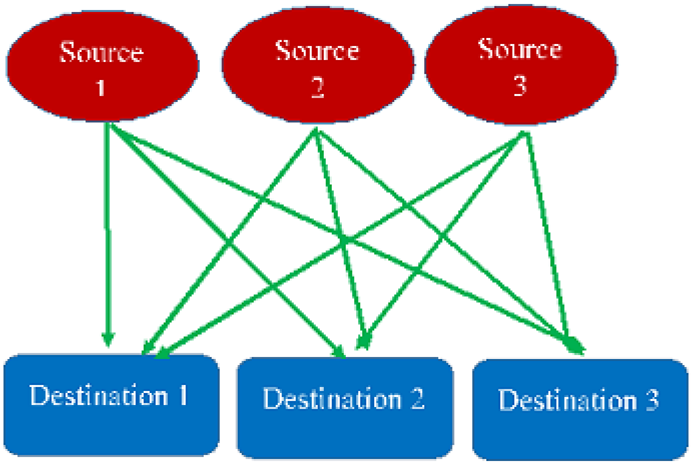
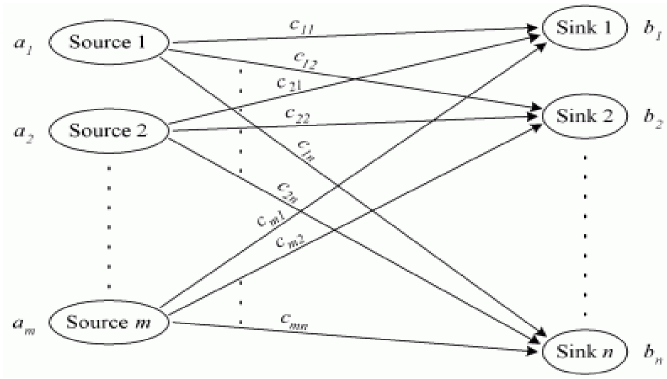
Chapter 7
TRANSPORTATION PROBLEM



Chapter 7 **Transportation**

Transportation problem is a special case of linear programming which aims to minimize the transportation cost to supply goods from various sources to different destinations, while satisfying the supply limit and demand requirement.



<u>Mathematical representation of a transportation problem</u>

In general, if there be m sources and n destinations with a_i availability in i^{th} source and

b_j requirement in j^{th} destination. Also 'c_{ij}' is the cost of transportation from i^{th} source to j^{th} destination, then a transportation problem seeks to determine non- negative values of 'x_{ij}', so as to

Minimize Z = $\sum_{i=1}^{m}\sum_{j=1}^{n} x_{ij}c_{ij}$

such that $\sum_{j=1}^{n} x_{ij} = a_i$ for $i = 1,\ldots,m$

$\sum_{i=1}^{m} x_{ij} = b_j$ for $j = 1,\ldots,n$

$x_{ij} \geq 0 \quad \forall\, i, j$

Also for a balanced transportation problem

$$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$$

To / From	D_1	D_2	...	D_n	Supply A_i
S_1	C_{11} x_{11}	C_{12} x_{12}	...	C_{1n}	A_1
S_2	C_{21} x_{21}	C_{22} x_{22}	...	C_{2n}	A_2
.	.	.	...	.	.
S_m	C_{m1} x_{m1}	C_{m2} x_{m2}	...	C_{mn}	A_m
B_j	B_1	B_2	...	B_n	$\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$

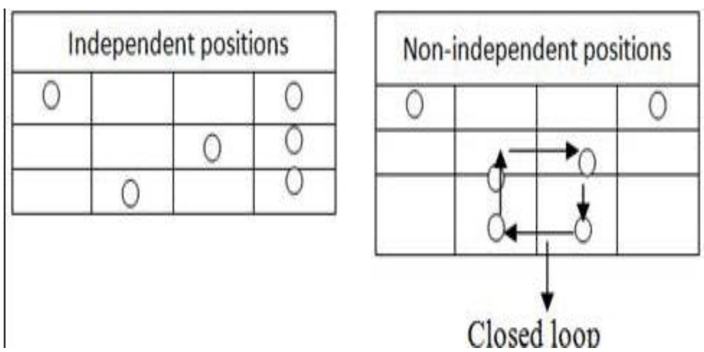
Basic feasible solution: A feasible solution of a $m \times n$ transportation problemin which allocations 'x_{ij}' are provided satisfying the conditions $\sum_{j=1}^{n} x_{ij} = a_i$ and $\sum_{i=1}^{m} x_{ij} = b_j$ for each i and j , is said to be a basic feasible solution.

Optimal solution : A basic feasible solution which minimizes the total transportation cost is known as an optimal solution.

