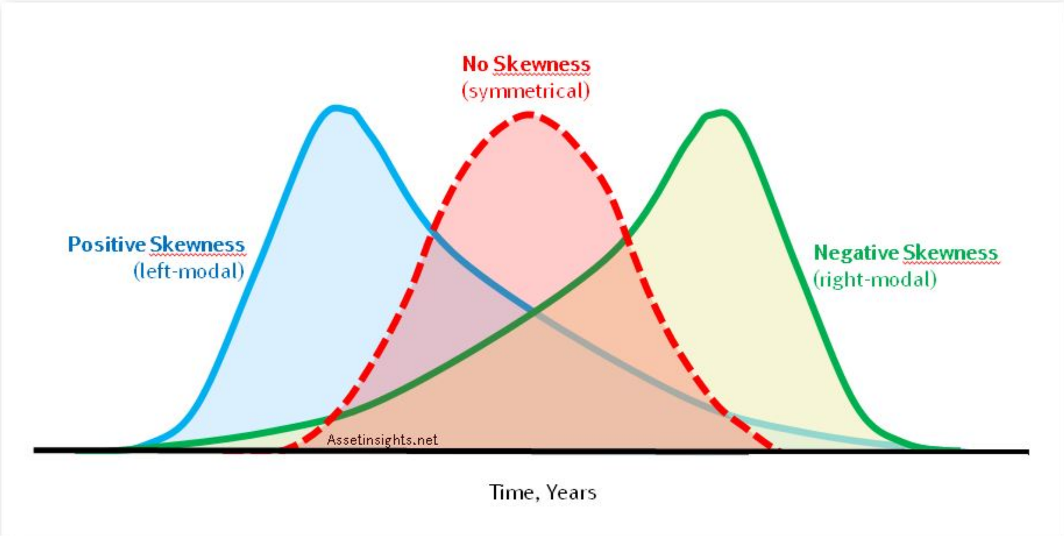
Probability Distributions



Contents

Chapter 1

Probability Distributions

Probability distributions are of two types viz. discrete probability distributions and continuous probability distributions. Binomial and Poission distributions are discrete distributions whereas Normal distribution is a continuous probability distribution.

1.1 Binomial Distribution

Let there be n independent finite trials in an experiment such that

- each trial has only two possible outcomes success and failure
- probability of success(p) and probability of failure(q) are constant for all the trials and $p + q = 1$

If a random variable X denotes the number of successes in n trials then

$P(X = r) = {}^nC_r \; p^r \; q^{n-r}$

or $P(r) = {}^nC_r \; q^{n-r} \; p^r$

$\therefore$ Distribution may be given as $(q + p)^n$

1.1.1 Mean of a Binomial Distribution

$$
\begin{aligned}
\text{Mean} &= \sum_{r=0}^{n} r \; P(r) \\
&= {}^nC_1 \; q^{n-1} \; p + 2 \; {}^nC_2 \; q^{n-2} \; p^2 + 3 \; {}^nC_3 \; q^{n-3} \; p^3 + \cdots + np^n \\
&= nq^{n-1} \; p + \tfrac{2n(n-1)}{2!} q^{n-2} \; p^2 + \tfrac{3n(n-1)(n-2)}{3!} q^{n-3} \; p^3 + \cdots + np^n \\
&= np \left[q^{n-1} + (n-1) q^{n-2} \; p + \tfrac{(n-1)(n-2)}{2!} q^{n-3} \; p^2 + \cdots + p^{n-1} \right] \\
&= np \left[q^{n-1} + {}^{n-1}C_1 \; q^{n-2} \; p + {}^{n-1}C_2 \; q^{n-3} \; p^2 + \cdots + p^{n-1} \right]
\end{aligned}
$$

$$= np(q+p)^{n-1}$$
$$= np \qquad \text{as } (q+p) = 1$$

1.1.2 Variance of a Binomial Distribution

Variance $= \sum_{r=0}^{n} r^2 P(r) -$ (mean)^2

Now $\sum_{r=0}^{n} r^2 \; P(r) = 1^2 \; {}^nC_1 \; q^{n-1} \; p + 2^2 \; {}^nC_2 \; q^{n-2} \; p^2 + 3^2 \; {}^nC_3 \; q^{n-3} \; p^3 + \cdots + n^2 p^n$

