio test fails, applying Raabe's test, we get

$$\lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{2n+3}{2n+2}-1\right)=\lim_{n\to\infty}\left(\frac{n}{2n+2}\right)=\frac{1}{2}<1$$

Hence by Raabe's test, the given series diverges.

*ii*. Ignoring the first term, $u_n=\frac{3.6.9...3n\,x^n}{7.10.13...(3n+4)}$, $\Rightarrow u_{n+1}=\frac{3.6.9...3n(3n+3)}{7.10.13...(3n+4)(3n+7)}x^{n+1}$

Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{3n+7}{3n+3}.\frac{1}{x}=\frac{1}{x}$

Hence by Ratio test, the given series converges if $\frac{1}{x}>1$ i.e. $x<1$, diverges if $\frac{1}{x}<1$ i.e. $x>1$ and the test fails if $x=1$

When $x=1$, $\frac{u_n}{u_{n+1}}=\frac{3n+7}{3n+3}$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{3n+7}{3n+3}-1\right)=\lim_{n\to\infty}\frac{4n}{3n+3}=\frac{4}{3}>1$$

Hence by Raabe's test, the given series converges if $x=1$

∴ the given series converges if $x\le 1$ and diverges if $x>1$.

*iii*. Ignoring the first term, $u_n=\frac{1.3.5...(2n-1)}{2.4.6....(2n)}\frac{x^{2n+1}}{2n+1}$, $\Rightarrow u_{n+1}=\frac{1.3.5...(2n+1)}{2.4.6....(2n+2)}\frac{x^{2n+3}}{2n+3}$

Then $\lim_{n\to\infty}\frac{u_n}{u_{n+1}}=\lim_{n\to\infty}\frac{2n+2}{2n+1}.\frac{2n+3}{2n+1}.\frac{1}{x^2}=\frac{1}{x^2}$

Hence by Ratio test, the given series converges if $\frac{1}{x^2}>1$ i.e. $x^2<1$, diverges if $\frac{1}{x^2}<1$ i.e. $x^2>1$ and the test fails if $x^2=1$

When $x^2=1$, $\frac{u_n}{u_{n+1}}=\frac{2n+2}{2n+1}.\frac{2n+3}{2n+1}=\frac{4n^2+10n+6}{4n^2+4n+1}$

$$\Rightarrow \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right)=\lim_{n\to\infty} n\left(\frac{4n^2+10n+6}{4n^2+4n+1}-1\right)=\lim_{n\to\infty}\frac{6n^2+5n}{4n^2+4n+1}=\frac{6}{4}=\frac{3}{2}>1$$

Hence by Raabe's test, the given series converges if $x^2=1$

$\therefore$ The given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$.

### 2.4.3 Logarithmic Test

Logarithmic test is applied after the failure of D' Alembert's raio test and generally when the term $\frac{u_n}{u_{n+1}}$ involves $e$.

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that $\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = l$

Then $(i)$ $\sum_{n=1}^{\infty} u_n$ converges if $l > 1$ $(ii)$ $\sum_{n=1}^{\infty} u_n$ diverges if $l < 1$ $(iii)$ Test fails if $l = 1$

**Example 19** Test the convergence of the series $x + \frac{2^2x^2}{2!} + \frac{3^3x^3}{3!} + \frac{4^4x^4}{4!} + \cdots$

**Solution**: Here $u_n = \frac{n^n x^n}{n!} \Rightarrow u_{n+1} = \frac{(n+1)^{n+1}x^{n+1}}{(n+1)!}$

Then $\lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}x} = \lim_{n\to\infty} \frac{n^n}{(n+1)^n x} = \lim_{n\to\infty} \frac{1}{\left(1+\frac{1}{n}\right)^n} \frac{1}{x} = \frac{1}{ex}$

Hence by Ratio test, the given series converges if $\frac{1}{ex} > 1$ $i.e.$ $x < \frac{1}{e}$, diverges if $\frac{1}{ex} < 1$ $i.e. x > \frac{1}{e}$ and the test fails if $ex = 1$, $i.e. x = \frac{1}{e}$