Non-degenerate Basic feasible solution: A basic feasible solution of a $m \times n$ transportation problem is said to be non degenerate if

(i) total number of allocations ' x_{ij}' are exactly equal to $m + n - 1$.

(ii) these allocations are in independent position i.e. they do not form a loop within themselves, horizontally or vertically.



7.1 Methods to find basic feasible solution

An initial basic feasible solution to
a transportation problem can be found by any one of thefollowing methods:

(i) North West Corner Rule
(ii) Least Cost Method or Matrix Minima method
(iii) Row / Column Minima Method
(iv) Vogel's Approximation Method (VAM)

(i) North West Corner Rule

1. Balance the transportation problem if not originally by adding a dummy source or destination making $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$,with zero transportation cost in added cells.
2. Start with the cell in the upper left hand corner which is north west corner (1,1) and allocate the maximum possible amount x_{ij} = Min (a_i, b_j) in the cell (i, j), such that either the availability of the source S_i is exhausted or the requirement at destination D_j is satisfied or both.
3. Adjust supply and demand across the row and column in which allocation x_{ij} has been made.

4. Move to right hand cell (1,2) if there is still any available quantity left otherwise move down to cell (2,1).
5. Continue the procedure until all the available quantity is exhausted.

(ii) Least cost method or Matrix minima method

1. Balance the transportation problem if not originally by adding a dummy source or destination making $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$,with zero transportation cost in added cells.
2. Choose the cell with lowest cost and allocate the maximum feasible amount x_{ij} = Min (a_i, b_j) in the cell (i, j), such that either the availability of the source S_i is exhausted or the requirement at destination D_j is satisfied or both. If such cell of lowest cost is not unique, select the least cost cell where we allocate more amount.
3. Adjust supply and demand across the row and column in which allocation x_{ij} has been made.
4. Repeat the process until all the available quantity is exhausted.

(iii) Row / Column Minima method

1. Balance the transportation problem if not originally by adding a dummy source or destination making $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$,with zero transportation cost in added cells.
2. Select the smallest cost in the first row/ column of the transportation table and allocate the maximum feasible amount x_{ij} = Min(a_i, b_j) in the cell (i, j), such that either the availability of the source S_i is exhausted or the requirement at destination D_j is satisfied or both.
3. Adjust supply and demand across the row and column in which allocation x_{ij} has been made.
4. Move to next row/ column and repeat the process until all the available quantity is exhausted.

(iv) Vogel's approximation method (VAM)

1. Balance the transportation problem if not originally by adding a dummy source or destination making $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$,with zero transportation cost in added cells.
2. For each row and column of the transportation table, write the difference between smallest and the next to smallest cost below each column and on the right of the corresponding row. These differences are known as penalties.
3. Row or column having largest penalty is identified and the minimum cost cell in that particular row or column is allocated with the largest possible amount x_{ij} = Min (a_i, b_j) in the cell (i, j), such that either the availability of the source S_i is exhausted or the requirement at destination D_j is satisfied or both.In case of tie for maximum penalties, choose arbitrarily.

4. Adjust supply and demand across the row and column in which allocation x_{ij} has been made.
5. Re-compute the row and column penalties for the reduced transportation table and make the allocations.
6. Repeat the procedure until all the requirements are satisfied.

- Transportation cost using VAM is not unique due to arbitrary choosing of penalties in case of tie.
- VAM determines an initial basic feasible solution which is very close to the optimum solution.

Example 1. Obtain an initial basic feasible solution to the following transportation problem by

(i) North West Corner rule (ii) Row Minima method

(iii) Matrix minima method (iv) Vogel Approximation method

	W_1	W_2	W_3	W_4	$a_i\downarrow$
S_1	19	30	50	10	7
S_2	70	30	40	60	9
S_3	40	8	70	20	18
$b_j\rightarrow$	5	8	7	14	

Solution: **(i) NorthWest Corner rule**

Given transportation problem is already balanced, $\therefore$ allocating min [5,7] =5 to cell (1,1) and adjusting supply and demand across the row and column.

	$\mathbf{W_1}$	$\mathbf{W_2}$	$\mathbf{W_3}$	$\mathbf{W_4}$	$\mathbf{a_i\downarrow}$
$\mathbf{S_1}$	19 $\boxed{5}$	30	50	10	$\cancel{7}$,2
$\mathbf{S_2}$	70×	30	40	60	9
$\mathbf{S_3}$	40×	8	70	20	18
$\mathbf{b_j\rightarrow}$	$\cancel{5}$	8	7	14	

Moving to right hand cell (1,2), allocating min [2,8] =2 and adjusting supply&demand across the row and column.

