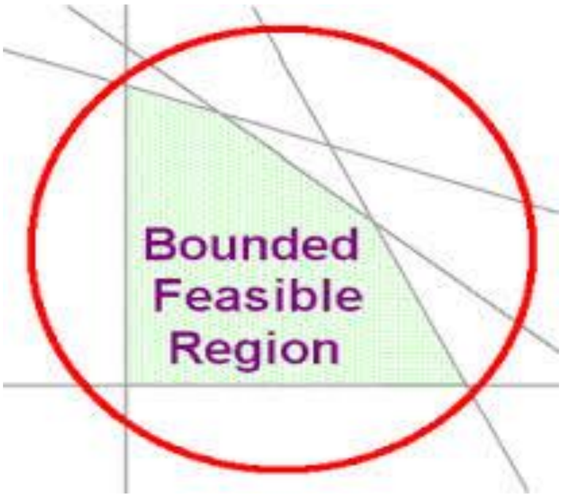
UNIT-4
Chapter6
Linear Programming



Linear Programming

6.1 Introduction

Operations Research is a scientific approach to problem solving for executive management. It came into existence in England during the Second World War, to make decisions regarding the best utilization of war material. In India, it was first introduced in 1949 at regional research laboratory in Hyderabad for planning and organizing research projects .

Linear Programming is the most important optimization (maximization or minimization) technique developed in the field of Operations Research. In practice, linear programming is the process of optimizing an objective subject to a given set of constraints. A general linear programming problem (lpp) is of the form:

Max (or Min) $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

Subject to

$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \leq$ (or = or $\geq$) b_1

$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \leq$ (or = or $\geq$) b_2

$\vdots$

$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n \leq$ (or = or $\geq$) b_m

$x_1, x_2, \ldots\ldots\ldots. x_n \geq 0$

Here $x_1, x_2, \ldots\ldots\ldots., x_n$ are called decision variables

Feasible Solution: A solution of a linear programming problem which also satisfies the non-negativity restrictions is called its feasible solution.

Optimum Solution: A feasible solution which optimizes (maximizes or minimizes) the objective function of the linear programming problem is called an optimal solution.

6.2 Formulation of Linear Programming Problem (LPP)

Example1 A firm produces 3 types of clothes say A, B and C. 3 kind of wools viz. red, yellow and blue are required for production. One unit length of type A cloth requires 3 meters red wool, 4 meters yellow wool and 5 meters blue wool. One unit length of type B cloth requires 2 meters red wool, 5 meters yellow wool and 6 meters blue wool, whereas one unit length of type C cloth requires 4 meters red wool and 7 meters blue wool. The firm has only a stock of 8 meters red wool, 10 meters yellow wool and 14 meters blue wool. Also the profit obtained from producing one unit length of type A cloth is Rs.3/−, of type B is Rs.4/−, and that of type C is Rs.2/− per unit length. Formulate how the firm should use the available material, so as to maximize the income from the finished cloth.

Formulation: Let x_1, $x_2$and x_3 be the quantity (in meters), to be produced of type A, B and C clothes respectively. Table 1.1 shows the objective of the lpp and requirements (availabilities) of material in concise form.

Table 1.1: Maximization of Profit

Wool(meters)	A(x_1)	B(x_2)	C(x_3)	Availability
Red	3	2	4	8
Yellow	4	5	0	10
Blue	5	6	7	14
Profit	$Rs.3/-$	$Rs.4/-$	$Rs.2/-$	Maximize Profit (Objective)

∴Lpp may be formulated as

Maximize Z = $3x_1 + 4x_2 + 2x_3$

Subject to

$3x_1 + 2x_2 + 4x_3 \leq 8$

$4x_1 + 5x_2 \leq 10$

$5x_1 + 6x_2 + 7x_3 \leq 14$

$x_1, x_2, x_3 \geq 0$

Example 2 A farmer has a 100 acre farm. He can sell all tomatoes, potatoes and radish he raises. The price he can obtain is Rs.3/- per kg. for tomatoes, Rs.2/- per kg. for potatoes and Rs.1/- per kg. for radish. The average yield per acre is 2,000 kg. of tomatoes, 3,000 kg. of potatoes and 1,000 kg. of radish. Fertilizer is available at Rs.1.5/- per kg. and amount required per acre is 100kg each for tomatoes and radish and 50kg for per acre of potatoes. Labour required for sloughing, cultivating and harvesting per acre is 7 man days for tomatoes, 6 man days for potatoes and 5 man days for radish. A total of 500 man days of labour are available at Rs.50 per man day. Formulate the problem as lpp to maximize farmer's total profit .

Formulation: Let x_1, $x_2$and x_3 acres of land be allotted for tomatoes, potatoes and radish respectively. Table 1.2 shows the objective of the lpp and requirements (availabilities) of material in concise form.

Table 1.2: Maximization of Profit

	Tomatoes (x_1) acres	Potatoes (x_2) acres	Radish (x_3) acres	Availability 100acres
S.P. (per kg)	$Rs.$3/-	Rs2 /-	Rs1/-	
Yield (per acre)	2,000kg	3,000kg	1,000kg	
Fertilizer used (per acre)	$Rs.$(1.5×100)	$Rs.$(1.5×50)	Rs.(1.5×100)	
Man-Days(per acre)	7	6	5	500

Profit = S.P − Expenditure

= Rs.[3(2, 000)x_1 + 2(3, 000)x_2 + 1(1, 000)x_3]

− Rs.[150x_1 + 75x_2 + 150x_3] − Rs.50[7x_1 + 6x_2 + 5x_3]

⇒ Profit = Rs.[5500x_1 + 5625x_2 + 600x_3]

∴ LPP may be formulated as

Maximize Z = 5500x_1 + 5625x_2 + 600x_3

Subject to

$x_1 + x_2 + x_3 \leq 100$

$x_1 + 6x_2 + 5x_3 \leq 500$

$x_1, x_2, x_3 \geq 0$

6.2 <u>Graphical method for solution of Linear Programming Problem (LPP)</u>

Graphical method for solving lpp is applicable to those problems which involve only 2 variables. There are 2 methods of solving a lpp graphically.

i. Corner-point method
ii. Iso-profit or Iso-cost method

In corner point method, feasible region is located by marking common region bounded by constraint inequations. The vertices of the corner points in the feasible region are feasible solutions. Optimum solution is found out by checking value of objective function at these points. In iso-profit or iso-cost method, profit or cost line is a line drawn parallel to objective function in the feasible region.