$$= nq^{n-1} \; p + 4 \; \tfrac{n(n-1)}{2!} \; q^{n-2} \; p^2 + 9 \tfrac{n(n-1)(n-2)}{3!} \; q^{n-3} \; p^3 + \cdots + n^2 p^n$$
$$= np \left[q^{n-1} + 2(n-1) q^{n-2} \; p + \tfrac{3(n-1)(n-2)}{2!} q^{n-3} \; p^2 + \cdots + np^{n-1} \right]$$
$$= np \left[q^{n-1} + (n-1) q^{n-2} \; p + \tfrac{(n-1)(n-2)}{2!} q^{n-3} \; p^2 + \cdots + p^{n-1} \right]$$
$$+ np \left[(n-1) q^{n-2} \; p + (n-1)(n-2) q^{n-3} \; p^2 + \cdots + (n-1) p^{n-1} \right]$$
$$= np \left\{ \left[(q+p)^{n-1} \right] + (n-1)p \left[q^{n-2} + (n-2) q^{n-3} \; p + \cdots + p^{n-2} \right] \right\}$$
$$= np \left\{ \left[(q+p)^{n-1} \right] + (n-1)p \left[(q+p)^{n-2} \right] \right\}$$
$$= np \left\{ 1 + (n-1)p \right\}$$

$\therefore Variance = np \left\{ 1 + (n-1)p \right\} - n^2 p^2$

$$= np \left\{ 1 - p \right\}$$
$$= npq$$

$\therefore$ for Binomial Distribution

- Mean $= np$
- $\mu_1 = 0$
- $\mu_2 = \sigma^2 = npq$
- $\mu_3 = npq(q-p)$
- $\mu_4 = npq \left[1 + 3pq(n-2) \right]$
- $\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(q-p)^2}{npq}, \quad \gamma_1 = \frac{(q-p)}{\sqrt{npq}}$
- $\beta_2 = \frac{\mu_4}{\mu_2^2} = 3 + \frac{1-6pq}{npq}, \;\; \gamma_2 = \frac{1-6pq}{npq}$

Note:

1. Binomial Distribution is symmetrical if $\beta_1 = 0$ i.e. if $\frac{(q-p)^2}{npq} = 0$ or $q = p = \frac{1}{2}$
2. Binomial Distribution is positively skewed if $\gamma_1 > 0$ i.e. $q-p > 0$ or $1-2p > 0$ or $p < \frac{1}{2}$

3. Binomial Distribution is negatively skewed if $p > \frac{1}{2}$

4. Since $0 < q < 1 \therefore$ for Binomial Distribution $npq < np$ i.e. Mean < Variance

Example 1. To prove Variance of a Binomial Distribution $\leq \frac{n}{4}$

Solution: Variance $= \sigma^2 = npq = np(1-p)$

$= n(p-p^2) = f(p)$ say

For $f(p)$ to be maximum

$f'(p) = 0$ and $f''(p) < 0$

Now $f(p) = n(p - p^2)$

$f'(p) = n(1-2p) = 0 \Rightarrow p = \frac{1}{2}$

$f''(p) = n(0-2) = -2n < 0$

$\therefore f(p)$ is maximum at $p = \frac{1}{2}$

$\therefore$ Maximum variance is at $p = \frac{1}{2}, q = \frac{1}{2}$

i.e. Maximum Variance $= n.\frac{1}{2}.\frac{1}{2} = \frac{n}{4}$

$\therefore$ Variance ≤ 4

Example 2. 6 dice are thrown 729 times. How many times would you expect at least 3 dice to show 1 or 2 ?

Solution: Here the Binomial Distribution is given by:

$N(q+p)^n$, where $n = 6, p = \frac{2}{6} = \frac{1}{3}, q = \frac{2}{3}, N = 729$

$\therefore$ B.D. is given by $729(\frac{2}{3} + \frac{1}{3})^6$

Now $P(X \geq 3) = 729\left[\ {}^6C_3(\frac{2}{3})^3(\frac{1}{3})^3 + {}^6C_4(\frac{2}{3})^2(\frac{1}{3})^4 + {}^6C_5(\frac{2}{3})(\frac{1}{3})^5 + {}^6C_6(\frac{1}{3})^6\right]$

$= \frac{729}{3^6}[160 + 60 + 12 + 1]$

$= 233$

Example 3. If the sum of mean and variance of Binomial Distribution is 4.8 for 5 trials. Find the distribution.