	$\mathbf{W_1}$	$\mathbf{W_2}$	$\mathbf{W_3}$	$\mathbf{W_4}$	$\mathbf{a_i\downarrow}$
$\mathbf{S_1}$	19 $\boxed{5}$	30 $\boxed{2}$	50×	10×	$\cancel{7}$,$\cancel{2}$
$\mathbf{S_2}$	70×	30	40	60	9
$\mathbf{S_3}$	40×	8	70	20	18
$\mathbf{b_j\rightarrow}$	$\cancel{5}$	$\cancel{8}$,6	7	14	

Moving down to cell (2,2), allocating min [6,9] =6 and adjusting supply and demand across the row & column.

	W_1	W_2	W_3	W_4	$a_i\downarrow$
S_1	19 $\boxed{5}$	30 $\boxed{2}$	50×	10×	~~7,2~~
S_2	70 ×	30 $\boxed{6}$	40	60	~~9~~,3
S_3	40×	8×	70	20	18
$b_j\rightarrow$	~~5~~	~~8,6~~	7	14	

Moving to right hand cell (2,3), allocating min [3,7] =3 and adjusting supply & demand across the row and column.

	W_1	W_2	W_3	W_4	$a_i\downarrow$
S_1	19 $\boxed{5}$	30 $\boxed{2}$	50×	10×	~~7,2~~
S_2	70 ×	30 $\boxed{6}$	40 $\boxed{3}$	60×	~~9,3~~
S_3	40×	8×	70	20	18
$b_j\rightarrow$	~~5~~	~~8,6~~	~~7~~,4	14	

Moving down to cell (3,3), allocating min [4,18] =4 and adjusting supply and demand across the row & column.

	W_1	W_2	W_3	W_4	$a_i\downarrow$
S_1	19 $\boxed{5}$	30 $\boxed{2}$	50×	10×	~~7,2~~
S_2	70 ×	30 $\boxed{6}$	40 $\boxed{3}$	60×	~~9,3~~
S_3	40×	8×	70 $\boxed{4}$	20	~~18~~,14
$b_j\rightarrow$	~~5~~	~~8,6~~	~~7,4~~	14	

Allocating the balance supply/demand i.e. '14' in the cell (3,4), the initial basic feasible solution using north west corner rule is given by

	W_1	W_2	W_3	W_4	$a_i\downarrow$
S_1	19 $\boxed{5}$	30 $\boxed{2}$	50×	10×	~~7,2~~
S_2	70 ×	30 $\boxed{6}$	40 $\boxed{3}$	60×	~~9,3~~
S_3	40×	8×	70 $\boxed{4}$	20 $\boxed{14}$	~~18,14~~
$b_j\rightarrow$	~~5~~	~~8,6~~	~~7, 4~~	~~14~~	

Transportation cost = 19(5) + 30(2) + 30(6) + 40(3) + 70(4) + 20(14) = 1015units

- Note: All the above computations may be done in a single table practically. Separate tables are being taken for demonstration purpose only.

(ii) Row minima method

Allocating min [7, 14] =7 to minimum cost cell (1,4) in row S_1 and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	$10\,\boxed{7}$	$\cancel{7}$
S_2	70	30	40	60	9
S_3	40	8	70	20	18
$b_j\rightarrow$	5	8	7	$\cancel{14}$,7	

Supply in row S_1 has been exhausted, so moving to row $S_2$and allocating min [8, 9] =8 to minimum cost cell (2,2) and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	$10\,\boxed{7}$	$\cancel{7}$
S_2	70	$30\ \boxed{8}$	40	60	$\cancel{9}$,1
S_3	40	8×	70	20	18
$b_j\rightarrow$	5	$\cancel{8}$	7	$\cancel{14}$,7	

Supply in row S_2 is still remaining, ∴ allocating remaining supply min [1, 7]= 1 to next minimum cost cell (2,3) and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	$10\,\boxed{7}$	$\cancel{7}$
S_2	70×	$30\ \boxed{8}$	$40\,\boxed{1}$	60×	$\cancel{9},\cancel{1}$
S_3	40	8×	70	20	18
$b_j\rightarrow$	5	$\cancel{8}$	$\cancel{7}$, 6	$\cancel{14}$,7	

Supply in row S_2 has been exhausted, so moving to row $S_3$and allocating min [7,18] =7 to minimum cost cell (3,4) and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	$10\,\boxed{7}$	$\cancel{7}$
S_2	70×	$30\ \boxed{8}$	$40\,\boxed{1}$	60×	$\cancel{9}, \cancel{1}$
S_3	40	8×	70	$20\,\boxed{7}$	$\cancel{18}$,11
$b_j\rightarrow$	5	$\cancel{8}$	$\cancel{7}$, 6	$\cancel{14}, \cancel{7}$	