As profit increases, the iso-profit lines move farther to the right, away from the origin while iso-cost lines moves towards the origin for minimization purpose. Here we will be using Corner-point method to solve a lpp.

Type I : Maximization Problem

Example 3 A firm manufactures 2 types of hats A and B and sells them at a profit of Rs.10 and Rs.6 respectively. Each hat of type A requires twice as much labour time as of type B. If all the hats are of type B only, firm can produce a total of 500 hats a day. The market limits daily sales of type A and B to 200 and 250 respectively. Determine the number of hats of each type to be produced so as to maximize the profit.

Formulation: Let x_1 hats of type A and x2 hats of type B be produced daily and t be the time required to produce a hat of type B.

Table 1.3 shows the objective of the LPP and requirements (availabilities) of material in concise form.

Table 1.3: Maximization of Profit

	A(x_1)	B(x_2)	
Daily Sales	Maximum 200	Maximum 250	
Time	2t	T	Maximum 500t
Profit	Rs.10	Rs.6	Maximize Profit (Objective)

∴ LPP may be formulated as:

Maximize Z = $10x_1 + 6x_2$

Subject to

$x_1 \leq 200$

$x_2 \leq 250$

$2x_1 + x_2 \leq 500$

$x_1, x_2 \geq 0$

Figure 1.1 shows the solution of given LPP using corner point method. Dotted line shows the iso-profit line.

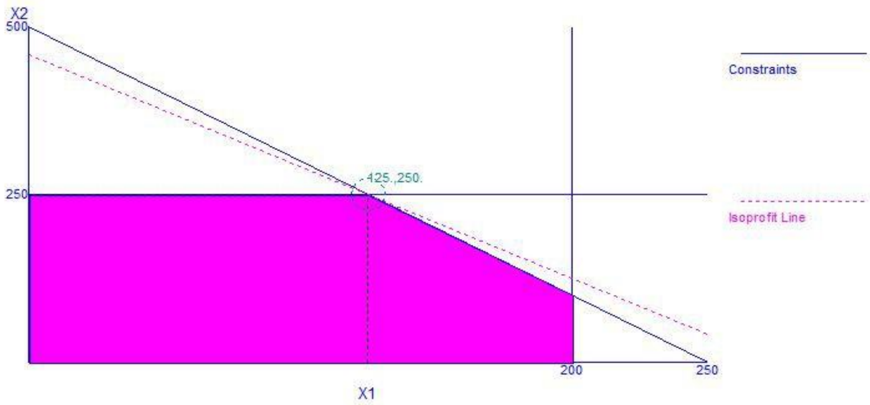


Figure 1.1: Solution of LPP using graphical method

Corner Point	Z = $10x_1 + 6x_2$
(0, 0)	0
(200, 0)	2000
(0, 250)	1500
(200, 100)	2600
(125, 250)	2750

∴Maximum profit is *Rs.*2,750 by producing 125 hats of type A and 250 hats of type B.

Type II: Minimization Problem

Example 4 A firm produces scientific and graphic calculators. Due to limitations in production capacity, not more than 150 scientific and 170 graphic calculators can be made daily. Speculations indicate an expected demand of at least 100 scientific and 80 graphing calculators each day. To satisfy a shipping contract, a total of at least 200 calculators much be shipped each day. If each scientific calculator costs *Rs.*150 and each graphic calculator costs *Rs.*100, how many of each type should be made daily to minimize net cost?

Solution: Let x_1 scientific and x_2 graphic calculators be produced daily.

Table 1.4 shows the objective of the lpp and requirements (availabilities) of material in concise form.

Table 1.4: Minimization of Cost

	Scientific calculators (x_1)	Graphic calculators (x_2)	
Production Limit	Maximum 150	Maximum 170	
Daily Demand	Minimum 100	Minimum 80	
Shipping Requirement	x_1	x_2	Minimum 200
Cost	Rs.150	Rs. 100	Minimize Cost (Objective)

∴lpp may be formulated as

Minimize $Z = 150x_1 + 100x_2$

Subject to

$x_1 \leq 150$

$x_2 \leq 170$

$x_1 \geq 100$

$x_2 \geq 80$

$x_1 + x_2 \geq 200$

$x_1 , x_2 \geq 0$

Figure1.2 shows the solution of given lpp using corner point method. Dotted line shows the iso-cost line.