Solution: Given $np + npq = 4.8, \quad n = 5$

$\Rightarrow np(1+q) = 4.8$

$\Rightarrow 5(1-q)(1+q) = 4.8$

$\Rightarrow 1 - q^2 = 0.96$

$\Rightarrow q^2 = 1 - \frac{96}{100}$

$\Rightarrow q^2 = \frac{1}{25}$

$\Rightarrow q = \frac{1}{5}$

$\therefore p = \frac{4}{5}$ and the distribution is $(\frac{1}{5} + \frac{4}{5})^5$

1.1.3 Moments of Binomial Distribution using Moment Generating Function (MGF)

The MGF about origin is expected value of e^{tr}

$M_o(t) = E(e^{tr})$

$= \sum p(r)e^{tr}$

$= \sum^n C_r p^r q^{n-r} e^{tr}$

$= \sum^n C_r (pe^t)^r q^{n-r}$

$= (q + pe^t)^n$

Now $\left[\frac{d}{dt} M_o(t)\right]_{t=0} = [n(q + pe^t)^{n-1} pe^t]_{t=0}$

$= n(q + p)^{n-1} p = np$

$\Rightarrow \acute{\mu_1} = np$

Now $M_a(t) = e^{-at} M_o(t)$

$\Rightarrow M_m(t) = e^{-npt}(q + pe^t)^n),$ as $mean = m = np$

$= (qe^{-pt} + pe^{qt})^n$

$= \left[(q + p) + (-qpt + pqt) + (q\frac{p^2t^2}{2!} + p\frac{q^2t^2}{2!}) + (-q\frac{p^3t^3}{3!} + p\frac{q^3t^3}{3!}) + (q\frac{p^4t^4}{4!} + p\frac{q^4t^4}{4!}) + \cdots\cdots\right]^n$

$= \left[1 + pq\frac{t^2}{2!} + pq(q^2 - p^2)\frac{t^3}{3!} + pq(q^3 + p^3)\frac{t^4}{4!} + \cdots\cdots\right]^n$

$= 1 + npq\frac{t^2}{2!} + npq(q - p)\frac{t^3}{3!} + npq(q^3 + p^3)\frac{t^4}{4!} + \cdots + \frac{n(n-1)}{2!}p^2q^2\frac{t^4}{4} + \cdots\cdots$

or $1 + \mu_1 t + \mu_2\frac{t^2}{2!} + \mu_3\frac{t^3}{3!} + \mu_4\frac{t^4}{4!} + \cdots\cdots$

$= 1 + npq\frac{t^2}{2!} + npq(q - p)\frac{t^3}{3!} + npq(1 + 3pq(n - 2))\frac{t^4}{4!} + \cdots\cdots$

Equating the coefficients of like powers of t on both sides

$\mu_2 = npq, \quad \mu_3 = npq(q - p), \quad \mu_4 = npq\,[1 + 3pq(n - 2)]$

1.2 Poisson Distribution

Poisson Distribution is a limiting case of Binomial Distribution when

1. $n \to \infty$
2. $p \to 0$
3. $np = \lambda$ is finite.

Under these conditions, Binomial Distribution is extended to Poisson Distribution with $P(r) = \frac{e^{-\lambda}\lambda^r}{r!}$

Proof: In Binomial Distribution

$$\begin{aligned}
P(r) &= {}^nC_r\ q^{n-r}\ p^r \\
&= {}^nC_r\ (1-p)^{n-r}\ p^r \\
&= {}^nC_r\ (1-\tfrac{\lambda}{n})^{n-r}\ (\tfrac{\lambda}{n})^r \qquad \because np = \lambda \\
&= \tfrac{n(n-1)(n-2)\cdots(n-(r-1))}{r!}\ (1-\tfrac{\lambda}{n})^{n-r}\ (\tfrac{\lambda}{n})^r \\
&= \tfrac{1(1-\frac{1}{n})(1-\frac{2}{n})\cdots(1-\frac{(r-1)}{n})}{(1-\frac{\lambda}{n})^r\ r!}\ (1-\tfrac{\lambda}{n})^n\ \lambda^r
\end{aligned}$$

