Chapter8
ASSIGNMENT PROBLEM

Chapter 8 **Assignment Problem**

8.1 Introduction

An assignment problem is a particular case of transportation problem in which a number of operations are to be assigned to an equal number of operators, where each operator performs only one operation. The objective is to minimize overall cost or to maximize the overall profit for a given assignment schedule.

Mathematical Representation of an Assignment Problem

If there are *n* jobs to be performed and *n* persons are available for doing this job. Assume all persons can do each job at a time with different payoffs. Let c_{ij} be the cost associated with i^{th} person assigned with j^{th} job. Then the problem is to find the values ' x_{ij} ' ($x_{ij} = 0$ or 1), so that total cost for performing all the jobs is minimum.

Minimize Z = $\sum_{i=1}^{m} \sum_{j=1}^{n} c_{ij} \; x_{ij}$

Such that

$\sum_{j=1}^{n} x_{ij} = 1$ for $i = 1,\ldots,n$

$\sum_{i=1}^{n} x_{ij} = 1$ for $j = 1,\ldots,n$

$x_{ij} = 1$ if i^{th} person is assigned with j^{th} job, 0 otherwise

Connection Between Transportation and Assignment Problem

An assignment problem is a special case of transportation problem in which $m = n$, all a_i and b_j are unity and each x_{ij} is limited to either 0 or 1.

Hungarian Method for Solving an Assignment Problem

1. Prepare a square $n \times n$ matrix. If not, make it square by adding suitable number of dummy rows (or columns) with zero cost elements.
2. Subtract the minimum element of each row from every element of that row.

3. Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns, so that each row and column is having a zero element.
4. Now start making assignments row - wise. Examine each row one by one and assign a '□' to '0' in the rows having single zeros. Then, mark a '×' to all zeroes in the column in which assignment has been made so that no other assignment can be made in the same column
5. Repeat the procedure for the columns.
6. Stop if all *n* assignments have been made, otherwise go to step 7.
7. (i) Mark '✓' to rows having no assignments i.e. in which all zeros have been crossed.

   (ii) Mark '✓' to columns having crossed zeros i.e. ⌧ in the marked row.

   (iii) Mark '✓' to rows having an assignment i.e. '[0]' in the marked column.

   (iv) Draw horizontal and vertical lines to unmarked rows and marked columns.
8. Select the smallest element among all the uncovered elements and subtract this value from all the elements in the matrix not covered by lines and add this value to all the elements that lie at the intersection of two lines.
9. Form a new matrix and repeat step 4 until all assignments have been made.

Hungarian method provides a convenient method of solving assignment problems comparative to other available methods. Solution of assignment problems can be classified in five types as shown below.

8.2 .1 Type I (Single step assignments)

Example1. A department head has four subordinates and four tasks have to be performed. Time (hours) each man would take to perform each task is given below.

How the tasks should be allocated to each subordinate so as to minimize the total man-hours?

Subordinates

Tasks		S_1	S_2	S_3	S_4
	T_1	5	6	8	9
	T_2	6	8	10	6
	T_3	9	5	8	5
	T_4	9	8	7	10

Solution: Given assignment problem is balanced i.e. number of rows and columns are equal. Subtracting minimum element in each row from all the elements of the row, hence inducing a zero element in each row

	S_1	S_2	S_3	S_4
T_1	0	1	3	4
T_2	0	2	4	0
T_3	4	0	3	0
T_4	2	1	0	3

Each column is having a zero element, so matrix is unchanged by subtracting minimum element in each column. Now making assignments on zeros to rows having single zeros and crossing out remaining zeros in same columns and repeating the process with columns if any allocations are left in row assignments

	S_1	S_2	S_3	S_4
T_1	[0]	1	3	4
T_2	~~0~~	2	4	[0]
T_3	4	[0]	3	~~0~~
T_4	2	1	[0]	3

All the assignments have been made in single step.

Optimal assignments: T_1 - S_1 , T_2 - S_4 , T_3 - S_2 , T_4 - S_3

Minimum man hours = (5+6+5+7) hours = 23 hours

8.2 .2 Type II (Multiple step simple assignments)

Example2. There are 4 machines and 4 operators with their respective payoffs (in 100 rupees) as shown below. How the machines should be allocated to each operator so as to minimize the total payoff ?

