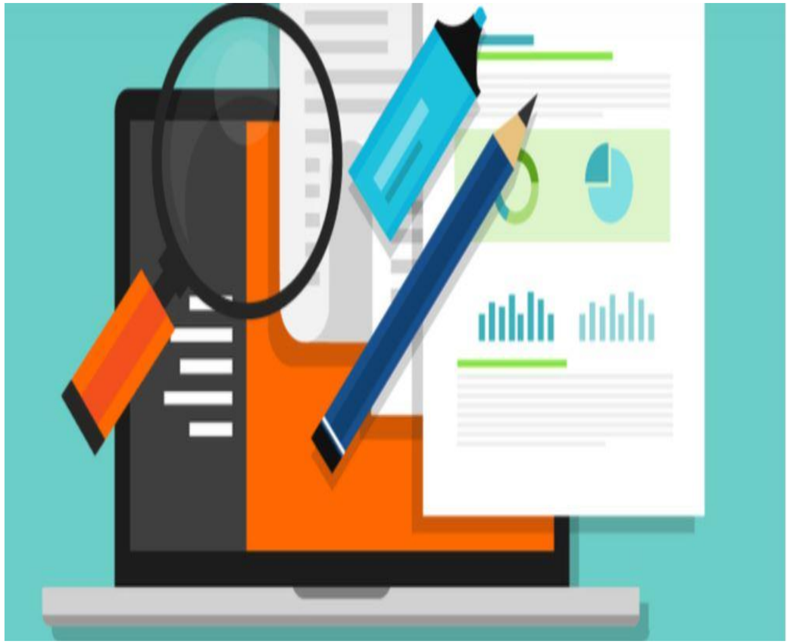

SUCCESSIVE DIFFERENTIATION & LEIBNITZ'S THEOREM

[The process of differentiating a function successively is called Successive Differentiation and the resulting derivatives are known as successive derivatives.]

Commonly used notations for higher order derivatives of a function $y = f(x)$

1^{st} Derivative: $f'(x)$ or y' or y_1 or $\frac{dy}{dx}$ or Dy

2^{nd} Derivative: $f''(x)$ or y'' or y_2 or $\frac{d^2y}{dx^2}$ or D^2y

$\vdots$

n^{th} Derivative: $f^{(n)}(x)$ or $y^{(n)}$ or y_n or $\frac{d^ny}{dx^n}$ or D^ny

Calculation of n^{th} Derivatives

1. n^{th} derivative of e^{ax}

Let $y = e^{ax}$

Then $y_1 = ae^{ax}$

$y_2 = a^2e^{ax}$

$\vdots$

$y_n = a^n e^{ax}$

2. n^{th} derivative of $(ax + b)^m$

Case1: when $n < m$

Let $y = (ax + b)^m$

Then $y_1 = m\, a(ax + b)^{m-1}$

$y_2 = m(m-1)a^2(ax + b)^{m-2}$

$\vdots$

$$y_n = m(m-1) \dots (m-n+1)a^n(ax + b)^{m-n} = \frac{m!}{(m-n)!}a^n(ax + b)^{m-n}$$

Case2: when $m = n$

$$y_n = \frac{n!}{(n-n)!} a^n (ax + \mathrm{b})^{n-n} = n!\, a^n$$

Case3: when $n > m$

$$y_n = 0$$

Case4: when $m = -1$

$$y = (ax + \mathrm{b})^{-1}$$

Then $y_1 = -a(ax + \mathrm{b})^{-2}$

$$y_2 = 2\, a^2 (ax + \mathrm{b})^{-3}$$

$$y_3 = -6\, a^3 (ax + \mathrm{b})^{-4}$$

$$\vdots$$

$$y_n = (-1)^n n!\, a^n (ax + b)^{-n-1} = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