Taking limits as $n \to \infty$

$P(r) = \frac{\lambda^r}{r!} \lim_{n\to\infty}(1-\frac{\lambda}{n})^n$

$P(r) = \frac{\lambda^r}{r!} \lim_{n\to\infty}\left[(1-\frac{\lambda}{n})^{\frac{-n}{\lambda}}\right]^{-\lambda}$

$\Rightarrow P(r) = \frac{e^{-\lambda}\lambda^r}{r!} \qquad \because\ \lim_{x\to\infty}(1+\frac{1}{x})^x = e$

1.2.1 Mean of a Poisson Distribution

$$\begin{aligned}
\text{Mean} &= \sum_{r=0}^{\infty} r\ P(r) \text{ as} \qquad n \to \infty \\
&= \sum_{r=0}^{\infty} r\ \tfrac{e^{-\lambda}\lambda^r}{r!} \\
&= \tfrac{e^{-\lambda}\lambda^1}{1!} + 2\tfrac{e^{-\lambda}\lambda^2}{2!} + 3\tfrac{e^{-\lambda}\lambda^3}{3!} + \cdots\cdots \\
&= e^{-\lambda}\lambda\left(1+\lambda+\tfrac{\lambda^2}{2!}+\cdots\cdots\right) \\
&= e^{-\lambda}\ \lambda\ e^{\lambda} \\
&= \lambda
\end{aligned}$$

1.2.2 Variance of a Poisson Distribution

$\text{Variance} = \sum_{r=0}^{\infty} r^2\ P(r) - (\text{mean})^2$

$$\begin{aligned}
\text{Now} \sum_{r=0}^{n} r^2\ P(r) &= 1^2\tfrac{e^{-\lambda}\lambda^1}{1!} + 2^2\tfrac{e^{-\lambda}\lambda^2}{2!} + 3^2\tfrac{e^{-\lambda}\lambda^3}{3!} + 4^2\tfrac{e^{-\lambda}\lambda^4}{4!} + \cdots\cdots \\
&= e^{-\lambda}\lambda\left(1+2\lambda+3\tfrac{\lambda^2}{2!}+4\tfrac{\lambda^3}{3!}+\cdots\cdots\right) \\
&= e^{-\lambda}\lambda\left[\left(1+\lambda+\tfrac{\lambda^2}{2!}+\tfrac{\lambda^3}{3!}+\cdots\right)+\left(\lambda+\lambda^2+3\tfrac{\lambda^2}{3!}+\cdots\right)\right] \\
&= e^{-\lambda}\lambda\left[e^{\lambda}+\lambda\left(1+\lambda+\tfrac{\lambda^2}{2!}+\cdots\cdot\right)\right] \\
&= e^{-\lambda}\lambda\left[e^{\lambda}+\lambda e^{\lambda}\right]
\end{aligned}$$

$$\begin{aligned}
\Rightarrow \sum_{r=0}^{n} r^2\ P(r) &= e^{-\lambda}\ \lambda\ e^{\lambda}\,(1+\lambda) \\
&= \lambda+\lambda^2
\end{aligned}$$

$\Rightarrow$ Variance $= \lambda + \lambda^2 - \lambda^2$

$= \lambda$

1.2.3 Moments of a Poisson Distribution

Since Poisson Distribution is a limiting case of Binomial Distribution, therefore mean and moments may be obtained from Binomial Distribution by taking $np = \lambda$, $p \to 0$ and $q \to 1$ as $\lim_{n\to\infty}$.

- Mean $= \lim_{n\to\infty} np = \lambda$
- $\mu_2 = \lim_{n\to\infty} npq = \lambda$
- $\mu_3 = \lim_{n\to\infty} npq(q-p) = \lambda.1(1-0) = \lambda$
- $\mu_4 = \lim_{n\to\infty} npq\,[1 + 3pq(n-2)]$

  $= \lim_{n\to\infty} npq\,[1 + 3npq - 6pq]$

  $= \lambda.1\,[1 + 3\lambda.1 - 6.0.1]$

  $= \lambda + 3\lambda^2$
- $\beta_1 = \frac{1}{\lambda}$
- $\beta_2 = 3 + \frac{1}{\lambda}$

Example 1. If the standard deviation of a poisson variate X is $\sqrt{3}$, then find the probability that X is strictly positive.