Since $\frac{u_n}{u_{n+1}}$ involves $e$ $\therefore$ putting $x = \frac{1}{e}$ and applying logarithmic test

$$\lim_{n\to\infty} n \log \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} n \log \frac{e}{\left(1+\frac{1}{n}\right)^n} = \lim_{n\to\infty} n \left[\log e - \log\left(1 + \frac{1}{n}\right)^n\right]$$

$$= \lim_{n\to\infty} n \left[1 - n \log\left(1 + \frac{1}{n}\right)\right] = \lim_{n\to\infty} n \left[1 - n\left(\frac{1}{n} - \frac{1}{2n^2} + \frac{1}{3n^3} - \cdots\right)\right]$$

$$= \lim_{n\to\infty} n \left[\frac{1}{2n} - \frac{1}{3n^2} + \cdots\right] = \lim_{n\to\infty} \left[\frac{1}{2} - \frac{1}{3n} + \cdots\right] = \frac{1}{2} < 1$$

$\therefore$ By logarithmic test, the series diverges for $x = \frac{1}{e}$.

Hence the series $\sum \frac{n^n x^n}{n!}$ converges for $x < \frac{1}{e}$ and diverges for $x \geq \frac{1}{e}$.

### 2.4.4 Cauchy's n$^{th}$ Root Test

Let $\sum_{n=1}^{\infty} u_n$ be a positive term series such that $\lim_{n\to\infty} (u_n)^{1/n} = l$

Then (*i*) $\sum_{n=1}^{\infty} u_n$ converges if $l < 1$

(*ii*) $\sum_{n=1}^{\infty} u_n$ diverges if $l > 1$

(*iii*) Test fails if $l = 1$

**Example 20** Test the convergence of the following series:

(*i*) $1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots$ (*ii*) $\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ (*iii*) $\sum_{n=1}^{\infty} 5^{-n-(-1)^n}$

**Solution**: (*i*) Here $u_n = \frac{1}{n^n}$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{1}{n^n}\right)^{1/n} = \lim_{n\to\infty} \frac{1}{n} = 0 < 1$$

Hence by Cauchy's root test, the given series converges.

(*ii*) Here $u_n = \left(\frac{n}{n+1}\right)^{n^2} \Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n$

$$= \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \frac{1}{e} < 1$$

Hence by Cauchy's root test, the given series converges.

(*iii*) Here $u_n = 5^{-n-(-1)^n} \Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} 5^{-\{n+(-1)^n\}.1/n}$

$$= \lim_{n\to\infty} 5^{-\left\{1+\frac{(-1)^n}{n}\right\}} = 5^{-1} = \frac{1}{5} < 1$$

Hence by Cauchy's root test, the given series converges.

**Example 21** Test the convergence of the following series:

$$\left(\frac{2^2}{1^2} - \frac{2}{1}\right)^{-1} + \left(\frac{3^3}{2^3} - \frac{3}{2}\right)^{-2} + \left(\frac{4^4}{3^4} - \frac{4}{3}\right)^{-3} + \cdots$$

**Solution**: Here $u_n = \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-n}$

$$\Rightarrow \lim_{n\to\infty} (u_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{n+1}{n}\right)^{n+1} - \frac{n+1}{n}\right]^{-1}$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^{-1} \left[\left(\frac{n+1}{n}\right)^{n} - 1\right]^{-1}$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^{-1} \left[\left(1 + \frac{1}{n}\right)^{n} - 1\right]^{-1} = [e-1]^{-1} = [2.7183 - 1]^{-1} < 1$$

Hence by Cauchy's root test, the given series converges.