3. n^{th} Derivative of $y = \log(ax + b)$

Let $y = \log(ax + b)$

Then $y_1 = \frac{a}{(ax+b)}$

$$y_2 = -\frac{a^2}{(ax+b)^2}$$

$$y_3 = \frac{2!\, a^3}{(ax+b)^3}$$

$$\vdots$$

$$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$$

4. n^{th} Derivative of $y = \sin(ax+b)$

Let $y = \sin(ax+b)$

Then $y_1 = a\cos(ax+b)$

$$= a \sin\left(ax + b + \frac{\pi}{2}\right)$$

$$y_2 = a^2 \cos\left(ax + b + \frac{\pi}{2}\right)$$

$$= a^2 \sin\left(ax + b + \frac{\pi}{2} + \frac{\pi}{2}\right)$$

$$\vdots$$

$$y_n = a^n \sin\left(ax + b + \frac{n\pi}{2}\right)$$

5. n^{th} Derivative of $y = \cos(ax+b)$

$$y_n = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

By computing in similar manner as of result **4**

6. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = e^{ax}\sin(ax+b)}$

Let $\quad y = e^{ax}\sin(bx+c)$

Then $y_1 = a\,e^{ax}\sin(bx+c) + e^{ax}b\cos(bx+c)$

$$= e^{ax}\,[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}\,[\,r\,cos\alpha\,\sin(bx+c) + r\,sin\alpha\,\cos(bx+c)]$$

by putting $a = r\,cos\alpha,\ b = r\,sin\alpha$

$$= e^{ax}\ r\ \sin(bx+c+\alpha)$$

Similarly, $y_2 = e^{ax}\ r^2\ \sin(bx+c+2\alpha)$

$\vdots$

$$y_n = e^{ax}\ r^n\ \sin(bx+c+n\alpha)$$

where $r^2 = a^2 + b^2$ and $\tan\alpha = \frac{b}{a}$

$$\therefore y_n = e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

7. $\boldsymbol{n^{th}}$ **Derivative of** $\boldsymbol{y = e^{ax}\cos(ax+b)}$

$$y_n = e^{ax}\ r^n\ \cos(bx+c+n\alpha) = e^{ax}\ (a^2+b^2)^{\frac{n}{2}}\ \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$$

By computing in similar manner as of result **6**

Summary of Results

Function	n^{th} derivative
$y = e^{ax}$	$y_n = a^n e^{ax}$
$y = (ax+b)^m$	$y_n = \begin{cases} \frac{m!}{(m-n)!} a^n (ax+\mathrm{b})^{m-n}, & m>0, m> n \\ 0, \quad m>0, & m< n, \\ n!\ a^n, & m = n \\ \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}, & m=-1 \end{cases}$
$y = \log(ax+b)$	$y_n = (-1)^{n-1} \frac{(n-1)!\, a^n}{(ax+b)^n}$
$y = \sin(ax+b)$	$y_n = a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$
$y = cos(ax+b)$	$y_n = a^n \cos\left(ax+b+\frac{\mathrm{n}\pi}{2}\right)$
$y = e^{ax} \sin(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \sin\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$
$\mathrm{y} = e^{ax} \cos(bx+c)$	$y_n = e^{ax} (a^2+b^2)^{\frac{n}{2}} \cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

Example 1 Find the n^{th} derivative of $\frac{1}{1-5x+6x^2}$

Solution: Let $y = \frac{1}{1-5x+6x^2}$

Resolving into partial fractions

$$y = \frac{1}{(1-3x)(1-2x)}$$

$$= \frac{3}{1-3x} - \frac{2}{1-2x}$$

$$\therefore y_n = \frac{3(-1)^n n!\,(-3)^n}{(1-3x)^{n+1}} - \frac{2(-1)^n n!\,(-2)^n}{(1-2x)^{n+1}}$$

$\because$ n^{th} derivative of $(ax+b)^{-1}$ is given by $\frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$

$$\Rightarrow y_n = (-1)^{2n}\, n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right] = n! \left[\left(\frac{3}{1-3x}\right)^{n+1} - \left(\frac{2}{1-2x}\right)^{n+1}\right]$$

Example 2 Find the n^{th} derivative of $\sin 6x \cos 4x$

Solution: Let y $= \sin 6x \cos 4x$

$$= \frac{1}{2}(\sin 10x + \sin 2x)$$

$$\therefore y_n = \frac{1}{2}\left[10^n \sin\left(10x + \frac{n\pi}{2}\right) + 2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

$\because n^{th}$ derivative of $\sin(ax+b)$ is given by $a^n \sin\left(ax+b+\frac{n\pi}{2}\right)$

Example 3 Find n^{th} derivative of $sin^2x cos^3x$

Solution: Let y = $sin^2x cos^3x$

$$= sin^2x cos^2x\ cosx = \frac{1}{4} sin^2 2x\ cosx\ = \frac{1}{8}(1-cos4x)cosx$$

$$= \frac{1}{8}cosx - \frac{1}{8}cos4x\ cosx = \frac{1}{8}cosx - \frac{1}{16}(cos5x + cos3x)$$

$$= \frac{1}{16}(2cosx - cos5x - cos3x)$$

$$\therefore y_n = \frac{1}{16}\left[2cos\left(x+\frac{n\pi}{2}\right) - 5^n cos\left(5x+\frac{n\pi}{2}\right) - 3^n cos\left(3x+\frac{n\pi}{2}\right)\right]$$

