Chapter 2

Theory of Probability

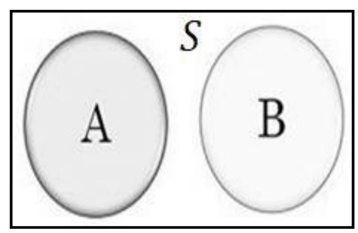
2.1 Introduction

The numerical measure of certainty of an event is called probability. The probability of any event lies between 0 and 1. Probability of a sure event is 1 while that of an impossible event is 0.

Sample Space: The set of all possible outcomes associated with an experiment is called sample space. For example while tossing a coin, the sample space is $S = \{H, T\}$and while tossing two coins $S = \{HH, HT, TH, TT\}$, whereas while rolling a die $S = \{1,2,3,4,5,6\}$.

Event: An event is the subset of a sample space. For example getting an odd number while rolling a die is an event $E = \{1,3,5\}$.

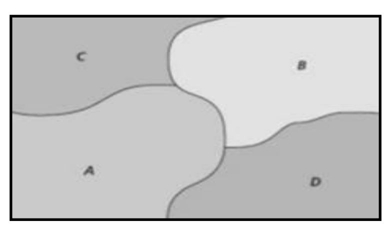
Mutually Exclusive Events



Mutually Exclusive Events: Two or more events $E_1, E_2, \dots\dots, E_n$ in a sample space are mutually exclusive if they have no point in common i.e. if $E_1 \cap E_2 \cap \dots \cap E_n = \phi$. Getting an odd number and getting an even number while rolling a die are two mutually exclusive events.

Exhaustive Events: Two or more events are called exhaustive events if at least one of them occurs when an experiment is performed. For example while tossing a coin; there are two exhaustive cases head and tail. Also while rolling a die; getting an odd number and getting an even number are two exhaustive events. Whereas getting a number less than two and getting a number greater than two, in case of rolling a die are non-exhaustive events.

Mutually Exclusive & Exhaustive Events



Mutually Exclusive and Exhaustive Events:

If events $E_1, E_2, \dots, E_n$ in a sample space are mutually exclusive and exhaustive then

$$P(E_1) + P(E_2) + \dots + P(E_n) = 1$$

Equally Likely Events: Two or more events are equally likely if they have same probability of occurrence. Getting an odd number and getting an even number while rolling an unbiased dice are two mutually exclusive and equally likely events.

Independent Events: Two events are said to be independent if the occurrence or non – occurrence of one does not affect the probability of occurrence of the other. Mathematically the events E_1 and E_2 are independent if and only if

$$P(E_1 \cap E_2) = P(E_1) \times P(E_2)$$

If two events are independent; they cannot be mutually exclusive and vice-versa.

2.2 Mathematical Definition of Probability

If a trial results in n exhaustive, mutually exclusive and equally likely events and m of them are favorable to happening of an event E; then probability of happening of E is given by: $P(E) = \frac{\text{Favorable number of cases}}{\text{Exaustive number of cases}} = \frac{n(E)}{n(S)} = \frac{m}{n}$

Example1 Give an example of two events which are mutually exclusive but not exhaustive.

Solution: In an experiment of tossing two coins,

Let E_1: Getting two heads

E_2: Getting two tails

Also $S = \{HH, HT, TH, TT\}$

$E_1 = \{HH\}$ and $E_2 = \{TT\}$

$\therefore E_1 \cap E_2 = \phi$ but $P(E_1) + P(E_2) \neq 1$

Hence the events $E_1 and E_2$ are mutually exclusive but not exhaustive.

Example2 Give an example of two events in an experiment of tossing two coins, which are mutually exclusive and exhaustive.

Solution: In an experiment of tossing two coins, let

E_1: Getting at least one head

E_2: Getting two tails

Now S= $\{HH, HT, TH, TT\}$

$E_1 = \{HH, HT, TH\}$and$E_2 = \{TT\}$

$\therefore E_1 \cap E_2 = \phi$ and $P(E_1) + P(E_2) = 1$

Hence the events $E_1 and E_2$ are mutually exclusive and exhaustive.

Example 3 An urn contains 5 white, 6 red and 4 blackballs. Two balls are drawn at random. Find the probability that both are red. Also find the probability of one white and one black ball.

Solution: White Balls: 5, Red Balls: 6, Black Balls: 4

Let E: Both balls are red

$P(E) \equiv P(\text{red ball})$ and $P(\text{red ball})$

$\Rightarrow P(E) = \frac{6}{15} \times \frac{5}{14} = \frac{1}{7}$

Let F: One white and one black ball are drawn

$$P(F) \equiv \big(P(\text{white ball}) \text{ and} P(\text{black ball})\big) \text{or} \big(P(\text{black ball}) \text{and} P(\text{white ball})\big)$$

$\Rightarrow P(F) = \frac{5}{15} \times \frac{4}{14} + \frac{4}{15} \times \frac{5}{14} = \frac{4}{21}$

Note: Questions in which replacement is not allowed can be attempted in a better manner using combinations.

Using combinations, $P(E) = \frac{6_{C_2}}{15_{C_2}} = \frac{6\times5}{15\times14} = \frac{1}{7}$

$P(F) = \frac{5_{C_1}\times 4_{C_1}}{15_{C_2}} = \frac{5\times4\times2}{15\times14} = \frac{4}{21}$

Example4 An urn contains 4 white, 5 red and 6 black balls. Three balls are drawn at random. Find the probability that balls are white, red and black.

Solution: White Balls: 4, Red Balls: 5, Black Balls: 6

Let E: Balls are white, red and black

$$P(E) = \frac{4_{C_1}\times 5_{C_1}\times 6_{C_1}}{15_{C_3}} = \frac{4\times5\times6\times3!}{15\times14\times13} = \frac{24}{91}$$

Example5 Four cards are drawn without replacement from a well shuffled pack of 52 cards. Find the probability that

(i) All cards are spades
(ii) There are two spades and two hearts
(iii) All cards are black

Also compute the probabilities if four cards are drawn with replacement.

Solution: Let E_1: All cards are spades

E_2: There are two spades and two hearts

E_3: All cards are black

$$P(E_1) = \frac{13_{C_4}}{52_{C_4}} = \frac{13\times12\times11\times10\times4!}{52\times51\times50\times49\times4!} = \frac{11}{4165}$$

$$P(E_2) = \frac{13_{C_2}\times13_{C_2}}{52_{C_4}} = \frac{13\times12\times13\times12\times4!}{52\times51\times50\times49\times2!\times2!} = \frac{468}{20825}$$

$$P(E_3) = \frac{26_{C_4}}{52_{C_4}} = \frac{26\times25\times24\times23\times4!}{52\times51\times50\times49\times4!} = \frac{46}{833}$$

Again if the four cards are drawn with replacement

$$P(E_1) = \frac{13}{52}\times\frac{13}{52}\times\frac{13}{52}\times\frac{13}{52} = \frac{1}{256}$$

$$P(E_2) = \frac{13}{52}\times\frac{13}{52}\times\frac{13}{52}\times\frac{13}{52}\times 6 = \frac{3}{128}$$

($\because$ Favorable events are SSHH or HHSS or SHSH or SHHS or HSHS or HSSH where S denotes Spade and H denotes Heart)

$$P(E_3) = \frac{26}{52}\times\frac{26}{52}\times\frac{26}{52}\times\frac{26}{52} = \frac{1}{16}$$

Example6 A pair of dice is rolled. What is the probability of sum 7?