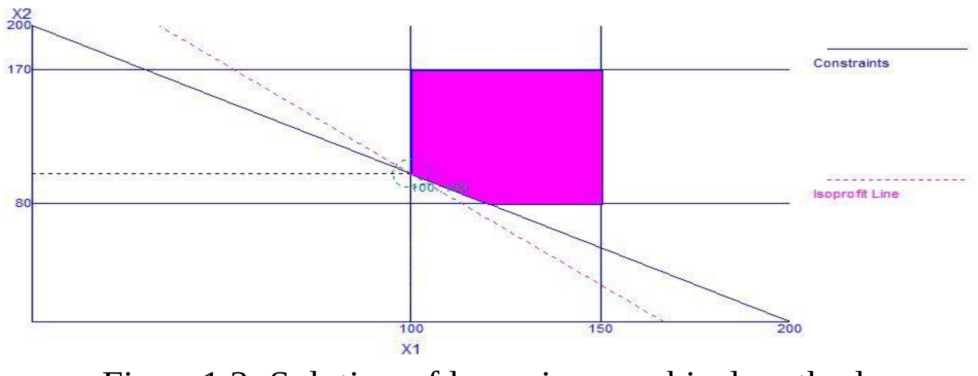


Figure1.2: Solution of lpp using graphical method

Corner Point	Z=150x_1+100x_2
(100,100)	25,000
(100,170)	32,000
(120,80)	26,000
(150,80)	30,500
(150,170)	39,500

∴Minimum cost is $Rs.$25,000 by producing 100 scientific and 100 graphic calculators.

Type III: Multiple Optimal Solutions

If in a given lpp, coefficients in the objective function are multiples of a constraint inequation i.e. slope of objective function line is same as slope of a constraint equation line, then the lpp has more than one (multiple optimal) solutions.

Example5 Solve the lpp

Maximize $Z = 4x_1+6x_2$

Subject to

$x_1 \leq 3$

$x_2 \geq 1.5$

$2x_1+3x_2 \leq 12$

$x_1, x_2 \geq 0$

Figure 1.3 shows the solution of given lpp using corner point method.

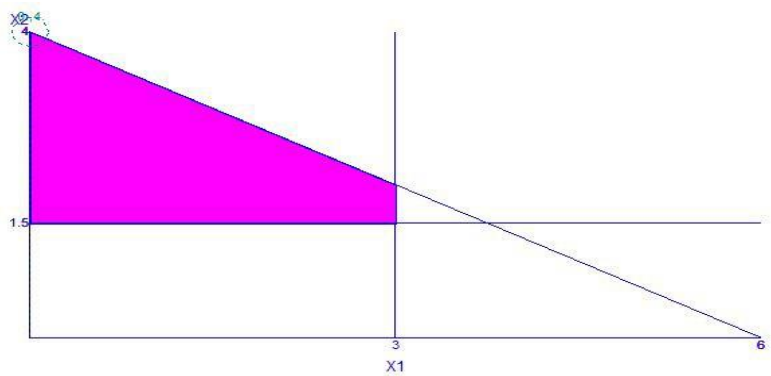


Figure1.3: Solution of lpp using corner point method

Corner Point	$Z = 4x_1+6x_2$
(0,4)	24
(0,1.5)	9
(3,2)	24
(3,1.5)	21

∴Maximum Z is 24 at (0,4) and (3,2)

Type IV: Unbounded Solution

If in a linear programming problem with maximization objective, feasible region is unbounded i.e. maximum value is infinite then it is said to have an unbounded solution.

Example 6 Solve the lpp

Maximize $Z = 2x_1+5x_2$

Subject to

$2x_1+x_2 \geq 12$

$5x_1+7x_2 \geq 35$

$x_1,x_2 \geq 0$

Figure 1.4 shows the graph of given lpp using corner point method.

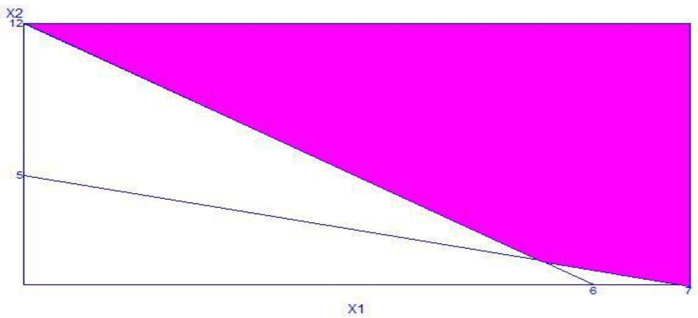


Figure 1.4: Solution of lpp using corner point method

As seen in graph, Maximum Z is infinite and hence the solution is unbounded.

Type V: Infeasible Solution

If in the given lpp, there is no common solution region in the first quadrant, then the lpp is said to have infeasible solution

Example7 Solve the lpp

Maximize $Z = x_1 + x_2$

Subject to

$x_1 + 2x_2 \geq 83$

$x_1 + 4x_2 \leq 12$

$x_1, x_2 \geq 0$

Figure 1.5 shows the graph of given lpp using corner point method.

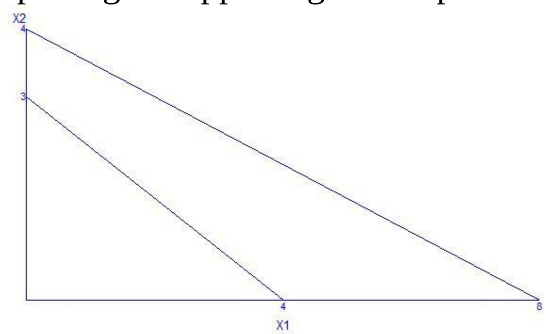


Figure1.5: Solution of lpp using corner point method

As shown in graph, there is no common solution space in the feasible region.

∴the solution is infeasible.

6.3 <u>Solving Linear Programming Problems Having Two or More Variables</u>

We can solve two variables lpp easily using graphical methods as explained in the previous section but for three or more variable problems, more advanced methods are required. Here are some more definitions required to explore these methods.

<u>Slack Variables</u> : A slack variable represents deficient quaintly of resources. It is added to $' < '$ or $' \leq '$ type constraint in order to get an equality constraint.

<u>Surplus Variables</u> :A surplus variable is the amount by which solution values exceed a resource. It is added to $' > '$ or $' \geq '$ type constraint in order to get an equality constraint.