$\because n^{th}$ derivative of $\cos(ax+b)$ is given by $a^n \cos\left(ax+b+\frac{n\pi}{2}\right)$

Example 4 Find the n^{th} derivative of $e^{3x}\cos x \sin^2 2x$

Solution: Let $y = e^{3x}\cos x \sin^2 2x$

$$= e^{3x}\cos x.\frac{1}{2}(1 - \cos 4x)\ ) \quad \because \sin^2 2x\ = \frac{1}{2}(1 - \cos 4x)$$

$$= \frac{1}{2}e^{3x}\ (\cos x - \cos x \cos 4x)\ = \frac{1}{2}e^{3x}\left(\cos x - \frac{1}{2}(\cos 5x\ + \cos 3x\right)$$

$$\Rightarrow\ y\ = \frac{1}{2}e^{3x}\cos x - \frac{1}{4}e^{3x}\cos 5x - \frac{1}{4}e^{3x}\cos 3x$$

$$\therefore y_n\ = \frac{1}{2}e^{3x}\ (9+1)^{\frac{n}{2}}\ \cos\left(x + n\tan^{-1}\frac{1}{3}\right) - \frac{1}{4}e^{3x}\ (9+25)^{\frac{n}{2}}\ \cos\left(5x + n\tan^{-1}\frac{5}{3}\right)$$

$$-\frac{1}{4}e^{3x}\,(9+9)^{\frac{n}{2}}\;\cos\left(3x+n\tan^{-1}\frac{3}{3}\right)$$

$\because n^{th}$ derivative of $e^{ax}\cos(bx+c)$ is given by $e^{ax}\,(a^2+b^2)^{\frac{n}{2}}\;\cos\left(bx+c+n\tan^{-1}\frac{b}{a}\right)$

$$=\frac{1}{2}e^{3x}\,10^{\frac{n}{2}}\;\cos\left(x+n\tan^{-1}\frac{1}{3}\right)-\frac{1}{4}e^{3x}\,34^{\frac{n}{2}}\;\cos\left(5x+n\tan^{-1}\frac{5}{3}\right)-\frac{1}{4}e^{3x}\,18^{\frac{n}{2}}\;\cos(3x+n\tan^{-1}1)$$

Example 5 Find the n^{th} derivative of $\log(4x^2-9)$

Solution: Let $y=\log(4x^2-9)=\log(2x+3)+\log(2x-3)$

$$y_n=(-1)^{n-1}(n-1)!\;2^n\left[\frac{1}{(2x+3)^n}+\frac{1}{(2x-3)^n}\right]$$

$\because n^{th}$ derivative of $\log(ax+b)$ is given by $(-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$

Example 7 If $y=\sin ax+\cos ax$, prove that $y_n=a^n[1+(-1)^n\sin 2ax\,]^{\frac{1}{2}}$

Solution: We have $y_n=a^n\left[\sin\left(ax+\frac{n\pi}{2}\right)+\cos\left(ax+\frac{n\pi}{2}\right)\right]$

$$=a^n\left[\left\{\sin\left(ax+\frac{n\pi}{2}\right)+\cos\left(ax+\frac{n\pi}{2}\right)\right\}^2\right]^{\frac{1}{2}}$$

$$=a^n\left[\sin^2\left(ax+\frac{n\pi}{2}\right)+\cos^2\left(ax+\frac{n\pi}{2}\right)+2\sin\left(ax+\frac{n\pi}{2}\right).\cos\left(ax+\frac{n\pi}{2}\right)\right]^{\frac{1}{2}}$$

$$=a^n[1+\sin(2ax+n\pi)]^{\frac{1}{2}}$$

$$= a^n[1 + \sin 2ax \cos n\pi + \cos 2ax \sin n\pi\,]^{\frac{1}{2}}$$

$$= a^n[1 + (-1)^n \sin 2ax\,]^{\frac{1}{2}} \quad \because \cos n\,\pi = (-1)^n \text{ and } \sin n\,\pi = 0$$