Solution: Let E: Getting a sum 7, when a pair of dice is rolled

$$S=\begin{Bmatrix}(1,1),(1,2),\dots,(1,6),(2,1),(2,2),\dots,(2,6),(3,1),(3,2),\dots,(3,6),\\ (4,1),(4,2),\dots,(4,6),(5,1),(5,2),\dots,(5,6),(6,1),(6,2),\dots,(6,6)\end{Bmatrix}$$

$$E = \{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\}$$

$$\text{P}(E) = \frac{n(E)}{n(S)} = \frac{6}{36} = \frac{1}{6}$$

Example7 Only 3 events A, B, C can happen. Given that chance of A is one-third that of B and odds against C are 2:1, find odds in favor of A.

Solution: Given $P(A) + P(B) + P(C) = 1$ ...1

Also $P(A) = \frac{1}{3}P(B) \Rightarrow P(B) = 3\,P(A)$ ...2

And $P(C) = \frac{1}{3}$ ...3

Using 2, 3 in 1

$\Rightarrow P(A) + 3P(A) + \frac{1}{3} = 1$

$\Rightarrow 4P(A) = \frac{2}{3}$

$\Rightarrow P(A) = \frac{1}{6}$ and $P(\bar{A}) = \frac{5}{6}$

Hence odds in favour of A are 1:5

Example 8 A problem in mathematics is given to three students A, B and C whose chances of solving are $\frac{2}{3}, \frac{1}{2}$ and $\frac{1}{3}$ respectively. What is the probability that the problem will be solved?

Solution: Probability of A solving the problem $P(A) = \frac{2}{3} \Rightarrow P(\bar{A}) = \frac{1}{3}$

Probability of B solving the problem $P(B) = \frac{1}{2} \Rightarrow P(\bar{B}) = \frac{1}{2}$

Also probability of C solving the problem $P(C) = \frac{1}{3} \Rightarrow P(\bar{C}) = \frac{2}{3}$

Now probability that A, B and C do not solve the problem is $\frac{1}{3} \times \frac{1}{2} \times \frac{2}{3} = \frac{1}{9}$

$\therefore$ The probability that the problem is solved $= 1 -$ (Probability that problem is not solved) $= 1 - \frac{1}{9} = \frac{8}{9}$

Example 9 A bag contains 50 tickets numbered from 1 to 50, out of which 5 are drawn at random and arranged in ascending order ($t_1 < t_2 < t_3 < t_4 < t_5$). Find the probability of t_4 carrying the number 45.

Solution: Exhaustive number of cases = 50_{C_5}.

To follow the given pattern, three tickets t_1, t_2, t_3 must be drawn out of tickets numbered from 1 to 44 with favorable number of cases 44_{C_3}, then t_4 is drawn bearing number 25 which has only one favourable case 1_{C_1}, and then t_5 is drawn out of remaining 5 tickets with favourable number of cases 5_{C_1}.

$\therefore$ Required probability is $\frac{44_{C_3} \times 1_{C_1} \times 5_{C_1}}{50_{C_5}} = \frac{44 \times 43 \times 42 \times 5 \times 5!}{50 \times 49 \times 48 \times 47 \times 46 \times 3!} = 0.03$

Example 10 A has two shares in lottery in which there is 2 prizes and 3 blanks; B has three shares in a lottery in which there are 3 prizes and 6 blanks. Compare the probability of $A's$ success to that of $B's$ success.

Solution: Probability of A getting not getting a prize in two shares $= \frac{3_{C_2}}{5_{C_2}} = \frac{3 \times 2!}{5 \times 4} = \frac{3}{10}$

$\therefore$ Probability of A getting a prize$= \frac{7}{10}$

Probability of B not getting a prize in three shares $= \frac{{}^{6}C_3}{{}^{9}C_3} = \frac{6\times5\times4}{9\times8\times7} = \frac{5}{21}$

$\therefore$ Probability of B getting a prize$= \frac{16}{21}$

Hence the probability of $A's$ success to that of $B's$ success $= \frac{7}{10} : \frac{16}{21}$

$= 147 : 160$

Example 11 What is the probability that a leap year selected at random will have 53 Mondays.

Solution: A leap year has 366 days, having 52 full weeks and 2 extra days.

$S = \{(\text{Monday}, \text{Tuesday}), (\text{Tuesday}, \text{Wednesday}), (\text{Wednesday}, \text{Thursday}),$

(Thursday, Friday), (Friday, Saturday), (Saturday, Sunday),

(Sunday, Monday)}

Let E: The leap year selected will have 53 Mondays.

$E = \{(\text{Monday}, \text{Tuesday}),\ (\text{Sunday}, \text{Monday})\}$

$\therefore P(E) = \frac{n(E)}{n(S)} = \frac{2}{7}$

Example 12 A and B alternatively throw a die until one gets a success and wins the game, where success is defined as getting a six. What are there respective probabilities of winning if A takes the first trial?

Solution: If success (S) is getting a six, then $P(S) = \frac{1}{6}$, and if F denotes failure, then $P(F) = \frac{5}{6}$.

Now A can win in 1st, 3rd, 5th , ⋯ trials, i.e. getting S or FFS or $FFFFS$, ⋯

$\therefore$ A's probability of winning $P(A) = \frac{1}{6} + \frac{5}{6}.\frac{5}{6}.\frac{1}{6} + \frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{5}{6}.\frac{1}{6} + \cdots$

This is a G.P. with $a = \frac{1}{6}$ and $r = \frac{5}{6}.\frac{5}{6} = \frac{25}{36}$

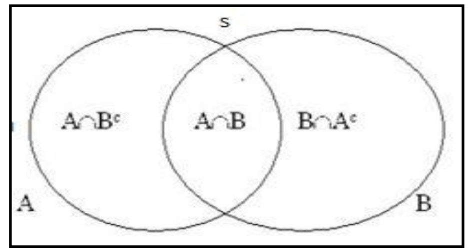
$\therefore P(A) = \frac{a}{1-r} = \frac{\frac{1}{6}}{1-\frac{25}{36}} = \frac{6}{11}$, $P(B) = 1 - \frac{6}{11} = \frac{5}{11}$

2.3 Addition Law of Probability

Statement: If A and B be any two events, then

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

Proof: $A \cup B = A \cup (B \cap A^C)$



$\Rightarrow P(A \cup B) = P(A \cup (B \cap A^C))$

$= P(A) + P(B \cap A^C)$

(∵Both A and $B \cap A^C$ are disjoint)

$= P(A) + [P(B \cap A^C) + P(A \cap B)] - P(A \cap B)$

$= P(A) + P[(B \cap A^C) \cup (A \cap B)] - P(A \cap B)$

$= P(A) + P(B) - P(A \cap B)$

$\because (B \cap A^C) \cup (A \cap B) = B$

Hence proved

- If A and B are mutually exclusive events, $A \cap B = \phi$

$\Rightarrow P\,(A \cup B) = P(A) + P(B)$

If A, B , C are 3 events, then

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) -$

$P(C \cap A) + P(A \cap B \cap C)$

Example13 Find the probability of getting a spade or an ace when a card is drawn from a well shuffled pack of 52 cards.

Solution: Let A: Getting a spade, $P(A) = \frac{13}{52}$

B: Getting an ace, $P(B) = \frac{4}{52}$

$A \cap B$: Getting an ace of spade, $P(A \cap B) = \frac{1}{52}$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$

Example14 Find the probability of getting neither heart nor king when a card is drawn from a well shuffled pack of 52 cards.