Supply in row S_3 is still remaining, ∴ allocating remaining supply min [5, 11]= 5 to next minimum cost cell (3,1) and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	10 $\boxed{7}$	~~7~~
S_2	70×	30 $\boxed{8}$	40 $\boxed{1}$	60×	~~9,1~~
S_3	40 $\boxed{5}$	8×	70	20 $\boxed{7}$	~~18,11~~, 6
$b_j\rightarrow$	~~5~~	~~8~~	~~7~~, 6	~~14, 7~~	

Allocating the balance supply/demand i.e. ‘6’ in the cell (3,3), the initial basic feasible solution using row minima method is given by

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	10 $\boxed{7}$	~~7~~
S_2	70×	30 $\boxed{8}$	40 $\boxed{1}$	60×	~~9,1~~
S_3	40 $\boxed{5}$	8×	70 $\boxed{6}$	20 $\boxed{7}$	~~18,11, 6~~
$b_j\rightarrow$	~~5~~	~~8~~	~~7 ,6~~	~~14, 7~~	

Transportation cost = 10(7) + 30(8) + 40(1) + 40(5) + 70(6) + 20(7) = 1110units

- Note: All the above computations may be done in a single table practically. Separate tables are being taken for demonstration purpose only.

(iii) Matrix minima method: Allocating min [8, 18] =8 to minimum cost cell (3,2) in the matrix and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19	30×	50	10	7
S_2	70	30 ×	40	60	9
S_3	40	8 $\boxed{8}$	70	20	~~18~~,10
$b_j\rightarrow$	5	~~8~~	7	14	

Allocating min [7, 14] =7 to next minimum cost cell (1,4) in the matrix and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	10 $\boxed{7}$	~~7~~
S_2	70	30 ×	40	60	9
S_3	40	8 $\boxed{8}$	70	20	~~18~~,10
$b_j\rightarrow$	5	~~8~~	7	~~14~~ ,7	

Allocating min [7, 10] =7 to next minimum cost cell (3,4) in the matrix and adjusting supply and demand across the row and column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	10 $\boxed{7}$	~~7~~
S_2	70	30 ×	40	60×	9
S_3	40	8 $\boxed{8}$	70	20 $\boxed{7}$	~~18,10~~ , 3
$b_j\rightarrow$	5	~~8~~	7	~~14 ,7~~	

Next minimum cost cell in the matrix are (2,3) and (3,1). Allocating min [7, 9] =7 to (2,3) as more supply amount can be assigned to this cell compared to (3,1).

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	10 $\boxed{7}$	~~7~~
S_2	70	30 ×	40 $\boxed{7}$	60×	~~9~~, 2
S_3	40	8 $\boxed{8}$	70×	20 $\boxed{7}$	~~18,10~~, 3
$b_j\rightarrow$	5	~~8~~	~~7~~	~~14 , 7~~	

Allocating remaining supply amount ‘2’ and ‘3’ respectively to cells (2,1) and (3,1), demand of $w_1$has been met and an initial basic feasible solution using matrix minima method is given by

	w_1	w_2	w_3	w_4	$a_i\downarrow$
S_1	19 ×	30×	50×	10 $\boxed{7}$	~~7~~
S_2	70 $\boxed{2}$	30 ×	40 $\boxed{7}$	60×	~~9,2~~
S_3	40 $\boxed{3}$	8 $\boxed{8}$	70×	20 $\boxed{7}$	~~18,10,3~~
$b_j\rightarrow$	~~5~~	~~8~~	~~7~~	~~14 ,7~~	

Transportation cost = 10(7) + 70(2) + 40(7) + 40(3) + 8(8) + 20(7) = 814units

(iv) Vogel's Approximation Method (VAM): Writing the difference of minimum cost and next minimum cost below each column and on the right of each row. Maximum penalty is 22 against w_2, so allocating min[8,18]=8 to minimum cost cell (3,2) in w_2 column.

	w_1	w_2	w_3	w_4	$a_i\downarrow$	
S_1	19	30×	50	10	7	(9)
S_2	70	30 ×	40	60	9	(10)
S_3	40	8 $\boxed{8}$	70	20	~~18~~ , 10	(12)
$b_j\rightarrow$	5	~~8~~	7	14		
	(21)	(22)	(10)	(10)		

Again writing the difference of minimum cost and next minimum cost, skipping allocated cells and crossed out cells, maximum penalty is 20 against S_2 and S_3. Taking S_3row arbitrarily and allocating min[10,14]=10 to minimum cost cell (3,4) in S_3, also adjusting corresponding supply and demand