Example 8 Find the n^{th} derivative of $\tan^{-1}\frac{x}{a}$

Solution: Let $y = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_1 = \frac{1}{a\left(1+\frac{x^2}{a^2}\right)} = \frac{a}{x^2+a^2} = \frac{a}{x^2-(ai)^2} = \frac{a}{(x+ai)(x-ai)} = \frac{a}{2ai}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right) = \frac{1}{2i}\left(\frac{1}{x-ai} - \frac{1}{x+ai}\right)$$

Differentiating $(n-1)$ times w.r.t. x, we get $y_n = \frac{1}{2i}\left[\frac{(-1)^{n-1}(n-1)!}{(x-ai)^n} - \frac{(-1)^{n-1}(n-1)!}{(x+ai)^n}\right]$

$\because (n-1)^{th}$ derivative of $(px+q)^{-1}$ is given by $\frac{(-1)^{n-1}(n-1)!\,p^{n-1}}{(px+q)^n}$

Substituting $x = r\,cos\theta,\ a = r\,sin\theta$ such that $\theta = \tan^{-1}\frac{x}{a}$

$$\Rightarrow y_n = \frac{(-1)^{n-1}(n-1)!}{2i}\left[\frac{1}{r^n(\cos\theta - i\,sin\theta)^n} - \frac{1}{r^n(\cos\theta + i sin\theta)^n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[(\cos\theta - i\,sin\theta)^{-n} - (\cos\theta + i\,sin\theta)^{-n}\right]$$

$$= \frac{(-1)^{n-1}(n-1)!}{2ir^n}\left[\cos n\theta + i\sin n\theta - \cos n\theta + i\sin n\theta\right], \text{ by De Moivre's theorem}$$

$$= \frac{(-1)^{n-1}(n-1)!}{r^n}\sin n\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{\left(\frac{a}{\sin\theta}\right)^n} \sin n\theta \qquad \because a = r\,sin\theta$$

$$= \frac{(-1)^{n-1}(n-1)!}{a^n} \sin n\theta \;\sin^n\theta \qquad \text{where } \theta = \tan^{-1}\frac{x}{a}$$

Example 9 Find the n^{th} derivative of $\frac{1}{1+x+x^2}$

Solution: Let $y = \frac{1}{1+x+x^2} = \frac{1}{(x-\omega)(x-\omega^2)}$ where $\omega = \frac{-1+i\sqrt{3}}{2}$ and $w^2 = \frac{-1-i\sqrt{3}}{2}$

$$\Rightarrow y = \frac{1}{\omega-\omega^2}\left(\frac{1}{x-\omega} - \frac{1}{x-\omega^2}\right)$$

$$= \frac{1}{i\sqrt{3}}\left(\frac{1}{x-\omega} - \frac{1}{x-\omega^2}\right) = \frac{-i}{\sqrt{3}}\left(\frac{1}{x-\omega} - \frac{1}{x-\omega^2}\right)$$

$$\Rightarrow y_n = \frac{-i}{\sqrt{3}}\left[\frac{(-1)^n n!}{(x-\omega)^{n+1}} - \frac{(-1)^n n!}{(x-\omega^2)^{n+1}}\right] \quad \because n^{th} \text{ derivative of } (ax+b)^{-1} \text{ is given by } \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

$$= \frac{-i\,(-1)^n n!}{\sqrt{3}}\left[\frac{1}{(x-\omega)^{n+1}} - \frac{1}{(x-\omega^2)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!}{\sqrt{3}}\left[\frac{1}{\left(x+\frac{1}{2}-\frac{i\sqrt{3}}{2}\right)^{n+1}} - \frac{1}{\left(x+\frac{1}{2}+\frac{i\sqrt{3}}{2}\right)^{n+1}}\right]$$

$$= \frac{i\,(-1)^{n+1} n!\, 2^{n+1}}{\sqrt{3}}\left[\frac{1}{\left(2x+1-i\sqrt{3}\right)^{n+1}} - \frac{1}{\left(2x+1+i\sqrt{3}\right)^{n+1}}\right]$$

$$\Rightarrow\ y_n = \frac{i\,(-2)^{n+1} n!}{\sqrt{3}\ r^{n+1}}\left[(cos\theta - isin\theta)^{-(n+1)} - \ (cos\theta + isin\theta)^{-(n+1)}\right]$$

by substituting $2x + 1 = r\ cos\theta,\ \sqrt{3} = r\ sin\theta$ such that $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