<u>Artificial Variables</u>: Artificial variables are added to those constraints with $' =',\ ' > '$ or $' \geq '$ signs. An Artificial variable is added to a constraint to get an initial solution to lpp. Artificial variables have no meaning in a physical sense and must be departed to attain a feasible solution to the lpp.

Basic & Non-Basic Variables: If there are $(m+n)$ variables and m constraints, only m variables can form a basic solution taking remaining n variables as zero. The m variables forming the solution are called basic variables and remaining n variables are called non-basic variables. Total number of solutions with $(m+n)$ variables and m constraints are ${}^{m+n}C_m$.

Basic Feasible solution: If the solution yields non-negative values to all basic variables, it is called basic feasible solution, otherwise it is infeasible.

Non-Degenerate Basic Feasible Solution: A basic solution is said to be non-degenerate if all the 'm' basic variables are having positive (non-zero) values. The solution is degenerate if one or more of the 'm' basic variables vanish. Degeneracy literally means that the problem will not generate new solution in further iterations.

Example 8 Find all basic feasible solutions of the equations:

$$2x_1 + 6x_2 + 2x_3 + x_4 = 3$$

$$3x_1 + 2x_2 + 2x_3 + 3x_4 = 1$$

Solution: Total number of basic solutions = ${}^4C_2 = 6$

Ist solution: Let x_3 and x_4 be basic variables

$$\therefore \left.\begin{matrix} x_1 = 0 \\ x_2 = 0 \end{matrix}\right] (non\ basic) \Rightarrow \left.\begin{matrix} 2x_3 + x_4 = 3 \\ 2x_3 + 3x_4 = 1 \end{matrix}\right] \Rightarrow \begin{matrix} x_3 = 2 \\ x_4 = -1 \end{matrix}$$ ∴ non feasible and degenerate

IInd solution: Let x_1 and x_4 be basic variables

$$\therefore \left.\begin{matrix} x_2 = 0 \\ x_3 = 0 \end{matrix}\right] (non\ basic) \Rightarrow \left.\begin{matrix} 2x_1 + x_4 = 3 \\ 3x_1 + 3x_4 = 1 \end{matrix}\right] \Rightarrow \begin{matrix} x_1 = \frac{-8}{3} \\ x_4 = \frac{-7}{3} \end{matrix}$$ ∴non feasible and degenerate

IIIrd solution: Let x_1 and x_2 be basic variables

$$\therefore \left.\begin{matrix} x_3 = 0 \\ x_4 = 0 \end{matrix}\right] (non\ basic) \Rightarrow \left.\begin{matrix} 2x_1 + 6x_2 = 3 \\ 3x_1 + 2x_2 = 1 \end{matrix}\right] \Rightarrow \begin{matrix} x_1 = 0 \\ x_2 = \frac{1}{2} \end{matrix}$$ ∴basic feasible and degenerate

IVth solution: Let x_2 and x_3 be basic variables

$$\therefore \left.\begin{matrix} x_1 = 0 \\ x_4 = 0 \end{matrix}\right] (non\ basic) \Rightarrow \left.\begin{matrix} 6x_2 + 2x_3 = 3 \\ 2x_2 + 2x_3 = 1 \end{matrix}\right] \Rightarrow \begin{matrix} x_2 = \frac{1}{2} \\ x_3 = 0 \end{matrix}$$ ∴basic feasible and degenerate

Vth solution: Let x_2 and x_4 be basic variables

$$\therefore \left.\begin{matrix} x_1 = 0 \\ x_3 = 0 \end{matrix}\right] (non\ basic) \Rightarrow \left.\begin{matrix} 6x_2 + x_4 = 3 \\ 2x_2 + 3x_4 = 1 \end{matrix}\right] \Rightarrow \begin{matrix} x_2 = \frac{1}{2} \\ x_4 = 0 \end{matrix}$$ ∴basic feasible and degenerate

VI[th] solution: Let x_1 and x_3 be basic variables

$$\therefore \left.\begin{matrix} x_2 = 0 \\ x_4 = 0 \end{matrix}\right] (non\ basic) \Rightarrow \left.\begin{matrix} 2x_1 + 2x_3 = 3 \\ 3x_1 + 2x_3 = 1 \end{matrix}\right] \Rightarrow \begin{matrix} x_1 = -2 \\ x_3 = \frac{7}{2} \end{matrix}$$ ∴not feasible and degenerate

6.3.1 <u>Simplex Method</u>

Simplex method is used for solving linear programming problems having two or more variables. Invented by George Dantzig in 1947, it tests adjacent vertices of the feasible set so that at each new vertex, the value of objective function is improved until an optimal solution is attained.

<u>Algorithm to solve lpp in which all constrains have ≤ signs, keeping *b_i's* +ve</u>

1. If the problem is of minimization, change it to maximization by putting Min$Z$= Max$\bar{Z}$, where $\bar{Z} = -Z$
2. Make all *b_i's* (*i=l,…….,m*) *+ve*, if not originally.
3. Rewrite the lpp by adding slack variables to all constraints inequations having$' \leq '$ signs.
4. Construct the initial table by putting decision variables ($x_1, x_2 \ldots x_n$) equal to zero.
5. Check $\Delta_j = C_B x_j - C_j \forall j$ and put the value below each variable column.
6. If all $\Delta_j \geq 0$, optimal solution has been attained, otherwise proceed to step 7.
7. Find the most negative $\Delta_j$value; corresponding column is the entering variable column.
8. Find the departing variable row where ratio $\frac{V_B}{x_k}, x_k > 0$ is minimum.
9. Find the pivotal entry at the intersection of entering variable column and departing variable row.
10. Make pivotal element one and remaining elements in the column zero, and return to step 5.