Solution: Let$A$: Getting a card of heart, $P(A) = \frac{13}{52}$

B: Getting a card of king, $P(B) = \frac{4}{52}$

$A \cap B$: Getting a king of heart, $P(A \cap B) = \frac{1}{52}$

Now $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$$= \frac{13}{52} + \frac{4}{52} - \frac{1}{52} = \frac{16}{52} = \frac{4}{13}$$

Again Probability of neither heart nor king is given by $P(A^C \cap B^C)$

$$P(A^C \cap B^C) = P(A \cup B)^C = 1 - P(A \cup B)$$

$$= 1 - \frac{4}{13} = \frac{9}{13}$$

Example15 Three newspapers A, B, C are published in a city and a survey on readers reveals the following information:

25% read A, 30% read B, 20% read C, 10% read both A and B, 5% read both A and C, 8% read both B and C, 3% read all three newspapers. For a person chosen at random, find the probability that he reads none of the newspapers.

Solution: $P(A) = \frac{25}{100}$, $P(B) = \frac{30}{100}$, $P(C) = \frac{20}{100}$, $P(A \cap B) = \frac{10}{100}$,

$$P(A \cap C) = \frac{5}{100},\ P(B \cap C) = \frac{8}{100},\ P(A \cap B \cap C) = \frac{3}{100}$$

Now $P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) -$

$$P(C \cap A) + P(A \cap B \cap C)$$

$$\Rightarrow P(A \cup B \cup C) = \frac{25}{100} + \frac{30}{100} + \frac{20}{100} - \frac{10}{100} - \frac{8}{100} - \frac{5}{100} + \frac{3}{100}$$

$$= \frac{55}{100} = \frac{11}{20}$$

$$\Rightarrow P(A \cup B \cup C)^C = 1 - P(A \cup B \cup C)$$

$$= 1 - \frac{11}{20} = \frac{9}{20}$$

Example 16 Discuss and comment on following:

$P(A) = \frac{1}{4}$, $P(B) = \frac{1}{3}$, $P(C) = \frac{2}{3}$ are probabilities of three mutually exclusive events A, B, C.

Solution: Since A, B, C are mutually exclusive events,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C)$$

$\Rightarrow P(A \cup B \cup C) = \frac{1}{4} + \frac{1}{3} + \frac{2}{3} = \frac{5}{4} > 1$

Which is not possible, hence it is a false statement.

2.4 Conditional Probability

The probability of occurrence of an event A, when event B has already occurred is called conditional probability of A and is denoted by $P(A|B)$.

If $P(A|B) = P(A)$, then event A is said to be independent of event B.

2.4.1 Multiplicative Law of Probability

Statement: The probability of simultaneous occurrence of two events is equal to the probability of event multiplied by conditional probability of the other, i.e. for two events A and B, $P(A \cap B) = P(A).\, P(B|A)$

Proof: Suppose a trial results in n exhaustive, mutually exclusive and equally likely outcomes, m of them being favourable to the happening of event A,

then $P(A) = \frac{m}{n}$ $\cdots$ ①

Out of m outcomes, favorable to the happening of event A, let m_1 be favourable to happening of event B

∴ Conditional probability of B given that A has already occurred is

$P(B|A) = \frac{m_1}{m}$ $\cdots$ ②

Again out of n exhaustive, mutually exclusive and equally likely outcomes, m_1 are favorable to happening of A and B

∴ Probability of simultaneous occurrence of A and B

$$= P(A \cap B) = \frac{m_1}{n}$$

$$= \frac{m_1}{n}.\frac{m}{m} = \frac{m}{n}.\frac{m_1}{m} = P(A).\, P(B|A) \quad \text{Using ① and ②}$$

$\therefore P(A \cap B) = P(A).\, P\left(\frac{B}{A}\right)$

- $P(A \cap B)$ is also written as $P(AB)$
- Interchanging A and B, $P(BA) = P(B).\, P(A|B)$
  or $P(AB) = P(B).\, P(A|B)$ $\quad \because A \cap B = B \cap A$
- If A and B are independent events, then $P(B|A) = P(B)$
  ∴ $P(AB) = P(A).\, P(B)$, if events A and B are independent

Formulae for conditional probability:

$$P(B|A) = \frac{P(A\cap B)}{P(A)}$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)}$$

Example17 An unbiased coin is tossed thrice. In which of the following cases events A and B are independent

(i) A: The first throw results in a tail
B: The last throw results in a head

(ii) A: The number of tails is two
B: The last throw results in a tail

Solution: $S = \{HHH, HHT, HTH, HTT, TTT, TTH, THT, THH\}$

(i) $A = \{TTT, TTH, THT, THH\}$, $P(A) = \frac{4}{8} = \frac{1}{2}$

$B = \{\mathrm{HHH}, \mathrm{HTH}, \mathrm{TTH}, \mathrm{THH}\}$, $P(B) = \frac{4}{8} = \frac{1}{2}$

$A \cap B = \{TTH, THH\}$, $P(A \cap B) = \frac{2}{8} = \frac{1}{4}$

Now $P(A \cap B) = \frac{1}{4}$, and $P(A). P(B) = \frac{1}{4}$

$\therefore P(A \cap B) = P(A). P(B)$

Hence events A and B are independent.

(ii) $A = \{\mathrm{HTT}, \mathrm{THT}, \mathrm{TTH}\}$, $P(A) = \frac{3}{8}$

$B = \{\mathrm{HHT}, \mathrm{HTT}, \mathrm{TTT}, \mathrm{THT}\}$, $P(B) = \frac{4}{8} = \frac{1}{2}$

$A \cap B = \{\mathrm{HTT}, \mathrm{THT}\}$, $P(A \cap B) = \frac{2}{8} = \frac{1}{4}$

Now $P(A \cap B) = \frac{1}{4}$, and$P(A). P(B)$= $\frac{3}{16}$

$\therefore P(A \cap B) \neq P(A). P(B)$

Hence the events A and B are not independent.

Example18 If $P(A) = 0.3$, $P(B) = 0.5$ and $P(A|B) = 0.4$

Find (i) $P(A \cap B)$, (ii) $(B|A)$, (iii) $P(A^C \cup B^C)$

Solution: (i) $P(A \cap B) = P(B).P\left(\frac{A}{B}\right) = 0.5 \times 0.4 = 0.2$

(ii) $P(B|A) = \frac{P(A\cap B)}{P(A)} = \frac{0.2}{0.3} = \frac{2}{3}$

$(iii)\ P(A^C \cup B^C) = P(A \cap B)^C$

$= 1 - P(A \cap B)$

$= 1 - 0.2 = 0.8$

Example19 Cards are dealt one by one from a well shuffled pack of 52 cards until an ace appears. Show that the probability that exactly n cards are dealt before the first ace appears is $\frac{4(51-n)(50-n)(49-n)}{52.51.50.49}$

Solution: Let A: Drawing n non ace cards

B: Drawing an ace in $(n+1)^{th}$ draw

$$P(A) = \frac{48_{C_n}}{52_{C_n}} = \frac{48!}{n!(48-n)!}.\frac{n!(52-n)!}{52!}$$

$$= \frac{(52-n)(51-n)(50-n)(49-n)}{52.51.50.49}$$

Also $P(B|A) = \frac{4_{C_1}}{(52-n)_{C_1}} = \frac{4}{52-n}$

$\therefore P(A \cap B) = P(A).P(B|A)$

$$= \frac{4(51-n)(50-n)(49-n)}{52.51.50.49}$$

Example20 Two dice are thrown and sum of numbers appearing is observed to be 6. Find the conditional probability that number 2 has appeared at least once.