$$\Rightarrow\ y_n = \frac{i\,(-2)^{n+1} n!}{\sqrt{3}\left(\frac{\sqrt{3}}{sin\theta}\right)^{n+1}}\ [cos(n+1)\theta + i\ sin(n+1)\theta - cos(n+1)\theta + isin(n+1)\theta]$$, using De Moivre's theorem

$$= \frac{i\,(-2)^{n+1} n!}{(\sqrt{3})^{n+2}}\ 2i\ sin(n+1)\theta\ sin^{n+1}\theta\ = \frac{(-2)^{n+2} n!}{\sqrt{3}^{n+2}}\ \ sin(n+1)\theta\ sin^{n+1}\theta$$, where $\theta = \tan^{-1}\frac{\sqrt{3}}{2x+1}$

Exercise 1 A

1. Find the n^{th} derivative of $\frac{x^4}{(x-1)(x-2)}$

2. Find the n^{th} derivative of $\cos x\ \cos 2x \cos 3x$

3. If $x = \sin t$, $y = \sin at$, show that $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2 y = 0$

4. If $p^2 = a^2 cos^2\theta + b^2 sin^2\theta$, show that $p + \frac{d^2p}{d\theta^2} = \frac{a^2b^2}{p^3}$

5. If $y = \frac{x}{x^2+a^2}$, find y_n i.e., the n^{th} derivative of y

6. If $y = e^x sin^2 x$, find y_n i.e., the n^{th} derivative of y

7. 7. Find n^{th} differential coefficient of $y = \log[(ax + b)(cx + d)]$

8. 8. If y = $x\log\frac{x-1}{x+1}$, show that $y_n = (-1)^{n-1}(n-2)!\left[\frac{x-n}{(x-1)^n} + \frac{x+n}{(x+1)^n}\right]$

9. 9. If $y = \tan^{-1}\frac{\sqrt{1+x^2}-1}{x}$, show that $y_n = \frac{1}{2}(-1)^{n-1}(n-1)!\ \sin n\,\theta\ sin^n\theta$

Answers

1. $(-1)^n n! \left[\frac{16}{(x-2)^{n+1}} - \frac{1}{(x-1)^{n+1}}\right]$
2. $\frac{1}{4}\left[2^n \cos\left(2x + \frac{n\pi}{2}\right) + 4^n \cos\left(4x + \frac{n\pi}{2}\right) + 6^n \cos\left(6x + \frac{n\pi}{2}\right)\right]$
5. $\frac{(-1)^n n!}{a^{n+1}} cos(n+1)\theta \, sin^{n+1}\theta$ where $\theta = \tan^{-1}\frac{x}{a}$
6. $\frac{1}{2} e^x \left[1 - 16\left(2x + \frac{n\pi}{2}\right)\right]$

Leibnitz's Theorem

If u and v are functions of x such that their n^{th}derivatives exist, then the n^{th}derivative of their product is given by

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

where u_r and v_r represent r^{th}derivatives of u and v respectively.

Remark: The term which vanishes after differentiating finitely should be taken as v.

Example10 Find the n^{th} derivative of $x \log x$

Solution: Let $u = \log x$ and $v = x$

Then $u_n = (-1)^{n-1}\frac{(n-1)!}{x^n}$ and $u_{n-1} = (-1)^{n-2}\frac{(n-2)!}{x^{n-1}}$

$\because n^{th}$ derivative of $\log(ax+b)$ is given by $(-1)^{n-1}\frac{(n-1)!\,a^n}{(ax+b)^n}$

Now, by Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x \log x)_n = (-1)^{n-1} \frac{(n-1)!}{x^n} x + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}} + 0$$

$$= (-1)^{n-1} \frac{(n-1)!}{x^{n-1}} + n(-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

$$= (-1)^{n-2} \frac{(n-2)!}{x^{n-1}} [-(n-1) + n] = (-1)^{n-2} \frac{(n-2)!}{x^{n-1}}$$

Example 11 Find the n^{th} derivative of $x^2 e^{3x} \sin 4x$

Solution: Let $u = e^{3x} \sin 4x$ and $v = x^2$

Then $u_n = e^{3x}\, 25^{\frac{n}{2}} \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) = e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right)$

$\because$ n^{th} derivative of $e^{ax} \sin(bx + c)$ is given by $e^{ax} (a^2 + b^2)^{\frac{n}{2}} \sin\left(bx + c + n \tan^{-1} \frac{b}{a}\right)$