	W_1	W_2	W_3	W_4	a_i↓		
S_1	19	30 ×	50	10	7	(9)	(9)
S_2	70	30 ×	40	60	9	(10)	(20)
S_3	40×	8 $\boxed{8}$	70×	20 $\boxed{10}$	~~18~~ , ~~10~~	(12)	(20)×
b_j→	5	~~8~~	7	~~14~~, 4			
	(21)	(22)	(10)	(10)			
	(21)	×	(10)	(10)			

Rewriting the difference of minimum cost and next minimum cost, skipping allocated cells and crossed out cells, maximum penalty is 51 against W_1.Allocating min[5,7]=5 to minimum cost cell (1,1) in w_1, also adjusting corresponding supply and demand

	W_1	W_2	W_3	W_4	a_i↓			
S_1	19 $\boxed{5}$	30 ×	50	10	~~7~~ , 2	(9)	(9)	(9)
S_2	70 ×	30 ×	40	60	9	(10)	(20)	(20)
S_3	40×	8 $\boxed{8}$	70×	20 $\boxed{10}$	~~18~~ , ~~10~~	(12)	(20)	×
b_j→	~~5~~	~~8~~	7	~~14~~ , 4				
	(21)	(22)	(10)	(10)				
	(21)	×	(10)	(10)				
	(51)		(10)	(50)				

Rewriting the difference of minimum cost and next minimum cost, skipping allocated cells and crossed out cells, maximum penalty is 50 against W_4.Allocating min[2,4]=2 to minimum cost cell (1,4) in S_1 , also adjusting corresponding supply and demand

	W_1	W_2	W_3	W_4	a_i↓				
S_1	19 $\boxed{5}$	30 ×	50	10 $\boxed{2}$	~~7~~ , ~~2~~	(9)	(9)	(9)	(40)
S_2	70 ×	30 ×	40	60	9	(10)	(20)	(20)	(20)
S_3	40×	8 $\boxed{8}$	70×	20 $\boxed{10}$	~~18~~ , ~~10~~	(12)	(20)	×	
b_j→	~~5~~	~~8~~	7	~~14~~ , ~~4~~, 2					
	(21)	(22)	(10)	(10)					
	(21)	×	(10)	(10)					
	(51)		(10)	(50)					
	(10)		(50)	(50)					

Now allocating remaining demands 7 and 2 of W_3 and W_4, supply in S_2 is exhausted.

	W_1	W_2	W_3	W_4	$a_i \downarrow$	
S_1	19 $\boxed{5}$	30 ×	50×	10 $\boxed{2}$	~~7~~ , ~~2~~	(9) (9) (9) (40)
S_2	70×	30 ×	40 $\boxed{7}$	60 $\boxed{2}$	~~9~~	(10) (20) (20) (20)
S_3	40×	8 $\boxed{8}$	70×	20 $\boxed{10}$	~~18~~ , ~~10~~	(12) (20) ×
$b_j \rightarrow$	~~5~~	~~8~~	~~7~~	~~14~~ , ~~4~~ , ~~2~~		
	(21)	(22)	(10)	(10)		
	(21)	×	(10)	(10)		
	(51)		(10)	(50)		
	×		(10)	(50)		

Transportation cost = 19(5) + 10(2) + 40(7) + 60(2) + 8(8) + 20(10) = 779units

- Note: All the above computations may be done in a single table practically. Separate tables are being taken for demonstration purpose only.

7.2 MODI (Modified Distribution)orTransportation method to find optimal solution

1. Find an initial basic feasible solution using any appropriate method preferably VAM.
2. Assign a set of numbers , u_i $(i= 1,2,\ldots,m)$ with each row and v_j $(j=1,2,\ldots..,n)$ with each column, such that for each **occupied** cell (r,s), $c_{r,s}= u_r + v_s$. Start with a $u_i =0$ for the row having maximum number of occupied cells and calculate remaining u_i and v_j using given relation for only **occupied** cells.
3. Enter $c_{ij}$for each **unoccupied** cell (i,j) in the upper left corner and enter $u_i + v_j$ for the same cell in the upper right corner.
4. Enter $d_{ij}= c_{ij} - (u_i + v_j)$for each **unoccupied** cell at lower right corner.
5. Examine d_{ij} for unoccupied cells.
   (i) If all $d_{ij} > 0$, solution is optimal and unique.
   (ii) If all $d_{ij} \geq 0$, solution is optimal and an alternate optimal solution exists.
   (iii) If least one $d_{ij} < 0$, solution is not optimal and go to step 6.
6. Form a new basic feasible solution by giving an allocation to the cell for which d_{ij} is most negative. For this **unoccupied** cell, generate a path forming a closed loop with **occupied** cells by drawing horizontal and vertical lines between them. Alternately put '+' and '-' signs at vertices of the loop starting with the **unoccupied** cell which is seeking an allocation. Add and subtract the minimum allocation where a subtraction is to be made so that one of the **occupied** cells is vacated in this process.
7. Repeat steps 2 to 5 until and optimal solution is obtained.