Operators

Machines		O_1	O_2	O_3	O_4
	M_1	5	4	2	7
	M_2	7	8	3	5
	M_3	5	3	4	6
	M_4	4	6	6	5

Solution: Given assignment problem is balanced i.e. number of rows and columns are equal. Subtracting minimum element in each row from all the elements of the row, hence inducing a zero element in each row

	O_1	O_2	O_3	O_4
M_1	3	2	0	5
M_2	4	5	0	2
M_3	2	0	1	3
M_4	0	2	2	1

Subtracting minimum element in each column from all the elements of each column and making assignments on zeros to rows having single zeros and crossing out remaining zeros in same columns and repeating the process with columns if any allocations are left in row assignments

	O_1	O_2	O_3	O_4
M_1	3	2	[0]	4
M_2	4	5	~~0~~	1
M_3	2	[0]	1	2
M_4	[0]	2	2	~~0~~

Putting a '✓' against 2^{nd} row in first step as it is unassigned, putting a '✓' against 3^{rd} column in second step corresponding to ~~0~~ in the ticked 2^{nd} row, again putting a '✓' against 1^{st} row in third step corresponding to [0] in the ticked 3^{rd} column. Also drawing horizontal and vertical lines to unticked rows and ticked columns

	O_1	O_2	O_3 ✓(ii)	O_4	
M_1	3	2	[0]	4	✓(iii)
M_2	4	5	~~0~~	1	✓(i)
M_3	2	[0]	1	2	
M_4	[0]	2	2	~~0~~	

Forming a new matrix by selecting the minimum element among all the uncovered elements i.e. min [3,2,4,5,1] =1, subtracting this value from all the elements in the matrix not covered by lines, adding this value to all the elements that lie at the intersection of two lines and making assignments rowise and then columnwise at single zeros

	O_1	O_2	O_3	O_4
M_1	2	1	[0]	3
M_2	3	4	~~0~~	[0]
M_3	2	[0]	2	2
M_4	[0]	2	3	~~0~~

Optimal assignments: M_1 - O_3 , M_2 - O_4 , M_3 - O_2 , M_4 - O_1

Minimum payoff = Rs. (2+5+3+4) 100 = Rs. 1400/-

8.2 .3 Type III (Alternate optimum assignments)

In this type of problems, after giving assignments at zeros, uncovered zeros (not having '□' or '×' marks) are left in the matrix. We arbitrarily provide assignment to a '0' and cross out other zeros in same row and column. Multiple optimal solutions are found out in such kind of problems.

Example 3. Assign 4 trucks to 3 destinations so that the distance travelled is minimized. The matrix shows the distance in miles.

Trucks

Destinations		T_1	T_2	T_3	T_4
	D_1	28	35	31	33
	D_2	20	25	22	24
	D_3	15	20	24	26

Solution: Given assignment problem is unbalanced i.e. number of rows and columns are not equal. Thereby adding a dummy destination D_4 with zero distances and subtracting minimum element in each row from all the elements of the row, hence inducing a zero element in each row

	T_1	T_2	T_3	T_4
D_1	0	7	3	5
D_2	0	5	2	4
D_3	0	5	9	11
D_4	0	0	0	0

Each column is having a zero element, so matrix is unchanged by subtracting minimum element in each column. Now making assignments on zeros to rows having single zeros and crossing out remaining zeros in same columns and repeating the process with columns if any allocations are left in row assignments

	T_1	T_2	T_3	T_4
D_1	$\boxed{0}$	7	3	5
D_2	$\not{0}$	5	2	4
D_3	$\not{0}$	5	9	11
D_4	$\not{0}$	$\boxed{0}$	$\not{0}$	$\not{0}$

Putting a '✓' against 2nd and 3rd rows in first step as they are unassigned, putting a '✓' against 1st column in second step corresponding to $\not{0}$ in the ticked 2nd and 3rd rows, again putting a '✓' against 1st row in third step corresponding to $\boxed{0}$ in the ticked 1st column. Also drawing horizontal and vertical lines to unticked rows and ticked columns

	✓(ii)				
	T_1	T_2	T_3	T_4	
D_1	$\boxed{0}$	7	3	5	✓(iii)
D_2	$\not{0}$	5	2	4	✓(i)
D_3	$\not{0}$	5	9	11	✓(i)
D_4	$\not{0}$	$\boxed{0}$	$\not{0}$	$\not{0}$	

Forming a new matrix by selecting the minimum element among all the uncovered elements i.e. min [7,3,5,2,4,9,11] =2, subtracting this value from all the elements in the matrix not covered by lines, adding this value to all the elements that lie at the intersection of two lines and making assignments rowise and then columnwise at single zeros