Solution: Let A: Number 2 has appeared at least once

B: Sum of numbers on two dice is 6

Now $P(A|B) = \frac{P(A\cap B)}{P(B)}$

$A \equiv \{(1,2),(2,2),(3,2),(4,2),(5,2),(6,2),(2,1),(2,3),(2,4),(2,5),(2,6)\}$

$B \equiv \{(1,5),(2,4),(3,3),(4,2),(5,1)\}$

$A \cap B \equiv \{(2,4), (4,2)\}$

Now $P(A \cap B) = \frac{2}{36}$ and $P(B) = \frac{5}{36}$ $\therefore P(A|B) = \frac{2}{5}$

Example21 A bag contains 7 red 5 black balls; another bag contains 5 red and 8 black balls. A ball is drawn from the first bag and without noticing its colour is put in the second bag and then a ball is drawn from second bag. Find the probability that ball drawn is red in colour.

Solution: Case 1 A: Red ball is drawn from first bag.

B: Red ball is drawn from second bag.

$$P(A \cap B) = P(A).P\left(\frac{B}{A}\right)$$

$$= \frac{7}{12} \cdot \frac{6}{14} = \frac{42}{168}$$

Case 2 C: Black ball is drawn from first bag.

B: Red ball is drawn from second bag.

Then $P(C \cap B) = P(C).P\left(\frac{B}{C}\right)$

$$= \frac{5}{12} \cdot \frac{5}{14} = \frac{25}{168}$$

Required probability $= P(A \cap B) + P(C \cap B)$

$$= \frac{42}{168} + \frac{25}{168} = \frac{67}{168}$$

2.5 Bay's Theorem

If $E_1, E_2, \dots \dots, E_n$ are n mutually exclusive and exhaustive events in a sample space such that $P(E_i) > 0$ for each $i, (i = 1,2, \dots, n)$ and A is an arbitrary event for which $P(A) \neq 0$, then the conditional probability of occurrence of E_i, given that A has already occurred is given by

$$P(E_i/A) = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{i=n} P(E_i)P(A/E_i)}, \qquad i = 1,2, \dots, n$$

Proof: As $E_1, E_2, \dots, E_n$ are n mutually exclusive and exhaustive events

$$\therefore E_1 \cup E_2 \cup \dots \cup E_n = S$$

Now $A = A \cap S = A \cap (E_1 \cup E_2 \cup \ldots \cup E_n)$

$\Rightarrow A = (A \cap E_1) \cup (A \cap E_2) \cup (A \cap E_3) \cup \ldots \cup (A \cap E_n)$

By addition law of probability

$P(A) = P(A \cap E_1) + P(A \cap E_2) + P(A \cap E_3) + \cdots + P(A \cap E_n)$

$\because (A \cap E_1), (A \cap E_2), (A \cap E_3)$ are all mutually exclusive

$\Rightarrow P(A) = P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3) + \cdots +$

$P(E_n)P(A/E_n)$

$= \sum_{i=1}^{i=n} P(E_i)P(A/E_i)$ ...1

$\therefore P(E_i/A) = \frac{P(E_i \cap A)}{P(A)} = \frac{P(E_i)P(A/E_i)}{\sum_{i=1}^{i=n} P(E_i)P(A/E_i)}$ using 1

Example22 Three urns I, II, III contain 6 red and 4 black balls, 2 red and 6 black balls, and 1 red and 8 black balls respectively. An urn is chosen randomly and a ball is drawn from the urn. If the ball drawn is red, find the probability that it was drawn from urn I.

Solution: Let A: A Red ball is drawn

E_1: Urn I is chosen, $P(E_1) = \frac{1}{3}$, P $(A/E_1) = \frac{6}{10} = \frac{3}{5}$

E_2: Urn II is chosen, $P(E_2) = \frac{1}{3}$, P $(A/E_2) = \frac{2}{8} = \frac{1}{4}$

E_3: Urn III is chosen, $P(E_3) = \frac{1}{3}$, P $(A/E_3) = \frac{1}{9}$

Required Probability $= P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1) + P(E_2)P(A/E_2) + P(E_3)P(A/E_3)}$

$$= \frac{\frac{1}{3} \cdot \frac{3}{5}}{\left(\frac{1}{3} \cdot \frac{3}{5}\right) + \left(\frac{1}{3} \cdot \frac{1}{4}\right) + \left(\frac{1}{3} \cdot \frac{1}{9}\right)} = \frac{108}{173}$$

Example23 Two urns I and II contain 3 red and 4 black balls, 2 red and 5 black balls respectively. A ball is transferred from urn I to urn II and then a ball is drawn from urn II. If the ball drawn is found to be red, find the probability that the ball transferred from urn I is red.

Solution: Let A: A Red ball is drawn from urn II

E_1: Ball transferred from urn I is red, $P(E_1) = \frac{3}{7}$, P $(A/E_1) = \frac{3}{8}$

E_2: Ball transferred from urn I is black, $P(E_2) = \frac{4}{7}$, P $(A/E_2) = \frac{2}{8}$

Required Probability $= P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$

$$= \frac{\frac{3}{7}\cdot\frac{3}{8}}{\left(\frac{3}{7}\cdot\frac{3}{8}\right)+\left(\frac{4}{7}\cdot\frac{2}{8}\right)} = \frac{9}{9+8} = \frac{9}{17}$$

Example24 An insurance company insured 2000 scooter drivers, 4000 car drivers and 6000 truck drivers. The probability of an accident involving a scooter driver is 0.01, a car driver is 0.03 and a truck driver is 0.15. If one of an insured person meets with an accident, what is the probability that he is a car driver?

Solution: Let A: An insured person meets with an accident

E_1: Person is an insured scooter driver

$$P(E_1) = \frac{2000}{2000+4000+6000} = \frac{2}{12}, \text{ P } (A/E_1) = 0.01$$

E_2: Person is an insured car driver

$$P(E_2) = \frac{4000}{2000+4000+6000} = \frac{4}{12}, \text{ P } (A/E_2) = 0.03$$

E_3: Person is an insured truck driver

$$P(E_3) = \frac{6000}{2000+4000+6000} = \frac{6}{12}, \text{ P } (A/E_3) = 0.15$$

Required Probability $= P(E_2/A) = \frac{P(E_2)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)+P(E_3)P(A/E_3)}$

$$= \frac{\frac{4}{12}(0.03)}{\frac{2}{12}(0.01)+\frac{4}{12}(0.03)+\frac{6}{12}(0.15)} = \frac{3}{26}$$

Example25 Ram speaks truth 2 out of 3 times and Shyam 4 out of 5 times; they agree in an assertion that from a bag containing 6 balls of different colour, a red ball has been drawn. Find the probability that the statement is true.