**Exercise 2B**

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2^n}{n^2+2}$ Ans. Convergent
2. $\sum_{n=1}^{\infty} \frac{n!}{2^{2n-1}}$ Ans. Divergent
3. $\sum_{n=1}^{\infty} \frac{1.2.3\ldots.n}{7.10\ldots.(3n+4)}$ Ans. Convergent
4. $1 + \frac{3}{2!} + \frac{5}{3!} + \frac{7}{4!}$ Ans. Convergent

5. $1+\frac{2}{5}x+\frac{6}{9}x^2+\frac{14}{17}x^3+\ldots..$ Ans. Convergent if $x<1$ , divergent if $x\geq 1$

6. $\sum_{n=2}^{\infty}\frac{1}{(\log n)^n}$ Ans. Convergent

7. $\sum_{n=1}^{\infty}\frac{n^{n^2}}{\left(n+\frac{1}{5}\right)^{n^2}}$ Ans. Convergent

8. $\sum_{n=1}^{\infty}\left(1+\frac{1}{\sqrt{n}}\right)^{-n^{3/2}}$ Ans. Convergent

9. $1+\frac{2^p}{2!}+\frac{3^p}{3!}+\frac{4^p}{4!}+\ldots..(p>0)$ Ans. Convergent

10. $\sum_{n=1}^{\infty}\sqrt{\frac{n-1}{n^3+1}}x^n\ (x>0)$ Ans.Convergent if $x<1$ , divergent if $x\geq 1$

**Cauchy's Integral Test**

If $u(x)$ is non-negative, integrable and monotonically decreasing function such that $u(n)=u_n$ , then if $\int_1^{\infty}u(x)\,d(x)$ converges then the series $\sum_{n=1}^{\infty}u_n$ also converges.

**Example 22** Test the convergence of the following series

$(i)\sum_{n=1}^{\infty}\frac{1}{n^2+1}$ $(ii)\ \sum_{n=2}^{\infty}\frac{1}{n(\log n)}$

**Solution**:$(i)$ Here $u_n=\frac{1}{n^2+1}$ .

Let $u(x)=\frac{1}{x^2+1}$

Clearly $u(x)$ is non-negative, integrable and monotonically decreasing function.

Consider $\int_1^{\infty}\frac{1}{x^2+1}\,d(x)=[tan^{-1}x]_1^{\infty}=tan^{-1}\infty-tan^{-1}1$

$=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$ , which is finite.

Hence $\int_1^{\infty} \frac{1}{x^2+1}\, d(x)$ converges, $\therefore \sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also converges.

$(ii)$ Here $u_n = \frac{1}{n(\log n)}$ .

Let $u(x) = \frac{1}{x(\log x)}$

Clearly $u(x)$ is non-negative, integrable and monotonically decreasing function.

Consider $\int_2^{\infty} \frac{1}{x(\log x)}\, d(x) = [\log\,(\log x)]_2^{\infty} = \log(log\infty) - \log\,(log 2) = \infty$, which is non-finite

Hence $\int_1^{\infty} \frac{1}{x(\log x)}\, d(x)$ diverges, $\therefore$ $\sum_{n=2}^{\infty} \frac{1}{n(\log n)}$ also diverges.

## Exercise C

Test the convergence of the following series:

1. $\sum_{n=1}^{\infty} \frac{2.4.6....(2n+2)}{3.5.7......(2n+3)} x^{n-1}$ $(x > 0)$ Ans. Convergent if $x < 1$ , divergent if $x \geq 1$
2. $\sum_{n=1}^{\infty} \frac{(2n!)}{(n!)^2} x^n$ $(x > 0)$ Ans. Convergent if $x < \frac{1}{4}$, divergent if $x \geq \frac{1}{4}$
3. $\sum_{n=1}^{\infty} \frac{1}{n^2+n}$ Ans. Convergent
4. $\sum_{n=1}^{\infty} \frac{1.3.5....(2n-1)}{2.4.6......2n} x^n (x > 0)$ Ans. Convergent if $x < 1$ , divergent if $x \geq 1$
5. $x^2 + \frac{2^2}{3.4} x^4 + \frac{2^2 4^2}{3.4.5.6} x^6 + \frac{2^2 4^2 6^2}{3.4.5.6.7.8} x^8 + \ldots..$ Ans. Convergent if $|x| \leq 1$ , divergent if $|x| > 1$
6. $1 + \frac{x}{2} + \frac{2! x^2}{3^2} + \frac{3! x^3}{4^3} + \frac{4! x^4}{5^4} + \cdots$ Ans. Convergent if $x < e$ , divergent if $x \geq e$.
7. $\sum_{n=2}^{\infty} \frac{1}{n(\log n)^2}$ Ans. Convergent