Now, by Leibnitz's theorem, we have

$$(u\,v)_n = u_n v + n_{C_1} u_{n-1} v_1 + n_{C_2} u_{n-2} v_2 + \cdots + n_{C_r} u_{n-r} v_r + \cdots + u v_n$$

$$\Rightarrow (x^2 e^{3x} \sin 4x)_n = x^2 e^{3x}\, 5^n \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) + 2nx e^{3x}\, 5^{n-1} \sin\left(4x + (n-1) \tan^{-1} \frac{4}{3}\right) +$$

$$n(n-1) e^{3x}\, 5^{n-2} \sin\left(4x + (n-2) \tan^{-1} \frac{4}{3}\right) + 0$$

$$= e^{3x} 5^n \left[x^2 \sin\left(4x + n \tan^{-1} \frac{4}{3}\right) + \frac{2nx}{5} \sin\left(4x + (n-1) \tan^{-1} \frac{4}{3}\right) + \frac{n(n-1)}{25} \sin\left(4x + (n-2) \tan^{-1} \frac{4}{3}\right)\right]$$

Example 12 If $y = a\cos(\log x) + b\,sin(\log x)$, show that

$$x^2 y_{n+2} + (2n+1)xy_{n+1} + n(n+1)y_n = 0$$

Solution: Here $y = a\cos(\log x) + b\sin(\log x)$

$$\Rightarrow y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$

$$\Rightarrow xy_1 = -a\sin(logx) + b\,cos(logx)$$

Differentiating both sides w.r.t. x , we get

$$xy_2 + y_1 = -\frac{a}{x}\cos(logx) + \frac{-b}{x}\ sin(logx)$$

$$\Rightarrow x^2y_2 + xy_1 = -\{a\cos(logx) + b\ sin(logx)\} = -y$$

$$\Rightarrow x^2y_2 + xy_1 + y = 0$$

Using Leibnitz's theorem, we get

$$(y_{n+2}x^2 + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2) + (y_{n+1}x + n_{C_1}y_n.1) + y_n = 0$$

$$\Rightarrow y_{n+2}x^2 + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n + y_n = 0$$

$$\Rightarrow x^2y_{n+2} + (2n+1)xy_{n+1} + (n^2+1)y_n = 0$$

Example 13 If $y = \log\left(x + \sqrt{1+x^2}\right)$, prove that $(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Solution: $y = \log\left(x + \sqrt{1+x^2}\right)$

$$\Rightarrow y_1 = \frac{1}{x+\sqrt{1+x^2}}\left(1 + \frac{1}{2\sqrt{1+x^2}}2x\right) = \frac{1}{\sqrt{1+x^2}}$$

$\Rightarrow (1+x^2){y_1}^2 = 1$

Differentiating both sides w.r.t. x, we get

$(1+x^2)2y_1y_2 + 2x{y_1}^2 = 0$

$\Rightarrow (1+x^2)y_2 + xy_1 = 0$

Using Leibnitz's theorem

$[y_{n+2}(1+x^2) + n_{C_1}y_{n+1}2x + n_{C_2}y_n.2] + \left(y_{n+1}x + n_{C_1}y_n.1\right) = 0$

$\Rightarrow y_{n+2}(1+x^2) + y_{n+1}2nx + n(n-1)y_n + y_{n+1}x + ny_n = 0$

$\Rightarrow (1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2y_n = 0$

Example14 If $y = \sin(m\,sin^{-1}x)$, show that

$(1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n$. Also find $y_n(0)$

Solution: Here $y = \sin(m\,sin^{-1}x)$ ...①

$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}}\cos(m\,sin^{-1}x)$ ...②

$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2\cos^2(m\,sin^{-1}x)$

$\Rightarrow (1-x^2){y_1}^2 = \mathrm{m}^2[1-\sin^2(m\,sin^{-1}x)]$

$\Rightarrow (1-x^2){y_1}^2 = m^2(1-y^2)$ ...③

$\Rightarrow (1- x^2){y_1}^2 + m^2y^2 = m^2$

Differentiating w.r.t. x, we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) + m^2 2yy_1 = 0$$

$$\Rightarrow (1-x^2)y_2 - xy_1 + m^2y = 0$$

Using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n 1\right) + m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) + m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} = (2n+1)xy_{n+1} + (n^2-m^2)y_n \qquad \text{...④}$$