	T_1	T_2	T_3	T_4
D_1	$\boxed{0}$	5	1	3
D_2	$\not{0}$	3	$\boxed{0}$	2
D_3	$\not{0}$	3	7	9
D_4	2	$\boxed{0}$	$\not{0}$	$\not{0}$

Putting a '✓' against 3^{rd} row in first step as it is unassigned, putting a '✓' against 1^{st} column in second step corresponding to $\not{0}$ in the ticked 3^{rd} row, again putting a '✓' against 1^{st} row in third step corresponding to $\boxed{0}$ in the ticked 1^{st} column. Also drawing horizontal and vertical lines to unticked rows and ticked columns

	✓(ii)				
	T_1	T_2	T_3	T_4	
D_1	$\boxed{0}$	5	1	3	✓(iii)
D_2	$\not{0}$	3	$\boxed{0}$	2	
D_3	$\not{0}$	3	7	9	✓(i)
D_4	2	$\boxed{0}$	$\not{0}$	$\not{0}$	

Forming a new matrix by selecting the minimum element among all the uncovered elements i.e. min [5,1,3,7,9] =1, subtracting this value from all the elements in the matrix not covered by lines, adding this value to all the elements that lie at the intersection of two lines and making assignments rowise and then columnwise at single zeros

	T_1	T_2	T_3	T_4
D_1	$\not{0}$	4	$\not{0}$	2
D_2	1	3	$\boxed{0}$	2
D_3	$\boxed{0}$	2	6	8
D_4	3	$\boxed{0}$	$\not{0}$	$\not{0}$

Putting a '✓' against 1^{st} row in first step as it is unassigned, putting a '✓' against 1^{st} and 3^{rd} columns in second step corresponding to $\not{0}$ in the ticked 1^{st} row, again putting a '✓' against 2^{nd} and 3^{rd} rows in the third step corresponding to $\boxed{0}$ in the ticked 1^{st} and 3^{rd} columns. Also drawing horizontal and vertical lines to unticked rows and ticked columns

	✓(ii)		✓(ii)		
	T_1	T_2	T_3	T_4	
D_1	$\not{0}$	4	$\not{0}$	2	✓(i)
D_2	1	3	$\boxed{0}$	2	✓(iii)
D_3	$\boxed{0}$	2	6	8	✓(iii)
D_4	3	$\boxed{0}$	$\not{0}$	$\not{0}$	

Forming a new matrix by selecting the minimum element among all the uncovered elements i.e. min [4,2,3,6,8] =2, subtracting this value from all the elements in the matrix not covered by lines, adding this value to all the elements that lie at the intersection of two lines and making assignments rowwise and then columnwise at single zeros

	T_1	T_2	T_3	T_4
D_1	0	2	0	0
D_2	1	1	0	0
D_3	0	0	6	6
D_4	5	0	2	0

Now trying to make assignments on zeros we see that no row or column is having single zero, thus multiple solutions can be found out to this assignment problem by arbitrarily providing assignment at a zero and crossing out other zeros in same row and column. Some of the solutions are shown below

1st solution:

	T_1	T_2	T_3	T_4
D_1	$\boxed{0}$	2	$\cancel{0}$	$\cancel{0}$
D_2	1	1	$\boxed{0}$	$\cancel{0}$
D_3	$\cancel{0}$	$\boxed{0}$	6	6
D_4	5	$\cancel{0}$	2	$\boxed{0}$

Optimal assignments: $D_1 - T_1$, $D_2 - T_3$, $D_3 - T_2$, $D_4 - T_4$

Minimum Distance = (28+ 22 + 20) miles = 70 miles

2nd solution:

	T_1	T_2	T_3	T_4
D_1	$\cancel{0}$	2	$\boxed{0}$	$\cancel{0}$
D_2	1	1	$\cancel{0}$	$\boxed{0}$
D_3	$\boxed{0}$	$\cancel{0}$	6	6
D_4	5	$\boxed{0}$	2	$\cancel{0}$

Optimal assignments: $D_1 - T_3$, $D_2 - T_4$, $D_3 - T_1$, $D_4 - T_2$

Minimum Distance = (31+ 24 + 15) miles= 70 miles

3rd solution:

	T_1	T_2	T_3	T_4
D_1	$\cancel{0}$	2	$\cancel{0}$	$\boxed{0}$
D_2	1	1	$\boxed{0}$	$\cancel{0}$
D_3	$\boxed{0}$	$\cancel{0}$	6	6
D_4	5	$\boxed{0}$	2	$\cancel{0}$

Optimal assignments: $D_1 - T_4$, $D_2 - T_3$, $D_3 - T_1$, $D_4 - T_2$

Minimum Distance = (33+ 22 + 15) miles= 70 miles

8.2 .4 Type IV (Restrictions on assignments)

In this type of problems, some assignments are restricted i.e. some particular assignments cannot be made. An infinite number is induced at such places and hence those positions are never assigned.