Example 9 Solve the lpp

Max $Z = 3x_1 + 2x_2$

Subject to

$x_1 + x_2 \leq 4$

$x_1 - x_2 \leq 2$

$x_1, x_2 \geq 0$

Solution: Given lpp is of maximization type and also all the b_i's are positive. Adding slack variables to constraints, lpp is given by:

Max Z= $3x_1 + 2x_2 + 0s_1 + 0s_2$

Subject to

$$x_1 + x_2 + s_1 = 4$$
$$x_1 - x_2 + s_2 = 2$$
$$x_1, x_2, s_1, s_2 \geq 0$$

The simplex table is given as:

B.V.	C_B	V_B	x_1 (C_j = 3)	x_2 (C_j = 2)	s_1 (C_j = 0)	s_2 (C_j = 0)	Min Ratio $\frac{V_B}{x_k}, x_k > 0$
s_1	0	4	1	1	1	0	4
s_2	0	2	[1]	-1	0	1	2 →
	Z = 0		-3 ↑	-2	0	0	$\Delta_j = C_B x_j - C_j$
			x_1 enters and s_2 leaves				
s_1	0	2	0	[2]	1	-1	→
x_1	3	2	1	-1	0	1	
	Z = 6		0	-5 ↑	0	3	$\Delta_j = C_B x_j - C_j$
			x_2 enters and s_1 leaves				
x_2	2	1	0	1	$\frac{1}{2}$	$-\frac{1}{2}$	
x_1	3	3	1	0	$\frac{1}{2}$	$\frac{1}{2}$	
	Z = 11		0	0	$\frac{5}{2}$	$\frac{1}{2}$	$\Delta_j \geq 0$

$\therefore$ Maximum Z = 11 at $x_1 = 3$ and $x_2 = 1$

Example 10: Solve the lpp

Min Z = $-x_1 + x_2 - 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$, $-2x_1 + x_3 \geq -2$, $2x_1 - 2x_2 + 3x_3 \leq 0$, $x_1, x_2, x_3 \geq 0$

Solution: Rewriting the lpp in standard form   Adding slack variables

Max $\bar{Z} = x_1 - x_2 + 3x_3$

Subject to

$x_1 + x_2 + x_3 \leq 10$

$2x_1 - x_3 \leq 2$

$2x_1 - 2x_2 + 3x_3 \leq 0$

$x_1, x_2, x_3 \geq 0$

Max $\bar{Z} = x_1 - x_2 + 3x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$x_1 + x_2 + x_3 + s_1 = 10$

$2x_1 - x_3 + s_2 = 2$

$2x_1 - 2x_2 + 3x_3 + s_3 = 0$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

The simplex table is given as:

			$C_j \rightarrow$	1	-1	3	0	0	0	
B.V.	C_B	V_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio $\frac{V_B}{x_k}, x_k > 0$
s_1	0	10	1	1	1	1	0	0	10
s_2	0	2	2	0	-1	0	1	0	
s_3	0	0	2	-2	[3]	0	0	1	0→
	$\bar{Z} = 0$		-1	1	-3 ↑	0	0	0	$\Delta_j = C_B x_j - C_j$
s_1	0	10	$\frac{1}{3}$	[$\frac{5}{3}$]	0	1	0	$\frac{-1}{3}$	→
s_2	0	2	$\frac{8}{3}$	$\frac{-2}{3}$	0	0	1	$\frac{1}{3}$	
x_3	3	0	$\frac{2}{3}$	$\frac{-2}{3}$	1	0	0	$\frac{1}{3}$	
	$\bar{Z} = 0$		1	-1↑	0	0	0	1	$\Delta_j = C_B x_j - C_j$
x_2	-1	6	$\frac{1}{5}$	1	0	$\frac{3}{5}$	0	$\frac{-1}{5}$	
s_2	0	6	$\frac{14}{5}$	0	0	$\frac{2}{5}$	1	$\frac{1}{5}$	
x_3	3	4	$\frac{4}{5}$	0	1	$\frac{2}{5}$	0	$\frac{1}{5}$	
	$\bar{Z} = 6$		$\frac{6}{5}$	0	0	$\frac{3}{5}$	0	$\frac{4}{5}$	$\Delta_j \geq 0$

$\therefore$ Minimum Z = $-$Maximum $\bar{Z}$ =-6 at $x_1 = 0$, $x_2 = 6$ and $x_3 = 4$

Multiple optima (Alternate Optimal Solution)

Once we get an optimal solution of a lpp, in final simplex table $\Delta_j > 0$ for all non-basic variables and specifically $\Delta_j = 0$ for basic variables. But if $\Delta_j = 0$ for any non-basic variable column in the optimal simplex table, then an alternate optimal solution exists.

Example 11 Solve the lpp

Max Z = $6x_1 + 10x_2 + 2x_3$

Subject to

$2x_1 + 4x_2 + 3x_3 \leq 40$

$x_1 + x_2 \leq 10$

$2x_2 + x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

Solution: Adding slack variables

Max Z = $6x_1 + 10x_2 + 2x_3 + 0s_1 + 0s_2 + 0s_3$

Subject to

$2x_1 + 4x_2 + 3x_3 + s_1 = 40$

$x_1 + x_2 + s_2 = 10$

$2x_2 + x_3 + s_3 = 12$

$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$

The simplex table is given as:

		$C_j \rightarrow$	6	10	2	0	0	0	
B.V.	C_B	V_B	x_1	x_2	x_3	s_1	s_2	s_3	Min Ratio $\frac{V_B}{x_k}, x_k > 0$
s_1	0	40	2	4	3	1	0	0	10
s_2	0	10	1	1	0	0	1	0	10
s_3	0	12	0	[2]	1	0	0	1	6→
	Z = 0		-6	-10↑	-2	0	0	0	$\Delta_j = C_B x_j - C_j$
			x_2 enters and s_3 leaves						

s_1	0	16	2	0	1	1	0	-2	8
s_2	0	4	$\boxed{1}$	0	$\frac{-1}{2}$	0	1	$\frac{-1}{2}$	4→
x_2	10	6	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	-
	Z = 60		-6↑	0	3	0	0	5	$\Delta_j = C_B x_j - C_j$
			x_1 enters and s_2 leaves						
s_1	0	8	0	0	$\boxed{2}$	1	-2	-1	4→
x_1	6	4	1	0	$\frac{-1}{2}$	0	1	$\frac{-1}{2}$	
x_2	10	6	0	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$	12
	Z = 84		0	0	0↑	0	6	2	$\Delta_j \geq 0$

Solution is optimal and x_3 is non- basic still $\Delta_3 = 0$, ∴ an alternate optimal solution exists. Forcing in x_3 , hence s_1 leaves.

x_3	2	4	0	0	1	$\frac{1}{2}$	-1	$\frac{-1}{2}$	
x_1	6	6	1	0	0	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{-3}{4}$	
x_2	10	4	0	1	0	$\frac{-1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	
	Z = 84		0	0	0	0	6	2	$\Delta_j \geq 0$

∴ Maximum Z = 84 at $x_1 = 4$, $x_2 = 6$ and $x_3 = 0$ or $x_1 = 6$, $x_2 = 4$ and $x_3 = 4$

Unbounded solution

If in a simplex table, corresponding to most negative Δ_j, all the entries in the corresponding column are negative or zero, then no variable can leave the basis and the problem has an unbounded solution.

Example 12 Solve the lpp

Max Z = $4x_1 + x_2 + 3x_3 + 5x_4$

Subject to

$-4x_1 + 6x_2 + 5x_3 + 4x_4 \leq 20$

$-3x_1 - 2x_2 + 4x_3 + x_4 \leq 10$

$-8x_1 - 3x_2 + 3x_3 + 2x_4 \leq 20$

$x_1, x_2, x_3, x_4 \geq 0$

Solution: Adding slack variables

Max z = $4x_1 + x_2 + 3x_3 + 5x_4 + 0s_1 + 0s_2 + 0s_3$

Subject to

$-4x_1 + 6x_2 + 5x_3 + 4x_4 + s_1 = 20$

$-3x_1 - 2x_2 + 4x_3 + x_4 + s_2 = 10$

$-8x_1 - 3x_2 + 3x_3 + 2x_4 + s_3 = 20$

$x_1, x_2, x_3, x_4, s_1, s_2, s_3 \geq 0$

The simplex table is given as:

		$C_j \rightarrow$	4	1	3	5	0	0	0	
B.V.	C_B	V_B	x_1	x_2	x_3	x_4	s_1	s_2	s_3	Min Ratio $\frac{V_B}{x_k}, x_k > 0$
s_1	0	20	-4	6	5	[4]	1	0	0	20/4 = 5→
s_2	0	10	-3	-2	4	1	0	1	0	10/1= 10
s_3	0	20	-8	-3	3	2	0	0	1	20/2 = 10
	$Z = 0$		-4	- 1	-3	-5↑	0	0	0	$\Delta_j = C_B x_j - C_j$
x_4	5	10	-1	$\frac{3}{2}$	$\frac{5}{4}$	1	$\frac{1}{4}$	0	0	
s_2	0	2	-2	$\frac{-7}{2}$	$\frac{11}{4}$	0	$\frac{-1}{4}$	1	0	
s_3	3	0	-6	-6	$\frac{1}{3}$	0	$\frac{-1}{2}$	0	1	
	Z= 50		-9↑	13/2	13/4	0	5/4	0	0	$\Delta_j = C_B x_j - C_j$

x_1 corresponding to $\Delta_1 = -9$ is the entering vector but all the elements in Δ_1 column are negative. $\therefore$ No vector can leave the basis. Hence the given lpp has an unbounded solution.

6.4 <u>Duality in Linear Programming Problems</u>

For every linear programming problem (called the primal), there is an associated problem (called its dual), involving the same data but different variables. The two problems are very closely related and the solution of dual gives solution of the primal and vice versa. If the primal has *m* constraints and *n* variables, then the dual will contain *m* variables and *n* constraints. Duality in linear programming has many practical applications.

(i) A problem having large number of constraint inequations with few variables can be converted into its dual with fewer constraints having more variables and computations can be considerably reduced by solving the dual and hence finding the solution of the primal.

(ii) Duality in linear programming has certain far reaching consequences of economic nature. This can help in finding alternative courses of action in various situations.

Working rule for converting a primal lpp into its dual:

Step 1 In the primal, for maximization objective, ensure all inequality signs are of '$\leq$' type and if the objective function is of minimization, make all inequality signs to '$\geq$' if not originally.

Step 2 If the constraints have an equality equation, change it to two inequality inequations. Thus $a = b \Leftrightarrow a \leq b$ and $a \geq b$.

Step 3 Unrestricted variables if any are replaced by the difference of two non-negative variables. Therefore if x_1 is given to be unrestricted, then $x_1 = x_1' - x_1''$, where $x_1', x_1'' \geq 0$.

Step 4 Finally the dual of given lpp is obtained by

(i) Changing the maximize function to minimize and vice-versa.

(ii) Transposing the rows and columns of constraint coefficients.

(iii) Transposing the coefficients ($c_1, c_2, \ldots\ldots, c_n$) of the objective function and right hand side constants ($b_1, b_2, \ldots\ldots\ldots, b_m$).

(iv) Changing the inequality signs of '$\leq$' to '$\geq$' and vice-versa.