Example 2. Find an optimal solution of the transportation problem given in example 1.

Solution: An initial basic feasible solution of the transportation problem as given by VAM is :

	W_1	W_2	W_3	W_4
S_1	19 $\boxed{5}$	30	50	10 $\boxed{2}$
S_2	70	30	40 $\boxed{7}$	60 $\boxed{2}$
S_3	40	8 $\boxed{8}$	70	20 $\boxed{10}$

To assign set of numbers u_i and v_j, putting $u_2 = 0$ arbitrarily as all the rows are having equal number of allocations. Calculating remaining u_i and v_j using the relation $c_{ij} = (u_i + v_j)$ for occupied cells.

	W_1	W_2	W_3	W_4	
S_1	19∗			10 ∗	$u_1 = -50$
S_2			40 ∗	60 ∗	$u_2 = 0$
S_3		8 ∗		20 ∗	$u_3 = -40$
	$v_1=69$	$v_2=48$	$v_3=40$	$v_4=60$	

Now entering c_{ij} for each unoccupied cell (i,j) in the upper left corner and $(u_i + v_j)$ for the same cell in the upper right corner. Also writing $d_{ij} = c_{ij} - (u_i + v_j)$ at lower right corner. We observe that $d_{22} = -18$, seeking an allocation which is fulfilled by forming a closed loop with other occupied cells travelling horizontally or vertically.

	W_1	W_2	W_3	W_4	
S_1	∗	30 -2 +32	50 -10 +60	∗	$u_1 = -50$
S_2	70 69 +1	30 ? 48 -18	∗	∗	$u_2 = 0$
S_3	40 29 +11	∗	70 0 +70	∗	$u_3 = -40$
	$v_1=69$	$v_2=48$	$v_3=40$	$v_4=60$	

Adding and subtracting min[2,8]=2 at vertices of the loop to adjust supply and demand requirements as given below

?+2 2-2

8-2 10+2

Modified distribution of allocations is as shown in the table below:

	W_1	W_2	W_3	W_4
S_1	19 $\boxed{5}$	30	50	10 $\boxed{2}$
S_2	70	30 $\boxed{2}$	40 $\boxed{7}$	60
S_3	40	8 $\boxed{6}$	70	20 $\boxed{12}$

Reduced transportation cost = 19(5) + 10(2) + 30(2)+ 40(7) + 8(6) + 20(12) = 743units

To check whether the new solution is optimal, assign set of numbers u_i and v_j, putting u_1 =0 arbitrarily as all the rows are having equal number of allocations. Calculating remaining u_i and v_j using the relation $c_{ij} = (u_i + v_j)$ for occupied cells.

	W_1	W_2	W_3	W_4	
S_1	19 ∗			10 ∗	u_1 =0
S_2		30 ∗	40 ∗		u_2 =32
S_3		8 ∗		20 ∗	u_3=10
	v_1=19	v_2=-2	v_3=8	v_4=10	

Now entering c_{ij} for each unoccupied cell (i,j) in the upper left corner and $(u_i + v_j)$ for the same cell in the upper right corner.

	W_1	W_2	W_3	W_4	
S_1	∗	30 -2 +32	50 8 +42	∗	u_1= 0
S_2	70 41 +29	∗	∗	60 42 +18	u_2=32
S_3	40 29 +11	∗	70 18 +52	∗	u_3=10
	v_1=19	v_2=-2	v_3=8	v_4=10	

All $d_{ij} > 0$, ∴ the solution is optimal and minimum transportation cost = 743 units

7.2.1 Unbalanced Transportation Problem

A transportation problem is unbalanced if sum of supplies from different sources is not equal to sum of requirements in various destinations

i.e. $\sum_{i=1}^{m} a_i \neq \sum_{j=1}^{n} b_j$

(i) If $\sum_{i=1}^{m} a_i > \sum_{j=1}^{n} b_j$, add a dummy destination

(ii) If $\sum_{j=1}^{n} b_j > \sum_{i=1}^{m} a_i$, add a dummy source

Example3. Solve the following transportation problem, where goods are to be transported from 3 factories to 3warehouses. Cost of transportation is given in terms of 100$ and quantity in tons.