Solution: Let A: Ram and Shyam agree in the assertion that a red ball has been drawn

E_1: Red ball is drawn. $P(E_1) = \frac{1}{6}$, P $(A/E_1) = \frac{2}{3}.\ \frac{4}{5}$

E_2: Non Red ball is drawn. $P(E_1) = \frac{5}{6}$, P $(A/E_2) = \frac{1}{3}.\ \frac{1}{5}$

Required Probability= $P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$

$$= \frac{\frac{1}{6}\cdot\frac{2}{3}\cdot\frac{4}{5}}{\left(\frac{1}{6}\cdot\frac{2}{3}\cdot\frac{4}{5}\right)+\left(\frac{5}{6}\cdot\frac{1}{3}\cdot\frac{1}{5}\right)} = \frac{8}{8+5} = \frac{8}{13}$$

Example26 A card from a deck of 52 cards is missing. 2 cards are drawn from the remaining deck of 51 cards and are found to be of spade. Find the probability that missing card is of spade.

Solution: Let A: 2 cards of spade are drawn from a deck of 51 cards.

E_1: Missing card is of spade, $P(E_1) = \frac{13}{52}$, $P(A/E_1) = \frac{12_{C_2}}{51_{C_2}} = \frac{12\times11}{51\times50}$

E_2: Missing card is a non- spade, $P(E_2) = \frac{39}{52}$, $P(A/E_2) = \frac{13_{C_2}}{51_{C_2}} = \frac{13\times12}{51\times50}$

Required Probability $= P(E_1/A) = \frac{P(E_1)P(A/E_1)}{P(E_1)P(A/E_1)+P(E_2)P(A/E_2)}$

$$= \frac{\frac{13\times12\times11}{52\times51\times50}}{\frac{13\times12\times11}{52\times51\times50}+\frac{39\times13\times12}{52\times51\times50}} = \frac{11}{11+39} = \frac{11}{50}$$

2.6 Random Variable and Probability Distributions

A random variable is often described as a variable whose values are determined by chance. The values taken by the variable may be countable or uncountable, based on which it is classified as discrete or continuous. Random variable is typically denoted using capital letters such as 'X'.

2.6.1 Discrete Probability Distributions

A discrete probability distribution '$P(X)$'describes the probability of occurrence of each value of a discrete random variable 'X'. A discrete random variable is a random variable that has countable values, such as a set of positive integers. The discrete probability distribution '$P(X)$' of an experiment provides the corresponding probabilities to all possible values of the random variable 'X' associated with it such that $\sum P(X) = 1$

Example27 Find the probability distribution of the number of aces when two cards are drawn at random with replacement from a well shuffled pack of 52 cards.

Solution: Let X be a random variable showing number of aces. Clearly X can take values $0, 1$ or 2. If S denotes success i.e. getting an ace and F denotes failure i.e. getting a non-ace card, then $P(S) = \frac{4}{52} = \frac{1}{13}$ and $P(F) = \frac{12}{13}$

X	Event	$P(X)$
0	FF	$\frac{12}{13} \cdot \frac{12}{13} = \frac{144}{169}$
1	SF or FS	$\left(\frac{1}{13} \cdot \frac{12}{13}\right) + \left(\frac{12}{13} \cdot \frac{1}{13}\right) = \frac{24}{169}$
2	SS	$\frac{1}{13} \cdot \frac{1}{13} = \frac{1}{169}$

Example28 Find the probability distribution of the number of successes in two tosses of a die, when success is defined as a number greater than 4.

Solution: Let X be a random variable showing number of successes in two tosses of a die. Clearly X can take values $0, 1$ or 2. If S denotes success i.e. getting a number greater than 4 and F denotes failure, then $P(S) = \frac{2}{6} = \frac{1}{3}$ and $P(F) = \frac{2}{3}$

X	Event	$P(X)$
0	FF	$\frac{2}{3} \cdot \frac{2}{3} = \frac{4}{9}$
1	SF or FS	$\left(\frac{1}{3} \cdot \frac{2}{3}\right) + \left(\frac{2}{3} \cdot \frac{1}{3}\right) = \frac{4}{9}$
2	SS	$\frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}$

Example29 3 bad articles are mixed with 7 good ones. Find the probability distribution of number of bad articles if 3 are drawn at random without replacement from the lot.

Solution: Let X be a random variable showing number of bad articles. Clearly X can take values $0, 1, 2$ or 3.

X	Event	$P(X)$
0	0 Bad 3 Good	$\frac{7_{C_3}}{10_{C_3}} = \frac{210}{720}$
1	1 Bad 2 Good	$\frac{3_{C_1} \cdot 7_{C_2}}{10_{C_3}} = \frac{378}{720}$

2	2 Bad 1 Good	$\frac{3_{C_2}.7_{C_1}}{10_{C_3}} = \frac{126}{720}$
3	3 Bad 0 Good	$\frac{3_{C_3}}{10_{C_3}} = \frac{6}{720}$

Note: Combination is being used as articles are drawn without replacement.

2.6.2 Mathematical Expectation

If X be a random variable which can assume any one of the values x_1, x_2, $\cdots$, x_n with respective probabilities p_1, p_2, $\cdots$, p_n; then the mathematical expectation of X usually called as expected value of X, denoted by $E(X)$ is defined as:

$E(X) = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = \sum p_i x_i$; where $\sum p_i = 1$

Physical interpretation of $E(X)$

If $\bar{x}$ denotes mean of set of observations x_1, x_2, $\cdots$, x_n with respective frequencies f_1, f_2, $\cdots$, f_n; then $\bar{x} = \frac{\sum f_i x_i}{N}$, $N = \sum f_i$

$\Rightarrow \bar{x} = \frac{f_1}{N} x_1 + \frac{f_2}{N} x_2 + \cdots + \frac{f_n}{N} x_n \qquad \cdots ①$

Now out of total N cases, f_i are favorable to x_i

$\therefore P(X = x_i) = \frac{f_i}{N} = p_i$, $i = 1, 2, \cdots, n$

$\Rightarrow \frac{f_1}{N} = p_1, \frac{f_2}{N} = p_2, \cdots, \frac{f_n}{N} = p_n \qquad \cdots ②$

$\therefore \bar{x} = p_1 x_1 + p_2 x_2 + \cdots + p_n x_n = \sum p_i x_i$ using ② in ①

Hence $\bar{x} = E(X)$

Hence mathematical expectation of a random variable is nothing but its arithmetic mean.

$\therefore$We conclude Mean ($\bar{x}$) = $E(X) = \sum p_i x_i$

Similarly Variance (σ^2) $= \frac{\sum f_i x_i^{\,2}}{N} - \left(\frac{\sum f_i x_i}{N}\right)^2$

$$= \sum p_i x_i^{\,2} - \left(\sum p_i x_i\right)^2 = E(X^2) - (E(X))^2$$

Example30 What is the expected number of heads appearing when a fair coin is tossed 3 times.

Solution: Let X be a random variable showing number of heads. Clearly X can take values 0, 1, 2 , 3.

x_i	Event	p_i	$p_i\, x_i$
0	TTT	$\frac{1}{8}$	0
1	HTT, THT, TTH	$\frac{3}{8}$	$\frac{3}{8}$
2	HHT, HTH, THH	$\frac{3}{8}$	$\frac{6}{8}$
3	HHH	$\frac{1}{8}$	$\frac{3}{8}$

$\therefore\ E(X) = \sum p_i\, x_i\ = \frac{12}{8} = 1.5$

Example31 A man draws 2 balls from a bag containing 3 white and 5 black balls. If he receives Rs70 for every white ball he draws and Rs. 35 for every black ball, what is his expectation?