Solution: Variance $= \lambda = 3$

$\therefore P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$

$= \frac{e^{-3}3^r}{r!}, \qquad r = 0, 1, 2, 3, \cdots$

The probability that X is strictly positive is :

$P(X > 0) = 1 - P(X = 0)$

$= 1 - e^{-3}$

Example 2. In a certain factory producing tyres, there is a small chance of 1 in 500 tyres to be defective. The tyres are supplied in lots of 10. Using poisson distribution, calculate the approximate number of lots containing

(1) no defective

(2) at least one defective

tyres in a consignment of 10,000 lots.

Solution: $p = \frac{1}{500}, \quad n = 10, \quad \lambda = np = \frac{1}{50} = 0.02$

$P(X = r) = \frac{e^{-\lambda}\lambda^r}{r!}$

(1) Probability of no defective tyre in a lot is given by:

$$P(X = 0) = \frac{e^{-0.02}(0.02)^0}{0!}$$
$$= e^{-0.02}$$
$$= 0.9802$$

$\therefore$ Number of lots containing no defective tyres $= 10,000 \times 0.9802 = 9802$ lots

(2) Probability of at least one defective tyres in a lot is given by:

$$P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - 0.9802 = 0.0198$$

$\therefore$ Number of lots containing at least one defective tyre $= 10,000 \times 0.0198 = 198$ lots

Example 3. A skilled typist kept a record of his mistakes made per day during 300 working days.

Mistakes (per day)	0	1	2	3	4	5	6
No. of days	143	90	42	12	9	3	1

Solution: Mean number of mistakes $= \frac{\sum f_i \ x_i}{\sum f_i}$

$= \frac{1}{300}[(143 \times 0) + (90 \times 1) + (42 \times 2) + (12 \times 3) + (9 \times 4) + (5 \times 3) + (6 \times 1)]$

$= \frac{267}{300} = 0.89 = \lambda$

Mistakes (per day)	$P(r) = \frac{e^{-0.89}(0.89)^r}{r!}$	Theretical frequency
0	$P(0) = \frac{e^{-0.89}(0.89)^0}{0!} = 0.411$	$0.411 \times 300 = 123.3 = 123$(say)
1	$P(1) = \frac{e^{-0.89}(0.89)^1}{1!} = 0.365$	$0.365 \times 300 = 109.5 = 110$(say)
2	$P(2) = \frac{e^{-0.89}(0.89)^2}{2!} = 0.0.163$	$0.163 \times 300 = 48.9 = 49$(say)
3	$P(3) = \frac{e^{-0.89}(0.89)^3}{3!} = 0.048$	$0.048 \times 300 = 14.4 = 14$(say)
4	$P(4) = \frac{e^{-0.89}(0.89)^4}{4!} = 0.011$	$0.011 \times 300 = 3.3 = 3$(say)
5	$P(5) = \frac{e^{-0.89}(0.89)^5}{5!} = 0.002$	$0.002 \times 300 = 0.6 = 1$(say)
6	$P(6) = \frac{e^{-0.89}(0.89)^6}{6!} = 0.0003$	$0.0003 \times 300 = 0.09 = 0$(say)

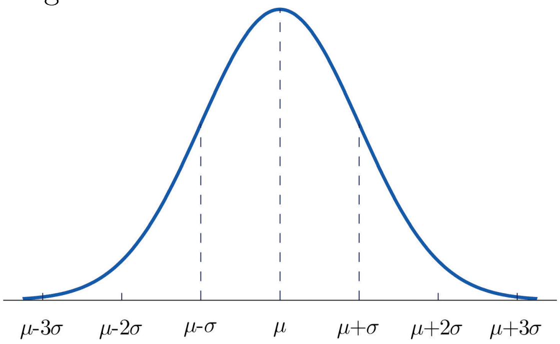
1.3 Normal Distribution

The probability curve of a normal variate x with mean μ and standard deviation σ is given by

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \qquad -\infty < x < \infty$$

Figure 1.1 shows normal distribution curve for variable x with mean μ and standard deviation σ.