	W_1	W_2	W_3	a_i↓
F_1	2	3	3	3
F_2	3	4	4	4
F_3	1	5	1	5
b_j→	3	4	3	

Solution: Here $\sum_{i=1}^{3} a_i = 12$, $\sum_{j=1}^{3} b_j = 10$, the problem is unbalanced.

Adding a dummy destination with demand 2 tones and zero transportation cost to make the problem balanced:

	W_1	W_2	W_3	W_4	$a_i\downarrow$
F_1	2	3	3	0	3
F_2	3	4	4	0	4
F_3	1	5	1	0	5
$b_j \rightarrow$3		4	3	2	

Now solving the given problem by VAM in single table by calculating penalties for each row and column and assigning maximum amount in minimum cost cell to row/ column with maximum penalty in each round.

	W_1	W_2	W_3	W_4	$a_i\downarrow$	
F_1	2×	3 $\boxed{3}$	3×	0 ×	~~3~~	(2)(1)(0) (0)
F_2	3×	4 $\boxed{1}$	4 $\boxed{1}$	0 $\boxed{2}$	~~4~~,~~2~~, 1	(3)(1) (0) (0)
F_3	1 $\boxed{3}$	5×	1 $\boxed{2}$	0×	~~5~~, ~~2~~	(1) (4) (4)×
$b_j \rightarrow$	~~3~~	~~4~~,1	~~3~~	~~2~~		
	(1)	(1)	(2)	(0)		
	(1)	(1)	(2)	×		
	×	(1)	(2)			
		(1)	(1)			

Transportation cost = 3(3) + 4(1) + 4(1) + 1(3) + 1(2) = 2200$

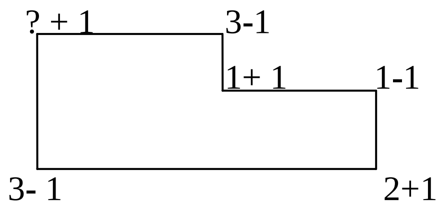
Now for optimality check using MODI method, finding set of numbers u_i and v_j, putting $u_2 = 0$ as 2nd row is having maximum number of allocations. Calculating remaining u_i and v_j using the relation $c_{ij} = (u_i + v_j)$ for occupied cells.

	W_1	W_2	W_3	W_4	
F_1		3 *			$u_1 = -1$
F_2		4 *	4 *	0 *	$u_2 = 0$
F_3	1 *		1 *		$u_3 = -3$
	$v_1=4$	$v_2=4$	$v_3=4$	$v_4=0$	

Now entering c_{ij} for each unoccupied cell (i,j) in the upper left corner and $(u_i + v_j)$ for the same cell in the upper right corner. Also writing $d_{ij} = c_{ij} - (u_i + v_j)$ at lower right corner. We observe that $d_{11} = -1$ and $d_{12} = -1$ which are equal. Choosing (1,1) arbitrarily for allocation by forming a closed loop with other occupied cells travelling horizontally or vertically.

	W_1	W_2	W_3	W_4	
F_1	2 ? 3 -1	∗	3 3 0	0 -1 1	u_1= -1
F_2	3 ? 4 -1	∗	∗	∗	u_2 = 0
F_3	∗	5 1 4	∗	0 -3 3	u_3 = -3
	v_1=4	v_2=4	v_3=4	v_4=0	

Adding and subtracting min[1,3]=1 at vertices of the loop to adjust supply and demand requirements as given below



Modified distribution of allocations is as shown in the table below:

	W_1	W_2	W_3	W_4
F_1	2 $\boxed{1}$	3 $\boxed{2}$	3	0
F_2	3	4 $\boxed{2}$	4	0 $\boxed{2}$
F_3	1 $\boxed{2}$	5	1 $\boxed{3}$	0

Reduced transportation cost = 2(1) + 3(2) + 4(2)+ 1(2) + 1(3) = 2100$

To check whether the new solution is optimal, assign set of numbers u_i and v_j, putting u_3 =0 arbitrarily as all the rows are having equal number of allocations. Calculating remaining u_i and v_j using the relation $c_{ij} = (u_i + v_j)$ for occupied cells.

	W_1	W_2	W_3	W_4	
F_1	2∗	3 ∗			u_1 =1
F_2		4 ∗		0 ∗	u_2 =2
F_3	1 ∗		1 ∗		u_3=0
	v_1=1	v_2=2	v_3= 1	v_4=-2	

Now entering c_{ij} for each unoccupied cell (i,j) in the upper left corner and $(u_i + v_j)$ for the same cell in the upper right corner. Also writing $d_{ij}= c_{ij} – (u_i + v_j)$ at lower right corner.