Solution: Following table shows the amount received by the man for each event:

Event	Probability (p_i)	Amount (x_i)	$p_i\, x_i$
2 black balls	$\frac{5_{C_2}}{8_{C_2}} = \frac{10}{28}$	35+35=70	$\frac{10}{28} \times 70$
1white 1 black ball	$\frac{3_{C_1} \times 5_{C_1}}{8_{C_2}} = \frac{15}{28}$	70+35=105	$\frac{15}{28} \times 105$
2 white balls	$\frac{3_{C_2}}{8_{C_2}} = \frac{3}{28}$	70+70=140	$\frac{3}{28} \times 140$

$\therefore\ E(X) = \sum p_i\, x_i\ = \frac{700}{28} + \frac{1575}{28} + \frac{420}{28}\ =\ \frac{385}{4} = 96.25$

Example32 For a random variable X, the probability mass function is

$f(x) = kx$, for $x = 1, 2, \cdots, n$

$= 0$, otherwise

Find expectation of X.

Solution: Here $f(x)$ denotes probability mass function

$\therefore \sum_{x=1}^{n} f(x) = \sum_{x=1}^{n} kx =\ k \sum_{x=1}^{n} x = 1$

$\Rightarrow k\,\frac{n(n+1)}{2} = 1$

$\Rightarrow k = \frac{2}{n(n+1)}$

$\therefore E(X) = \sum_{x=1}^{n} xf(x) = \sum_{x=1}^{n} xkx = \sum_{x=1}^{n} x^2 \frac{2}{n(n+1)}$

$= \frac{2}{n(n+1)} \sum_{x=1}^{n} x^2$

$= \frac{2}{n(n+1)} (1^2 + 2^2 + \cdots + n^2)$

$= \frac{2}{n(n+1)} \frac{n(n+1)(2n+1)}{6} = \frac{(2n+1)}{3}$

Example 33 A random variable X has the following probability function:

X	-2	-1	0	1	2	3
$P(X)$	k	0.1	0.3	2k	0.2	K

Calculate mean and variance.

Solution: For any probability function $\sum p_i$=1

$\Rightarrow$ k + 0.1 + 0.3 + 2k + 0.2 + k = 1

$\Rightarrow$ k = 0.1

x_i	p_i	$p_i x_i$	$p_i {x_i}^2$
-2	0.1	-0.2	0.4
-1	0.1	-0.1	0.1
0	0.3	0	0
1	0.2	0.2	0.2
2	0.2	0.4	0.8
3	0.1	0.3	0.9
		$\sum p_i\, x_i = 0.6$	$\sum p_i\, {x_i}^2 = 2.4$

$\therefore$Mean = $\sum p_i\, x_i = 0.6$

Variance (σ^2) $= \sum p_i\, {x_i}^2 - (\sum p_i\, x_i)^2$

= 2.4 − 0.36 = 2.04

2.6.3 Continuous Probability Distributions

The probability distribution $P(X)$ associated with a continuous random variable X is called a continuous distribution. A continuous random variable is having a set of infinite and uncountable values, for example set of real numbers in the interval (0,1) is uncountable.

If X be a continuous random variable taking values in the interval $[a,b]$, the function $f(x)$ is said to be the probability density function (PDF) of X, if it satisfies the following properties:

i. $f(x) \geq 0 \quad \forall x \in X$ in $[a,b]$.
ii. Total area under the probability curve is one, i.e. $P(a \leq x \leq b)$ =1.

For two distinct points c and d in the interval $[a,b]$; $P(c \leq x \leq d)$ is givenby area under the probability curve between the ordinates $x = c$ and $x = d$, $i.e.$ $P(c \leq x \leq$ *d= cdfxdx.*

Also $F(x) = P(X \leq x) = \int_{-\infty}^{x} f(x)dx$ is called cumulative distribution function or simply distribution function.

Example 34 Find whether the following is a probability density function:

$$f(x) = \begin{cases} x, & 0 \leq x \leq 1 \\ 2x, & 1 < x \leq 2 \end{cases}$$

Solution: For $f(x)$ to be a probability density function, $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{0} f(x)\,dx + \int_{0}^{1} f(x)\,dx + \int_{1}^{2} f(x)\,dx + \int_{2}^{\infty} f(x)\,dx$$

$$= \int_{-\infty}^{0} 0\,dx + \int_{0}^{1} x\,dx + \int_{1}^{2} 2x\,dx + \int_{2}^{\infty} 0\,dx$$

$$= 0 + \frac{1}{2}[x^2]_0^1 + [x^2]_1^2$$

$$= \frac{1}{2}[1-0] + [4-1] = \frac{7}{2} \neq 1$$

Hence $f(x)$ is not a probability density function.

Example 35 Let X be a random variable with PDF given by

$$f(x) = kx^4, \; |x| \leq 1$$

$$= 0, \text{ otherwise}$$

i. Find the value of the constant k

ii. Find $E(X)$ and $Var(X)$

iii. Find $P(X) \geq \frac{1}{2}$

Solution: $i.$ For $f(x)$ to be PDF, $\int_{-\infty}^{\infty} f(x)\, dx = 1$

$$\Rightarrow \int_{-1}^{1} kx^4\, dx = 1$$

$$\Rightarrow \quad \frac{k}{5}[x^5]_{-1}^{1} = 1 \qquad \Rightarrow \quad k = \frac{5}{2}$$

$$ii. \quad E(X) = \int_{-\infty}^{\infty} xf(x)\, dx$$

$$= \int_{-1}^{1} xkx^4\, dx$$

$$= k\int_{-1}^{1} x^5\, dx = 0 \quad (\because x^5 \text{ is an odd function})$$

Also $Var(X) = E(X^2) - (E(X))^2$

$$= \int_{-\infty}^{\infty} x^2 f(x)\, dx - 0$$

$$= k\int_{-1}^{1} x^6\, dx = \frac{5}{7}$$

$$iii. P(X) \geq \frac{1}{2} = \int_{\frac{1}{2}}^{1} kx^4\, dx = \frac{k}{5}[x^5]_{\frac{1}{2}}^{1} = \frac{31}{64}$$

Example 36 Show that the function $f(x)$ defined as

$f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$, is a probability density function and find the probability that the variate X having $f(x)$ as density function will lie in the interval $(1,2)$. Also find the probability distribution function $F(2)$.

Solution: We have $\int_{-\infty}^{\infty} f(x)dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{\infty} e^{-x}\, dx = -[e^{-x}]_0^{\infty} = 1$

$\therefore f(x)$ is a probability density function.