Figure 1.1: Normal Distribution Curve

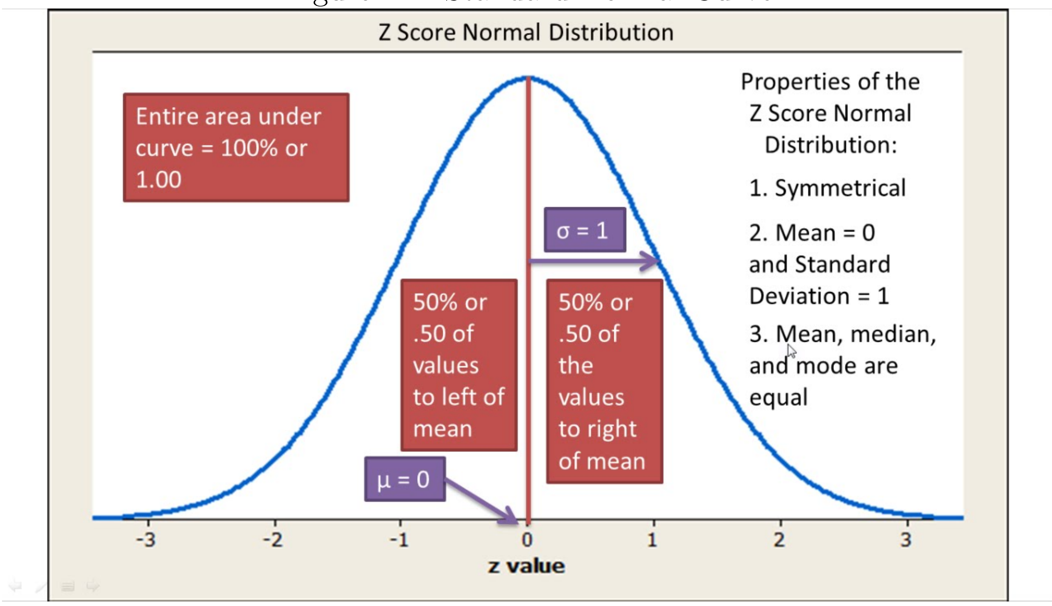


Any normal variable x with mean μ and standard deviation σ is changed to standard normal variate z with mean 0 and standard deviation 1, using the relation $z = \left(\frac{x-\mu}{\sigma}\right)$ and hence the probability density function of z is given by

$$\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \qquad -\infty < z < \infty$$

Figure 1.2 gives normal distribution curve for standard normal variate z.

Figure 1.2: Standard Normal Curve



Area under the curve between the ordinates $a < z < b$ gives the probability of variate z taking the values between a and b.

Note:

- The graph of $p(x)$ or $\phi(z)$ is a bell shaped curve.
- Since the distribution is symmetrical, mean, mode and median coincide at $x = \mu$ or $z = 0$. Also $\beta_1 = 0 \Rightarrow \gamma_1 = 0$ and $\beta_2 = 3 \Rightarrow \gamma_2 = 0$
- The ordinate at $x = \mu$ or $z = 0$, divides the whole area into two equal parts. Also since the total area under the probability curve is 1, area to the right of the ordinate as well as to the left of the ordinate $x = \mu$ or $z = 0$ is 0.5.
- Since the distribution is symmetrical, all moments of odd order about mean are zero

  i.e. $\mu_{2n+1} = 0, n = 0, 1, 2, 3, \cdots\cdots$
- The moments of even order are given by :

  $\mu_{2n} = 1.3.5\cdots\cdots(2n-1)\sigma^{2n}, n = 0, 1, 2, 3, \cdots\cdots$

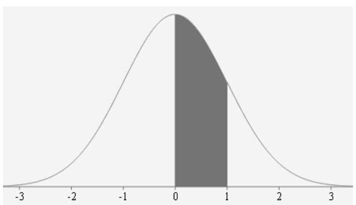
  Putting $n = 1$ and 2, $\mu_2 = \sigma^2$, $\mu_4 = 3\sigma^4$, $\beta_1 = \frac{\mu_3{}^2}{\mu_2{}^3} = 0$, $\beta_2 = \frac{\mu_4}{\mu_2{}^2} = \frac{3\sigma^4}{\sigma^4} = 3$

Example 1. The daily wages of 1000 workers are normally distributed with mean Rs.100 and with a standard deviation of Rs.5. Estimate the number of workers whose daily wages will be:

1. between Rs.100 and Rs.105
2. between Rs.96 and Rs.105
3. more than Rs.110
4. less than Rs.92
5. Also estimate the daily wages of 100 highest paid workers.

Solution: Let the random variable X denote the daily wages in rupees. Then X is a random variable with mean $\mu = 100$ and S.D. $\sigma = 5$.