	W_1	W_2	W_3	W_4	
F_1	*	*	3 2 1	0 -1 1	$u_1= 1$
F_2	3 ? 3 0	*	4 3 1	*	$u_2= 2$
F_3	*	5 2 3	*	0 -2 2	$u_3 = 0$
	$v_1=1$	$v_2=2$	$v_3=1$	$v_4=-2$	

All $d_{ij} \geq 0$, ∴ the solution is optimal and an alternate optimal solution exists as $d_{21}=0$. To find the alternate solution, giving an allocation to cell (2,1) by making a loop with other allocated cells and adding and subtracting min[1,2] =1 at the vertices of the loop.

1-1 2+1

? +1 2-1

Modified distribution of allocations is as shown in the table below:

	W_1	W_2	W_3	W_4
F_1	2	3 $\boxed{3}$	3	0
F_2	3 $\boxed{1}$	4 $\boxed{1}$	4	0 $\boxed{2}$
F_3	1 $\boxed{2}$	5	1 $\boxed{3}$	0

Transportation cost = 3(3) + 3(1) + 4(1)+ 1(2) + 1(3) = 2100$

7.2.2 Degeneracy in Transportation Problems

As stated earlier, degeneracy occurs in transportation problems if total number of allocations 'x_{ij}' are less than $(m + n - 1)$ or if these allocations form a loop within themselves. If degeneracy occurs while solving a transportation problem, all u_i and v_j cannot be found out while optimality check using MODI method.

To resolve degeneracy, an extremely small quantity Δ is allocated to a least cost unallocated cell such that it does not form any loop with allocated cells. If the allocation in least cost cell forms a loop with already allocated cells, the allocation Δ may be given to next least cost cell.

Example4. Solve the following transportation problem, where goods are to be transported from 3 sources to 3 destinations.

	D_1	D_2	D_3	a_i↓
S_1	1	4	3	5
S_2	4	2	2	4
S_3	5	2	4	6
b_j→	5	5	5	

Solution: Solving the given problem by VAM in single table by calculating penalties for each row and column and assigning maximum amount in minimum cost cell to row/ column with maximum penalty in each round.

	D_1	D_2	D_3	$a_i\downarrow$	
S_1	$1\ \boxed{5}$	$4\ \times$	$3\ \times$	~~5~~	(2) ×
S_2	$4\ \times$	$2\ \times$	$1\ \boxed{4}$	~~4~~	(1) (1)
S_3	$5\ \times$	$2\ \boxed{5}$	$4\ \boxed{1}$	~~6~~	(2) (2)
$b_j\rightarrow$	~~5~~	~~5~~	~~5~~, ~~1~~		
	(3)	(2)	(2)		
	×	(0)	(3)		

Transportation cost = 1(5) + 1(4) + 2(5) + 4(1) = 23 units

Total number of allocated cells are 4, which are less than (m+n-1) i.e. 5. Trying to allocate small quantity Δ to least cost unallocated cell which forms a loop with other allocated cells (2,3), (3,3) and (3,2). ∴ Allocating Δ to next least cost cell (1,3)

	D_1	D_2	D_3	$a_i\downarrow$
S_1	$1\ \boxed{5}$	$4\ \times$	$3\ \boxed{\Delta}$	~~5~~
S_2	$4\ \times$	$2\ \times$	$1\ \boxed{4}$	~~4~~
S_3	$5\ \times$	$2\ \boxed{5}$	$4\ \boxed{1}$	~~6~~
$b_j\rightarrow$	~~5~~	~~5~~	~~5~~, ~~1~~	

Now for optimality check using MODI method, finding set of numbers u_i and v_j, putting u_3 =0 and calculating remaining u_i and v_j using the relation $c_{ij} = (u_i + v_j)$ for occupied cells.

	D_1	D_2	D_3	
S_1	1 ∗		3 ∗	u_1 =-1
S_2			1 ∗	u_2 = -3
S_3		2 ∗	4 ∗	u_3 =0
	v_1 =2	v_2 =2	v_3 =4	

Entering c_{ij} for each unoccupied cell (i,j) in the upper left corner and ($u_i + v_j$) for the same cell in the upper right corner. Also writing $d_{ij}= c_{ij} – (u_i + v_j)$ at lower right corner

	D_1	D_2	D_3	
S_1	*	4 1 +3	*	u_1 =-1
S_2	4 -1 +5	2 -1 +3	*	u_2 = -3
S_3	5 2 +3	*	*	u_3 =0
	v_1 =2	v_2 =2	v_3 =4	

All $d_{ij} > 0$, ∴ the solution is optimal and minimum transportation cost = 23 units