Putting $x = 0$ in ①, ② and ③

$y(0) = 0,\ y_1(0) = m$ and $y_2(0) = 0$

Putting $x = 0$ in ④, we get

$$y_{n+2}(0) = (n^2-m^2)y_n(0)$$

Putting $n = 1,2,3$ ... in the above equation, we get

$$y_3(0) = (1^2-m^2)y_1(0) = (1^2-m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2-m^2)y_2(0) = 0 \qquad \because y_2(0) = 0$$

$$y_5(0) = (3^2-m^2)y_3(0) = m(1^2-m^2)(3^2-m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ even \\ m(1^2-m^2)(3^2-m^2)\ldots[(n-2)^2) - m^2], & if\ n\ is\ odd \end{cases}$$

Example 15 If $y = e^{msin^{-1}x}$, show that

$$(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0.$$ Also find $y_n(0)$.

Solution: Here $y = e^{msin^{-1}x}$ $\quad \ldots ①$

$$\Rightarrow y_1 = \frac{m}{\sqrt{1-x^2}} \text{e}^{m\, sin^{-1}x}$$

$$= \frac{my}{\sqrt{1-x^2}} \quad \ldots\ldots ②$$

$$\Rightarrow (1-x^2){y_1}^2 = m^2y^2$$

Differentiating above equation w.r.t. x , we get

$$(1-x^2)2y_1y_2 + {y_1}^2(-2x) = m^2 2yy_1$$

$$\Rightarrow (1-x^2)y_2 - xy_1 - m^2y = 0 \quad \ldots\ldots ③$$

Differentiating above equation n times w.r.t. x using Leibnitz's theorem, we get

$$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n 1\right) - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) - m^2y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0 \ldots\ldots ④$$

To find $y_n(0)$: Putting $x = 0$ in ①, ② and ③

$y(0) = 1,\ y_1(0) = m$ and $y_2(0) = m^2$

Also putting $x = 0$ in, we get

$$y_{n+2}(0) = (n^2 + m^2)y_n(0)$$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = (1^2 + m^2)y_1(0) = (1^2 + m^2)m \qquad \because y_1(0) = m$$

$$y_4(0) = (2^2 + m^2)y_2(0) = m^2(2^2 + m^2) \qquad \because y_2(0) = m^2$$

$$y_5(0) = (3^2 + m^2)y_3(0)$$

$$= m(1^2 + m^2)(3^2 + m^2)$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} m^2(2^2 + m^2) \ldots [(n-2)^2 + m^2] , & if\ n\ is\ even \\ m(1^2 + m^2)(3^2 + m^2) \ldots [(n-2)^2 + m^2] , & if\ n\ is\ odd \end{cases}$$

Example 16 If $y = tan^{-1}x$, show that

$(1 - x^2)y_{n+2} + 2(n+1)xy_{n+1} + n(n+1)y_n = 0$. Also find $y_n(0)$

Solution: Here $y = tan^{-1}x$ ...①

$$\Rightarrow y_1 = \frac{1}{1+x^2} \qquad \text{...②}$$

$$y_2 = \frac{-2x}{1+x^2}$$

$$\Rightarrow (1 + x^2)y_2 + 2xy_1 = 0 \qquad \text{...③}$$

Differentiating equation ③ n times w.r.t. x using Leibnitz's theorem

$$[y_{n+2}(1+x^2)+n_{C_1}y_{n+1}(2x)+n_{C_2}y_n(2)]+2\big(y_{n+1}x+n_{C_1}y_n1\big)=0$$

$$\Rightarrow y_{n+2}(1+x^2)+y_{n+1}2nx+n(n-1)y_n+2(y_{n+1}x+ny_n)=0$$

$$\Rightarrow (1+x^2)y_{n+2}+2(n+1)xy_{n+1}+n(n+1)y_n=0\ldots\ldots④$$

To find $y_n(0)$: Putting $x=0$ in ①, ②and ③, we get

$y(0)=0$, $y_1(0)=1$ and $y_2(0)=0$

Also putting $x=0$ in ④,we get

$y_{n+2}(0)=-n(n+1)y_n(0)$

Putting $n=1,2,3\ldots$ in the above equation, we get

$y_3(0)=-1(2)y_1(0)\quad=-2\qquad\qquad \because y_1(0)=1$

$y_4(0)=-2(3)y_2(0)\quad=0\qquad\qquad \because y_2(0)=0$

$y_5(0)=-3(4)y_3(0)\quad=-3(4)(-2)=4!$

$y_6(0)=-4(5)y_4(0)\quad=0$

$y_7(0)=-5(6)y_5(0)=-5(6)4!=-(6!)$

$\vdots$

$\Rightarrow\quad y_{2n+1}(0)=(-1)^n(2n)!$ and $\quad y_{2n}(0)=0$

Example17 If $y=(sin^{-1}x)^2$, show that $(1-x^2)y_{n+2}-(2n+1)xy_{n+1}-n^2y_n=0$. Also find $y_n(0)$