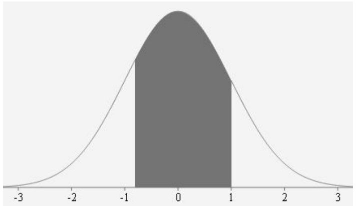
$$Z = \frac{X-\mu}{\sigma} = \frac{X-100}{5}$$



(1) $P(100 < X < 105)$

$= P(\frac{100-100}{5} < Z < \frac{105-100}{5})$

$= P(0 < Z < 1) = 0.3413$ using normal distribution table 1.3, given in the end.



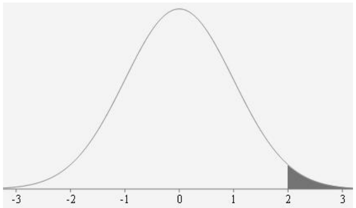
(2) $P(96 < X < 105)$

$= P(\frac{96-100}{5} < Z < \frac{105-100}{5})$

$= P(-0.8 < Z < 1)$

$= P(0 < Z < 0.8) + P(0 < Z < 1)$

$= 0.2881 + 0.3413 = 0.6294$

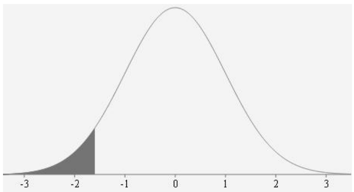


(3) $P(X > 110)$

$= P(Z > \frac{110-100}{5})$

$= P(Z > 2) = 0.5 - P(0 < Z < 2)$

$= 0.5 - 0.4772 = 0.0228$



(4) $P(X < 92)$

$= P(Z < \frac{92-100}{5})$

$= P(Z < -1.6) = P(Z > 1.6)$

$= 0.5 - P(0 < Z < 1.6)$

$= 0.5 - 0.4452 = 0.0548$

(5) Proportion of 100 highest paid workers is $\frac{100}{1000} = \frac{1}{10} = 0.1$

To determine $X = r$ such that $P(X > r) = 0.1$

When $X = r$, $Z = \frac{r-100}{5} = Z_1$ (say)

$\therefore P(Z > Z_1) = 0.1$

$\Rightarrow P(0 < Z < Z_1) = 0.5 - 0.1 = 0.4$

From Normal Distribution table 1.3, $Z_1 = 1.28$ approx.

$\therefore Z_1 = \frac{r-100}{5} = 1.28$

$\Rightarrow r = 100 + 5 \times 1.28 = 106.4$

Hence the lowest daily wages of 100 highest paid workers are Rs.106.4.

1.3.1 Moments of Normal Distribution

- **Odd order moments about mean of normal distribution are zero**

  r^{th} moment about mean $\bar{x}$ is given by $\mu_r = \int_{-\infty}^{\infty} \frac{f_i(x-\bar{x})^r}{n}\, dx$

  $\therefore$ odd order moments of normal distribution with mean μ are given by

$$\mu_{2n+1} = \int_{-\infty}^{\infty} (x-\mu)^{2n+1}\ p(x)dx \quad \because p(x) = \frac{f_i}{n}$$

$$= \frac{1}{\sigma\sqrt{2\pi}} \int_{-\infty}^{\infty} (x-\mu)^{2n+1} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad \because p(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (\sigma z)^{2n+1} e^{\frac{-z^2}{2}} dz \quad \text{By putting } z = \frac{x-\mu}{\sigma},\ dx = \sigma dz$$

$$= \frac{\sigma^{2n+1}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n+1} e^{\frac{-z^2}{2}} dz$$

$$= 0 \qquad z^{2n+1} e^{\frac{-z^2}{2}} \text{ being an odd function of } z$$

- **Even order moments about mean are given by:**

$\mu_{2n} = 1.3.5 \cdot \cdots \cdot (2n-1)\sigma^{2n}, n = 0, 1, 2, 3, \cdots$

Proof: $\mu_{2n} = \int_{-\infty}^{\infty} (x-\mu)^{2n}\ p(x)dx$

$$= \frac{\sigma^{2n}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z^{2n} e^{\frac{-z^2}{2}} dz$$

$$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_{0}^{\infty} z^{2n} e^{\frac{-z^2}{2}} dz \qquad \text{Being even function of } z$$

$$= \frac{2\sigma^{2n}}{\sqrt{2\pi}} \int_{0}^{\infty} (2t)^n e^{-t} (2t)^{-\frac{1}{2}} dt \quad \text{Putting } \frac{z^2}{2} = t \quad \Rightarrow zdz = dt$$

$$\Rightarrow \mu_{2n} = \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \int_0^{\infty} e^{-t} t^{n-\frac{1}{2}} dt$$

$$= \frac{2^n \sigma^{2n}}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \qquad \because \Gamma(n) = \int_{0}^{\infty} e^{-x} x^{n-1} dx$$