**Alternating Series**

An infinite series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}u_n = u_1 - u_2 + u_3 - u_4 + \cdots$ $(u_i > 0 \; \forall \; i)$ is called an alternating series.

**I Leibnitz's Test**

The alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}u_n$ converges, if it satisfies the following conditions:

$(i)$ $u_{n+1} \leq u_n$

$(ii)$ $\lim\limits_{n\to\infty} u_n = 0$

**Example 23** Test the convergence of the following series

$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ $\quad$ $(ii)$ $1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$

**Solution**: $(i)$ The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n}$. Here $u_n = \frac{1}{n}$

Since $\frac{1}{n+1} < \frac{1}{n}$ $\quad \therefore \; u_{n+1} \leq u_n$

Also $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{n} = 0$

Hence by Leibnitz's test, the given series converges.

$(ii)$ The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$. Here $u_n = \frac{1}{n^2}$

Since $\frac{1}{(n+1)^2} < \frac{1}{n^2} \; \therefore \; u_{n+1} \leq u_n$

Also $\lim\limits_{n\to\infty} u_n = \lim\limits_{n\to\infty}\frac{1}{n^2} = 0$

Hence by Leibnitz's test, the given series converges.

**II Absolute Convergence**

A series $\sum_{n=1}^{\infty} u_n$ is absolutely convergent if the series $\sum_{n=1}^{\infty}|u_n|$ is convergent.

For example, $\sum_{n=1}^{\infty} u_n = 1 - \frac{1}{2} + \frac{1}{2^2} - \frac{1}{2^3} + \cdots$ is absolutely convergent as $\sum_{n=1}^{\infty}|u_n| = 1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \cdots$ is a convergent series (since it is a geometric series whose common ratio $\frac{1}{2} < 1$ ).

**Result:** Every absolutely convergent series is convergent. But the converse may not be true.

**2.6.3 Conditional Convergence**

A series which is convergent but not absolutely convergent is called conditionally convergent series.

**Example 24** Test the convergence and absolute convergence of the following series:

$(i) 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ $(ii)\ 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \cdots$ $(iii) \sum_{n=2}^{\infty}(-1)^{n+1} \frac{1}{\log n}$

**Solution**: ($i$) The given series $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n}$ is convergent by Leibnitz's test.

Now, $\sum_{n=1}^{\infty}|u_n| = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n}$ is not convergent by p-series test.

Hence the given series is not absolutely convergent.

This is an example of conditionally convergent series.

($ii$) The given series $\sum_{n=1}^{\infty}(-1)^{n-1} \frac{1}{n^2}$ is convergent by Leibnitz's test.

Also, $\sum_{n=1}^{\infty}|u_n| = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent by p-series test.

Hence the given series is absolutely convergent.

($iii$)The given series $\sum_{n=2}^{\infty}(-1)^{n+1} u_n$

Here $u_n = \frac{1}{\log n}$ . Now $\log x$ is an increasing function $\forall x > 0$

$\therefore \log(n+2) > \log(n+1)$

$or\ \frac{1}{\log(n+2)} < \frac{1}{\log(n+1)}$

$\therefore u_{n+1} \leq u_n$

Also $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \frac{1}{log\ n} = 0$

Hence by Leibnitz's test, the given series is convergent.

Now for absolute convergence, consider $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\frac{1}{log\ n}$

It is a divergent series (see Example 9 (ii)).