Solution: Here $y = (sin^{-1}x)^2$ ...①

$\Rightarrow y_1 = 2sin^{-1}x.\frac{1}{\sqrt{1-x^2}}$ ...②

Squaring both the sides, we get

$(1-x^2){y_1}^2 = 4\,(sin^{-1}x)^2$

$\Rightarrow (1-x^2){y_1}^2 = 4\,(y)^2$

Differentiating the above equation w.r.t. x, we get

$(1-x^2)2y_1y_2 + {y_1}^2(-2x) - 4y_1 = 0$

$\Rightarrow (1-x^2)y_2 + y_1(-x) - 2 = 0$ ...③

Differentiating the above equation n times w.r.t. x using Leibnitz's theorem, we get

$[y_{n+2}(1-x^2) + n_{C_1}y_{n+1}(-2x) + n_{C_2}y_n(-2)] - \left(y_{n+1}x + n_{C_1}y_n1\right) = 0$

$\Rightarrow y_{n+2}(1-x^2) - y_{n+1}2nx - n(n-1)y_n - (y_{n+1}x + ny_n) = 0$

$\Rightarrow (1-x^2)y_{n+2} - (2n+1)xy_{n+1} - y_nn^2 = 0$ ...④

To find $y_n(0)$: Putting $x = 0$ in ①, ②and ③, we get

$y(0) = 0\,, y_1(0) = 0$ and $y_2(0) = 2$

Also putting $x = 0$ in ④, we get

$y_{n+2}(0) = n^2y_n(0)$

Putting $n = 1,2,3 \ldots$ in the above equation, we get

$$y_3(0) = 1^2 y_1(0) = 0 \qquad \because y_1(0) = 0$$

$$y_4(0) = 2^2 y_2 = 2^2 2 \qquad \because y_2(0) = 2$$

$$y_5(0) = 3^2 y_3(0) = 0$$

$$y_6(0) = 4^2 y_4(0) = 4^2 2^2 2$$

$$\vdots$$

$$\Rightarrow \quad y_n(0) = \begin{cases} 0, & if\ n\ is\ odd \\ 2.2^2.4^2 \ldots\ldots\ldots.(n-2)^2, & if\ n\ is\ even \end{cases}$$

Exercise 1 B

1. Find y_n, if $y = x^3 \cos x$

2. Find y_n, if $y = x^2 e^x \cos x$

3. If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$, prove that $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

4. If $y\sqrt{1+x^2} = \log\left(x + \sqrt{1+x^2}\right)$, prove that
$(1+x^2)y_{n+2} + (2n+3)xy_{n+1} + (n+1)^2 y_n = 0$

5. If $y = \left[x + \sqrt{1+x^2}\right]^m$, prove that $(x^2+1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

6. If $y = (sinh^{-1}x)^2$, show that
$(1+x^2)y_{n+2} + (2n+1)xy_{n+1} + n^2 y_n = 0$. Also find $y_n(0)$.

7. If $y = \cos(m\, sin^{-1}x)$, show that

$$(1 - x^2)y_{n+2} = (2n + 1)xy_{n+1} + (n^2 - m^2)y_n$$. Also find $y_n(0)$.

8. If $f(x) = \tan x$, prove that $f^n(0) - n_{C_2} f^{n-2}(0) + n_{C_4} f^{n-4}(0) - \cdots = sin\frac{n\pi}{2}$

Answers

1. $x^3 cos\left(x + \frac{n\pi}{2}\right) + 3nx^2 cos\left[x + \frac{1}{2}(n-1)\pi\right] + 3n(n-1)xcos\left[x + \frac{1}{2}(n-2)\pi\right] + n(n-1)(n-2)cos\left[x + \frac{1}{2}(n-3)\pi\right]$
2. $2^{\frac{n}{2}} e^x cos\left(x + \frac{n\pi}{4}\right)$

$y_{2n+1}(0) = 0$ and $y_{2n}(0) = (-1)^{n-1} 2.\, 2^2.\, 4^2 \ldots (2n-2)^2$