Also $P(1 \leq x \leq 2) = \int_{1}^{2} e^{-x}\, dx = -[e^{-x}]_1^2 = 2.33$

Again $F(2) = \int_{-\infty}^{2} f(x)dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{2} e^{-x}\, dx = -[e^{-x}]_0^2 = 1 - e^{-2}$

$$= 1 - 0.135 = 0.865$$

2.7 Moments

The expected values $E(x-a), E((x-a)^2)$, $E((x-a)^3)$,... , $E((x-a)^r)$ are called moments about any point a**.**

Thus r^{th} moment about any point 'a' of any distribution is denoted by μ_r' and is given by $\mu_r' = \sum p_i\,(x_i - a)^r = \frac{1}{N}\sum f_i\,(x_i - a)^r$, where $N = \sum f_i$

In particular r^{th} moment about mean $\bar{x}$ is given by $\mu_r = \frac{1}{N}\sum f_i\,(x_i - \bar{x})^r$

Some important results:

- $\mu_0 = \mu_0' = 1$
- $\mu_1 = \frac{1}{N}\sum f_i\,(x_i - \bar{x}) = \frac{1}{N}\sum f_i\,x_i - \frac{\bar{x}}{N}\sum f_i = \bar{x} - \bar{x} = 0$
- $\mu_2 = \frac{1}{N}\sum f_i\,(x_i - \bar{x})^2 = \frac{1}{N}\sum f_i\,{x_i}^2 + \frac{\bar{x}^2}{N}\sum f_i - 2\frac{\bar{x}}{N}\sum f_i\,x_i$
  $= \frac{1}{N}\sum f_i\,{x_i}^2 + \bar{x}^2 - 2\bar{x}^2 = \frac{1}{N}\sum f_i\,{x_i}^2 - \bar{x}^2 = \sigma^2$
- $\mu_1' = \frac{1}{N}\sum f_i\,(x_i - a) = \frac{1}{N}\sum f_i\,x_i - \frac{a}{N}\sum f_i = \bar{x} - a$

Relation between μ_r' and μ_r

$$\mu_r = \frac{1}{N}\sum f_i\,(x_i - \bar{x})^r$$

$$= \frac{1}{N}\sum f_i\,((x_i - a) - (\bar{x} - a))^r$$

$$= \frac{1}{N}\sum f_i\,(d - \mu_1')^r \text{ by putting } x_i - a = d$$

$$\Rightarrow \mu_r = \frac{1}{N}\sum f_i\left(d^r - r_{C_1} d^{r-1}\mu_1' + r_{C_2} d^{r-2}{\mu_1'}^2 + \cdots + (-1)^r {\mu_1'}^r\right)$$

$$= \frac{1}{N}\sum f_i\,d^r - r_{C_1}\mu_1' \frac{1}{N}\sum f_i\,d^{r-1} + r_{C_2}{\mu_1'}^2\frac{1}{N}\sum f_i\,d^{r-2} + \cdots + (-1)^r {\mu_1'}^r \frac{1}{N}\sum f_i$$

$$\Rightarrow \mu_r = \mu_{\boldsymbol{r}}' - r_{C_1}\mu_{\boldsymbol{r-1}}'\mu_1' + r_{C_2}\mu_{\boldsymbol{r-2}}'{\mu_1'}^2 + \cdots + (-1)^r {\mu_1'}^r$$

In particular

$$\mu_2 = \mu_{\boldsymbol{2}}' - 2\,{\mu_1'}^2 + \mu_0'{\mu_1'}^2 = \mu_2' - {\mu_1'}^2 \text{ as } \mu_0' = 1$$

$$\mu_3 = \mu_{\boldsymbol{3}}' - 3\mu_{\boldsymbol{2}}'\mu_1' + 3{\mu_1'}^3 - \mu_{\boldsymbol{0}}'{\mu_1'}^3 = \mu_{\boldsymbol{3}}' - 3\mu_{\boldsymbol{2}}'\mu_1' + 2{\mu_1'}^3$$

Similarly $\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4$

2.8 Moment Generating Function (MGF)

The moment generating function of the variable x about a point $x = a$ is defined as the expected value of $e^{t(x-a)}$ and is denoted by $M_a(t)$.

$\therefore\ M_a(t) = E\left[e^{t(x-a)}\right]$.

$\Rightarrow M_a(t) = \sum p_i\, e^{t(x_i-a)}$ ...①

$= \sum p_i + t \sum p_i\,(x_i - a) + \frac{t^2}{2!}\sum p_i\,(x_i - a)^2 + \ldots + \frac{t^r}{r!}\sum p_i\,(x_i - a)^r + \cdots$

$(\because e^x = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^r}{r!} + \cdots)$

$\Rightarrow M_a(t) = 1 + t\mu'_1 + \frac{t^2}{2!}\mu'_2 + \cdots + \frac{t^r}{r!}\mu'_r + \cdots$ ...②

$\because\ p_i = \frac{f_i}{N}$

where μ'_r is the moment of order r about a point a

Hence μ'_r = coefficient of $\frac{t^r}{r!}$

Thus $M_a(t)$ generates moments and therefore it is called moment generating function.

Again rewriting ① as $M_a(t) = e^{-at} \sum p_i\, e^{tx_i}$

$\Rightarrow M_a(t) = e^{-at} M_0(t)$

Thus (MGF about the point a) $= e^{-at}$ (MGF about origin)

Again if $f(y)$ be density function of a continuous variate Y, then the moment generating function of the continuous probability distribution about $y = a$ is given by:

$$M_a(t) = \int_{-\infty}^{\infty} e^{t(y-a)} f(y)\, dy$$

We can also generate moments by differentiating r times w.r.t. t and then putting $t = 0$

i.e. $\mu'_r = \left[\frac{d^r}{dt^r} M_a(t)\right]_{t=0}$ ...③

Thus the moments about any point $x = a$ can be found using ② or more conveniently using ③

If a moment-generating function exists for a random variable X, then:

- $M(0) = 1$
- $M'(0) = E(X),\ M''(0) = E(X^2),\ M'''(0) = E(X^3), \cdots$
- The mean and variance of X can be found by evaluating the first and second derivatives of the moment-generating function at $t = 0$.

  i.e. $\bar{x} = E(X) = M'(0),\ \ \sigma^2 = E(X^2) - (E(X))^2 = M''(0) - [M'(0)]^2$

Example37 Find the moment generating function for the probability distribution given by number of heads appearing when a fair coin is tossed 3 times and hence find mean and variance.

Solution: Let X be a random variable showing number heads. Clearly X can take values $0, 1, 2$,3. Probability distribution is given by:

x_i	0	1	2	3
p_i	$\frac{1}{8}$	$\frac{3}{8}$	$\frac{3}{8}$	$\frac{1}{8}$

$$M(t) = E(e^{tX}) = \sum p_i\, e^{tx_i}$$

$$\Rightarrow M(t) = \frac{1}{8} + \frac{3}{8}e^{t} + \frac{3}{8}e^{2t} + \frac{1}{8}e^{3t}$$

$$M'(t) = \frac{3}{8}e^{t} + \frac{6}{8}e^{2t} + \frac{3}{8}e^{3t}$$

$\therefore$ Mean $= M'(0) = \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = \frac{12}{8} = 1.5$

Also $M''(t) = \frac{3}{8}e^{t} + \frac{12}{8}e^{2t} + \frac{9}{8}e^{3t}$

$\Rightarrow M''(0) = \frac{3}{8} + \frac{12}{8} + \frac{9}{8} = \frac{24}{8} = 3$

$\therefore$ Variance $= M''(0) - [M'(0)]^2 = 3 - (1.5)^2 = 0.75$

Example 38 Let X be a random variable with PDF given by

$$f(x) = \begin{cases} ke^{-kx}, & x \in (0, \infty) \\ 0 & , \text{otherwise} \end{cases}$$

Find moment generating function of $f(x)$, hence find mean and variance.

Solution: $M(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f(x)\, dx$

$$= \int_{-0}^{\infty} e^{tx} k e^{-kx}\, dx = \mathrm{k} \int_{-0}^{\infty} e^{-(k-t)x}\, dx = k \left[\frac{e^{-(k-t)}}{t-k}\right]_0^{\infty} = \frac{k}{k-t}$$

Also $M'(t) = \frac{k}{(k-t)^2}$

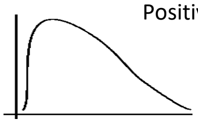
$\therefore$ Mean $= M'(0) = \frac{1}{k}$

Also $M''(t) = \frac{2k}{(k-t)^3} \Rightarrow M''(0) = \frac{2}{k^2}$

$\therefore$ Variance $= M''(0) - [M'(0)]^2 = \frac{2}{k^2} - \frac{1}{k^2} = \frac{1}{k^2}$

2.9 Skewness

Skewness is a measure of the asymmetry of the probability distribution of a random variable about its mean. In a symmetrical distribution, mean, mode and median coincide. The skewness value can be positive or negative.