Note: Equality equation in primal $\Leftrightarrow$ Unrestricted variable in dual

Example 13 Write the dual of the lpp

Max Z = $4x_1 + 5x_2 + 2x_3$

Subject to

$5x_1 - 4x_2 - 3x_3 \geq 30$

$x_1 + x_2 \leq 15$

$2x_1 + x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

Solution: Rewriting the lpp as required in maximization objective function

Max Z = $4x_1 + 5x_2 + 2x_3$

Subject to

$-5x_1 + 4x_2 + 3x_3 \leq -30$

$x_1 + x_2 + 0x_3 \leq 15$

$2x_1 + 0x_2 + x_3 \leq 12$

$x_1, x_2, x_3 \geq 0$

Now dual of the given lpp is:

Min $Z_w = -30w_1 + 15w_2 + 12w_3$

Subject to

$-5w_1 + w_2 + 2w_3 \geq 4$

$4w_1 + w_2 \geq 5$

$3w_1 + w_3 \geq 2$

$w_1, w_2, w_3 \geq 0$

Example 14 Write the dual of the lpp

Min Z = $x_1 - 2x_2$

Subject to

$5x_1 - 4x_2 \geq 10$

$-2x_1 + 3x_2 \leq 20$

$3x_1 - 5x_2 \leq 15$

$x_1, x_2 \geq 0$

Solution: Rewriting the lpp as required in minimization objective function

Min Z = $x_1 - 2x_2$

Subject to

$5x_1 - 4x_2 \geq 10$

$2x_1 - 3x_2 \geq -20$

$-3x_1 + 5x_2 \geq -15$

$x_1, x_2 \geq 0$

Now dual of the given lpp is:

Max $Z_w = 10w_1 - 20w_2 - 15w_3$

Subject to

$5w_1 + 2w_2 - 3w_3 \leq 1$

$-4w_1 + -3w_2 + 5w_3 \leq -2$

$w_1, w_2, w_3 \geq 0$

Example 15 Write the dual of the lpp

Max Z = $4x_1 + 3x_2$

Subject to

$x_1 + 3x_2 = 12$

$2x_1 + x_2 \leq 20$

$x_1, x_2 \geq 0$

Solution: Rewriting the lpp as required in maximization objective function and also an equality constraint in the lpp

Max Z = $4x_1 + 3x_2$

Subject to

$x_1 + 3x_2 \leq 12$

$-x_1 - 3x_2 \leq -12$

$2x_1 + x_2 \leq 20$

$x_1, x_2 \geq 0$

Now due to equality constraint at first place, dual will have first unrestricted variable.

∴ Dual of the given lpp is given by:

Min $Z_w = 12w_1' - 12\, w_1'' + 20w_2$ Min $Z_w = 12\, w_1 + 20w_2$

Subject to Subject to

$w_1' - w_1'' + 2w_2 \geq 4$ $\Rightarrow$ $w_1 + 2w_2 \geq 4$

$3w_1' - 3\, w_1'' + w_2 \geq 3$ $3w_1 + w_2 \geq 3$

$w_1', w_1'', w_2 \geq 0$ $w_2 \geq 0$, w_1 unrestricted

Example 16 Write the dual of the lpp

Max Z = $x_1 + 3x_2 + 2x_3$

Subject to

$x_1 + 3x_2 + 4x_3 \leq 18$

$2x_1 + 5x_2 - x_3 \leq 13$

$x_1, x_3 \geq 0$, x_2 unrestricted

Solution: Since x_2 is unrestricted, taking $x_2 = x_2' - x_2''$, where $x_2', x_2'' \geq 0$ and rewriting the lpp as required in maximization objective function

Max Z = $x_1 + 3x_2' - 3x_2'' + 2x_3$

Subject to

$x_1 + 3x_2' - 3x_2'' + 4x_3 \leq 18$

$2x_1 + 5x_2' - 5x_2'' - x_3 \leq 13$

$x_1, x_2', x_2'', x_3 \geq 0$

$\therefore$ Dual of the given lpp is given by:

Min $Z_w = 18w_1 + 13w_2$ Min $Z_w = 18\, w_1 + 13w_2$

Subject to Subject to

$w_1 + 2w_2 \geq 1$ $\Rightarrow$ $w_1 + 2w_2 \geq 1$

$3w_1 + 5w_2 \geq 3$ $3w_1 + 5w_2 = 3$

$-3w_1 - 5w_2 \geq -3$ $4w_1 - w_2 \geq 2$

$4w_1 - w_2 \geq 2$ $w_1, w_2 \geq 0$

$w_1, w_2 \geq 0$

Note: As the second variable x_2 in the primal lpp is unrestricted, hence the second constraint will be an equality equation.

Example 17 Write the dual of the lpp

Min Z = $x_1 + 2x_2$

Subject to

$-x_1 + x_2 \leq -16$

$2x_1 + 7x_2 = 21$

$3x_1 + 4x_2 \geq 11$

$x_2 \geq 0$, x_1 unrestricted

Solution: x_1 is unrestricted variable, thus rewriting $x_1 = x_1' - x_1''$, where x_1', $x_1'' \geq 0$. Also looking at second equality constraint, rewriting the lpp as required in minimization objective function

Min Z = $x_1' - x_1'' + 2x_2$

Subject to

$x_1' - x_1'' - x_2 \geq 16$

$-2x_1' + 2x_1'' - 7x_2 \geq -21$

$2x_1' - 2x_1'' + 7x_2 \geq 21$

$3x_1' - 3x_1'' + 4x_2 \geq 11$

$x_1', x_1''\ x_2 \geq 0$

∴ Dual of the given lpp is given by:

MaxZ_w= $16w_1 - 21w_2' + 21w_2'' + 11w_3$		Max $Z_w = 16w_1 - 21w_2 + 11w_3$
Subject to		Subject to
$w_1 - 2w_2' + 2w_2'' + 3w_3 \leq 1$	⇒	$w_1 - 2w_2 + 3w_3 = 1$
$-w_1 + 2w_2' - 2w_2'' - 3w_3 \leq -1$		$-w_1 - 7w_2 + 4w_3 \leq 2$
$-w_1 - 7w_2' + 7w_2'' + 4w_3 \leq 2$		$w_1, w_3 \geq 0$, w_2 unrestricted
$w_1, w_2', w_2'' \geq 0$		

Note: As the first variable x_1 in the primal lpp is unrestricted, hence the first constraint will be an equality equation. Also due to second equality constraint in primal, w_2 will be unrestricted.