Example4. There are 4 jobs and 4 workers with their respective payoffs (in dollars) as shown below. Somehow W_2 is incompetent for J_3 and W_3 is not interested in J_1. How the jobs should be assigned to each worker so as to minimize the total payoff ?

Workers

Jobs		W_1	W_2	W_3	W_4
	J_1	50	40	-	40
	J_2	30	20	50	40
	J_3	50	-	30	20
	J_4	40	30	20	50

Solution: Allocating infinite cost at restricted places and subtracting minimum element in each row from all the elements of the row, hence inducing a zero element in each row

	W_1	W_2	W_3	W_4
J_1	10	0	∞	0
J_2	10	0	30	20
J_3	30	∞	10	0
J_4	20	10	0	30

Subtracting minimum element in each column from all the elements of each column and making assignments on zeros to rows having single zeros and crossing out remaining zeros in same columns and repeating the process with columns if any allocations are left in row assignments

	W_1	W_2	W_3	W_4
J_1	0	0	∞	~~0~~
J_2	0	0	30	20
J_3	20	∞	10	[0]
J_4	10	10	[0]	30

There are unassigned and uncrossed zeros after making assignments in all the rows and columns on single zeros. Hence multiple optimal solutions exist as shown below

1st solution

	W_1	W_2	W_3	W_4
J_1	[0]	~~0~~	∞	~~0~~
J_2	~~0~~	[0]	30	20
J_3	20	∞	10	[0]
J_4	10	10	[0]	30

Optimal assignments: J_1 - W_1 , J_2 - W_2 , J_3 - W_4 , J_4 - W_3

Minimum payoff = (50 + 20 + 20 + 20)$= 110$

2nd solution

	W_1	W_2	W_3	W_4
J_1	~~0~~	[0]	∞	~~0~~
J_2	[0]	~~0~~	30	20
J_3	20	∞	10	[0]
J_4	10	10	[0]	30

Optimal assignments: J_1 - W_2 , J_2 - W_1 , J_3 - W_4 , J_4 - W_3

Minimum payoff = (40 + 30 + 20 + 20)$ = 110$

8.2 .5 Type V (Maximization Problems)

Here objective of assignments is maximization instead of minimization and to solve such assignment problems, we make each entry negative and then apply Hungarian method as usual.

Example5. A pharmaceutical firm has a team of 4 salesmen and there are 4 districts. Estimated profit (in 100 dollars) each salesman can yield per day in different districts is given below. How the districts should be assigned to each salesman so as to maximize the total profit?

Salesmen

		S_1	S_2	S_3	S_4
Districts	D_1	18	10	15	11
	D_2	16	12	14	14
	D_3	17	17	12	13
	D_4	15	12	13	16

Solution: Since objective function is of maximization, making each entry negative so that Hungarian method may be applied

	S_1	S_2	S_3	S_4
D_1	-18	-10	-15	-11
D_2	-16	-12	-18	-18
D_3	-17	-17	-12	-13
D_4	-15	-12	-13	-16

Subtracting minimum element (most negative) in each row from all the elements of the row, hence inducing a zero element in each row

	S_1	S_2	S_3	S_4
D_1	0	8	3	7
D_2	2	6	0	0
D_3	0	0	5	4
D_4	1	4	3	0

Each column is having a zero element, thereby making assignments in rows having single zeros, crossing out remaining zeros in respective columns and then repeating the process in columns

	S_1	S_2	S_3	S_4
D_1	[0]	8	3	7
D_2	2	6	[0]	~~0~~
D_3	~~0~~	[0]	5	4
D_4	1	4	3	[0]

Optimal assignments: D_1 - S_1 , D_2 - S_3, D_3 - S_2 , D_4 - S_4

Maximum Profit per day =(18+14+17+16)100$ = 6500$