Mass of distribution is concentrated to left

Mass of distribution is concentrated to the right

Coefficient of skewness $= \frac{\text{Mean} - \text{Mode}}{\text{Standard Deviation}}$

$\therefore S_k = \frac{M - M_o}{\sigma}$, S_k is called Karl Pearson's coefficient of skewness and lies between -3 and $+3$.

If mode is ill defined, then using $M_o = 3M_d - 2M$

$S_k = \frac{3(M - M_d)}{\sigma}$

Karl Pearson defined the following four coefficients based upon the first four moments about the mean:

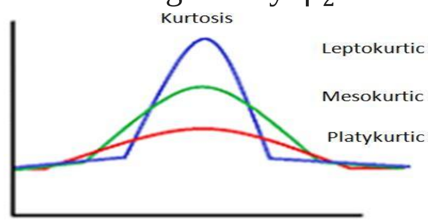
$$\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3}, \quad \gamma_1 = \sqrt{\beta_1}$$

$$\beta_2 = \frac{\mu_4}{{\mu_2}^2}, \quad \gamma_2 = \beta_2 - 3$$

β_1 gives a measure of departure from symmetry and β_2 is associated with skewness.

2.10 Kurtosis

Kurtosis measures the degree of peakness of a distribution and is given by β_2



If $\beta_2 > 3$, the curve is peaked or leptokurtic

If $\beta_2 = 3$, the curve is normal or mesokurtic

If $\beta_2 < 3$, the curve is flat topped or platykurtic

Example39 Calculate the first four moments of the following distribution about the mean and hence find β_1 and β_2 .

x	0	1	2	3	4	5	6	7	8
f	1	8	28	56	70	56	28	8	1

Solution: Taking 4 as assumed mean, let us first calculate moments about $x = 4$

$\mu'_r = \frac{1}{N}\sum f\,(x - a)^r = \frac{1}{N}\sum f\,d^r$, where $d = x - 4$

x	f	$d = x - 4$	fd	fd^2	fd^3	fd^4
0	1	-4	-4	16	-64	256
1	8	-3	-24	72	-216	648
2	28	-2	-56	112	-224	448
3	56	-1	-56	56	-56	56
4	70	0	0	0	0	0
5	56	1	56	56	56	56
6	28	2	56	112	224	448
7	8	3	24	72	216	648
8	1	4	4	16	64	256
	N= 256		0	512	0	2816

$\mu'_1 = \frac{1}{N}\sum f\,d = 0,\ \mu'_2 = \frac{1}{N}\sum f\,d^2 = \frac{512}{256} = 2,\ \mu'_3 = \frac{1}{N}\sum f\,d^3 = 0,$

$\mu'_4 = \frac{1}{N}\sum f\,d^4 = \frac{2816}{256} = 11$

Moments about mean:

$\mu_1 = 0$

$$\mu_2 = \mu_2' - {\mu_1'}^2 = 2$$

$$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + 2{\mu_1'}^3 = 0$$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'{\mu_1'}^2 - 3{\mu_1'}^4 = 11$$

$$\beta_1 = \frac{{\mu_3}^2}{{\mu_2}^3} = 0 \text{ , } \beta_2 = \frac{\mu_4}{{\mu_2}^2} = \frac{11}{4} = 2.75$$

Exercise 2

1. In a single throw of two dice, what is the probability of getting six as a product of numbers on two dice?
2. A bag contains 4 black, 2 red and 3 blue pens. If 2 pens are drawn at random from the pack and then another pen is drawn without replacement, what is the probability of drawing 2 black and 1 blue pens?
3. A purse contains 2 silver and 4 copper coins. A second purse contains 4 silver and 3 copper coins. If a coin is pulled out at random from one of the two purses, what is the probability that coin is a silver coin?
4. A problem in mathematics is given to three students A, $B$and C whose chances of solving are $\frac{1}{2}$, $\frac{1}{3}$ and $\frac{1}{4}$ respectively. What is the probability that the problem will not be solved?
5. If four whole numbers taken at random are multiplied together, find the probability that the last digit in the product is 1,3,7 or 9.
6. A speaks truth in 60% and B in 75% cases of the cases. In what percentage of cases they are likely to contradict each other in stating the same fact.
7. A and B take turns in throwing two dice and the first to throw 10 wins the game. If A has the first throw, find B's chances of winning.
8. A bag contains 10 white and 3 black balls. Another bag contains 3 white and 5 black balls. Two balls are transferred from the first bag and placed in the second and then one ball is drawn from the second bag. What is the probability that it is a white ball?
9. In each of a set of games it is 2 to 1in favour the winner of the previous game. What is the probability that who wins the first game shall win at least 3 games out of the next four games?
10. A and B take turns in throwing two dice with A having the first trial. The first to throw 10 being awarded a prize. Find their expectations if the prize money is Rs 460.
11. In a lottery m tickets are drawn out of n tickets numbered from 1 to n. Show that the expectation of sum of numbers drawn is $m\left(\frac{n+1}{2}\right)$.

12. Find the probability distribution of number of kings drawn when two cards are drawn one by one, without replacement, from a deck of 52 cards.
13. A random variable X has the following probability distribution:

X	0	1	2	3	4	5
$P(X)$	0.1	K	0.2	$2K$	0.3	K

Find $i. K$ $ii. P(X \leq 1)$ $iii. P(X > 3)$
14. Let$X$ be a random variable with PDF given by

$$f(x) = \frac{1}{4},\ |x| \leq 2$$

$$= 0, \text{ otherwise}$$

Obtain the values of $i.\ P(X < 1)$ $ii.\ P(|X| > 1)$ $iii.\ P(2X + 3) > 5)$
15. For the function $f(x)$ defined by $f(x) = ce^{-x}, 0 \leq x \leq \infty$, find the value of c which changes $f(x)$ to a probability density function.
16. If X be a random variable with PDF given by
$f(x) = \begin{cases} e^{-x}, & x \in (0, \infty) \\ 0 & , \text{ otherwise} \end{cases}$, find moment generating function of $f(x)$.
17. Find $E(X^3)$ using the moment generating function $(1 - 2t)^{-10}$

Answers

1. $\frac{1}{9}$
2. $\frac{1}{14}$
3. $\frac{19}{42}$
4. $\frac{1}{4}$
5. $\frac{16}{625}$
6. 45%
7. $\frac{11}{23}$
8. $\frac{59}{130}$
9. $\frac{4}{9}$
10. Rs.240, Rs. 220
12. $X = 0$; $P(X) = \frac{188}{221}$, $X = 1$; $P(X) = \frac{32}{221}$, $X = 2$; $P(X) = \frac{1}{221}$
13. $i.$ 0.1 $ii.$ 0.2 $iii.$ 0.4

14. $i.\ \frac{3}{4}$ $ii.\ \frac{1}{2}$ $iii.\ \frac{1}{4}$

15. 1

16. $\frac{1}{1-t}$

17. 10,560.