Hence the given series is not absolutely convergent. This is an example of conditionally convergent series.

**Example 25** Test the convergence of the series:

$(i\ )\sum_{n=1}^{\infty}(-1)^{n-1}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$ $\qquad (ii)\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n2^n}$

**Solution**:$(i)$ The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\,u_n$

Consider $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\left[\frac{1}{n^2} + \frac{1}{(n+1)^2}\right]$

Now, $\frac{1}{n^2} + \frac{1}{(n+1)^2} < \frac{1}{n^2} + \frac{1}{n^2} = \frac{2}{n^2}$

Since $\sum_{n=1}^{\infty}\frac{1}{n^2}$is convergent (by p-series test) $\therefore$ by Comparision test $\sum_{n=1}^{\infty}|u_n|$ is also convergent.

Hence the given series is absolutely convergent and so convergent also.

$(ii)$ The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\,u_n$

Consider $\sum_{n=1}^{\infty}|u_n| = \sum_{n=1}^{\infty}\frac{1}{n2^n}$

Here $|u_n| = \frac{1}{n2^n}$

Now $\lim_{n\to\infty}\left|\frac{u_n}{u_{n+1}}\right| = \lim_{n\to\infty}\left|\frac{(n+1)2^{n+1}}{n2^n}\right| = 2 > 1$

$\therefore$ by Ratio test $\sum_{n=1}^{\infty}|u_n|$ is convergent or the given series is absolutely convergent and hence convergent.

**Example 26** Find the values of $x$ for which the series

$x - \frac{x^3}{3} + \frac{x^5}{5} + \cdots$ is absolutely convergent and conditionally convergent.

**Solution**: The given series is $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^{2n-1}}{2n-1} = \sum_{n=1}^{\infty}(-1)^{n-1}u_n$

Then $|u_n| = \left|\frac{x^{2n-1}}{2n-1}\right|$

Now, $\lim_{n\to\infty}\left|\frac{u_{n+1}}{u_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+1}}{2n+1}.\frac{2n-1}{x^{2n-1}}\right| = \frac{1}{x^2}$

Thus , by Ratio test $\sum_{n=1}^{\infty}|u_n|$ converges if $x^2 < 1\ i.e. |x| < 1$ , diverges if $x^2 > 1\ i.e. |x| > 1$ and test fails if $|x| = 1$

When $|x| = 1\ i.e. x = 1\ or\ x = -1$, we have

For $x = 1$ ,

the given series is $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$, which is convergent by Leibnitz's test but not absolutely convergent.

For $x = -1$ ,

the given series is $-1 + \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots$, which is also convergent by Leibnitz's test but not absolutely convergent.

Hence the given series is absolutely convergent for $|x| < 1\ or$

$-1 < x < 1$ and conditionally convergent for $|x| = 1$  $i.e.\ x = 1\ or - 1.$

## Exercise D

1. Show that the series $1-\frac{(2!)^2}{4!}+\frac{(3!)^2}{6!}-\cdots$is convergent.
2. Show that the series $x-\frac{x^3}{3!}+\frac{x^5}{5!}-\cdots$is absolutely convergent.
3. Test the convergence and absolute convergence of the series $1-\frac{1}{2.3}+\frac{1}{2^2.5}-\frac{1}{2^3.7}$ ...

   Ans. Absolutely convergent
4. Show that the series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{2n+3}$ is conditionally convergent.
5. Test the absolute convergence of the series $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\sqrt{n^2+1}-n\right)$

   Ans. Not absolutely convergent
6. Show that the series $\frac{\sin x}{1^3}-\frac{\sin 2x}{2^3}+\frac{\sin 3x}{3^3}-\cdots$ converges absolutely.
7. Find the interval of convergence of the series $x-\frac{x^2}{\sqrt{2}}+\frac{x^3}{\sqrt{3}}-\frac{x^4}{\sqrt{4}}$ ...

   Ans. $0<x\leq 1$