6.4.1 **Dual Simplex Method**

Simplex method can only handle problems involving '≤' type constraints. For problems involving '≥' or '=' constraints big-M or two-phase methods are used, which require artificial variables for solution. Dual simplex method can solve linear programing problems involving '≤' and '≥' signs without the use of artificial variables in a convenient way. In this method ' ≥ ' signs are changed to ' ≤ ' to avoid using any surplus or artificial variables.

Dual Simplex Method Algorithm

1. If the problem is of minimization, change it to maximization by putting
   Min Z= Max $\bar{Z}$, where $\bar{Z} = -Z$
2. Change all inequality signs to '≤' type if not originally, multiplying by -1.
3. By introducing slack variables, change the lpp to standard form and construct an initial dual simplex table.
4. Check $\Delta_j = C_B x_j - C_j \forall j$ and put the value below each variable column in the dual simplex table.
   - (i) If all $\Delta_j \geq 0$ and V_B are non-negative, optimum basic feasible solution has been attained.
   - (ii) If all $\Delta_j \geq 0$ and at least one of the V_B is negative, go to step 5.
   - (iii) If at least one Δ_j is negative, method not applicable to given lpp.
5. Select the most negative V_B, the corresponding vector leaves the basis.
6. Check all x_j's for the leaving vector,

(i) If all x_j's ≥ 0, there does not exist any feasible solution to the given problem.

(ii) If at least one x_j is negative, calculate the ratio $\left[\frac{\Delta_j}{x_j}, x_j < 0\right]$, j = 1, 2, …… n.

The vector corresponding to Max $\left[\frac{\Delta_j}{x_j}, x_j < 0\right]$ enters the basis.

7. Find the pivotal entry at the intersection of entering variable column and departing variable row.

8. Make the pivotal element 1 and remaining elements in the column zero and return to step 4.

Note: In simplex method, entering variable is found first and leaving variable after that, but in dual simplex method firstly we find leaving variable and entering variable later on.

Example 18 Solve using dual simplex method.

Minimize Z $= 2x_1 + x_2$

Subject to

$3x_1 + x_2 \geq 3$

$4x_1 + 3x_2 \geq 6$

$x_1 + 2x_2 \geq 3$

$x_1, x_2 \geq 0$

Solution: Rewriting minimization problem in maximization form and changing all ' $\geq$ ' signs to ' $\leq$ ' , multiplying by '-1'.

Max $\bar{Z} = -2x_1 - x_2$

Subject to

$-3x_1 - x_2 \leq -3$

$-4x_1 - 3x_1 \leq -6$

$-x_1 - 2x_2 \leq -3$

$x_1, x_2 \geq 0$

Introducing slack variables

Max $\bar{Z} = -2x_1 - x_2$

Subject to

$-3x_1 - x_2 + s_1 = -3$

$-4x_1 - 3x_1 + s_2 = -6$

$-x_1 - 2x_2 + s_3 = -3$

$x_1, x_2, s_1, s_2, s_3 \geq 0$

The dual simplex table is given as:

			$C_j \rightarrow$ -2	-1	0	0	0	
B.V.	C_B	V_B	x_1	x_2	s_1	s_2	s_3	Max $\left[\frac{\Delta_j}{x_j}, x_j < 0\right]$
s_1	0	-3	-3	-1	1	0	0	Max $\left[\frac{2}{-4}, \frac{1}{-3}\right] = \frac{1}{-3}$
s_2	0	-6	-4	$\boxed{-3}$	0	1	0	→ x_2 enters
s_3	0	-3	-1	-2	0	0	1	
	$\bar{Z} = 0$		2	1 ↑	0	0	0	$\Delta_j = C_B x_j - C_j$
			s_2 leaves and x_2 enters					
s_1	0	-1	$\boxed{-\frac{5}{3}}$	0	1	$-\frac{1}{3}$	0	→ Max $\left[\frac{\frac{2}{3}}{-\frac{5}{3}}, \frac{\frac{1}{3}}{-\frac{1}{3}}\right] = -\frac{2}{5}$
x_2	-1	2	$\frac{4}{3}$	1	0	$-\frac{1}{3}$	0	x_1 enters
s_3	0	1	$\frac{5}{3}$	0	0	$-\frac{2}{3}$	1	
	$\bar{Z} = -2$		$\frac{2}{3}$ ↑	0	0	$\frac{1}{3}$	0	$\Delta_j = C_B x_j - C_j$
			s_1 leaves and x_1 enters					
x_1	-2	$\frac{3}{5}$	1	0	$-\frac{3}{5}$	$\frac{1}{5}$	0	
x_2	-1	$\frac{6}{5}$	0	1	$\frac{4}{5}$	$-\frac{3}{5}$	0	
s_3	0	0	0	0	1	-1	1	
	$\bar{Z} = -\frac{12}{5}$		0	0	$\frac{2}{5}$	$\frac{1}{5}$	0	$\Delta_j \geq 0$ and $V_B \geq 0$

Minimum Z = $-\left(-\frac{12}{5}\right) = \frac{12}{5}$ at $x_1 = \frac{3}{5}$ and $x_2 = \frac{6}{5}$