Again changing n to $n-1$

$$\mu_{2n-2} = \frac{2^{n-1}\sigma^{2n-2}}{\sqrt{\pi}} \Gamma\left(n - \frac{1}{2}\right)$$

$$\Rightarrow \frac{\mu_{2n}}{\mu_{2n}} = 2\sigma^2 \frac{\Gamma\left(n+\frac{1}{2}\right)}{\Gamma\left(n-\frac{1}{2}\right)} = 2\sigma^2 \frac{\left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right)}{\Gamma\left(n-\frac{1}{2}\right)} = 2\sigma^2\left(n - \frac{1}{2}\right)$$

$$\Rightarrow \mu_{2n} = \sigma^2 (2n-1)\, \mu_{2n-2}$$

$$= [\sigma^2 (2n-1)]\, [\sigma^2 (2n-3)]\, \mu_{2n-4}$$

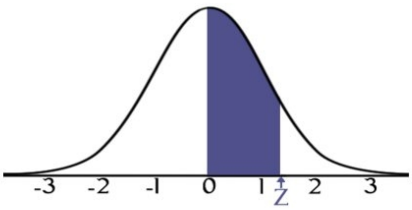
$$= [\sigma^2 (2n-1)]\, [\sigma^2 (2n-3)]\, [\sigma^2 (2n-5)]\, \mu_{2n-6}$$

$$\vdots$$

$$= [\sigma^2 (2n-1)]\, [\sigma^2 (2n-3)]\, [\sigma^2 (2n-5)] \cdots [\sigma^2 3]\, [\sigma^2 1]\, \mu_0$$

$$\Rightarrow \mu_{2n} = 1.3.5 \cdots (2n-5)(2n-3)(2n-1)\, \sigma^{2n}$$

Figure 1.3: Standard Normal Table



STANDARD NORMAL TABLE (Z)

Entries in the table give the area under the curve between the mean and *z* standard deviations above the mean. For example, for *z* = 1.25 the area under the curve between the mean (0) and *z* is 0.3944.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.0000	0.0040	0.0080	0.0120	0.0160	0.0190	0.0239	0.0279	0.0319	0.0359
0.1	0.0398	0.0438	0.0478	0.0517	0.0557	0.0596	0.0636	0.0675	0.0714	0.0753
0.2	0.0793	0.0832	0.0871	0.0910	0.0948	0.0987	0.1026	0.1064	0.1103	0.1141
0.3	0.1179	0.1217	0.1255	0.1293	0.1331	0.1368	0.1406	0.1443	0.1480	0.1517
0.4	0.1554	0.1591	0.1628	0.1664	0.1700	0.1736	0.1772	0.1808	0.1844	0.1879
0.5	0.1915	0.1950	0.1985	0.2019	0.2054	0.2088	0.2123	0.2157	0.2190	0.2224
0.6	0.2257	0.2291	0.2324	0.2357	0.2389	0.2422	0.2454	0.2486	0.2517	0.2549
0.7	0.2580	0.2611	0.2642	0.2673	0.2704	0.2734	0.2764	0.2794	0.2823	0.2852
0.8	0.2881	0.2910	0.2939	0.2969	0.2995	0.3023	0.3051	0.3078	0.3106	0.3133
0.9	0.3159	0.3186	0.3212	0.3238	0.3264	0.3289	0.3315	0.3340	0.3365	0.3389
1.0	0.3413	0.3438	0.3461	0.3485	0.3508	0.3513	0.3554	0.3577	0.3529	0.3621
1.1	0.3643	0.3665	0.3686	0.3708	0.3729	0.3749	0.3770	0.3790	0.3810	0.3830
1.2	0.3849	0.3869	0.3888	0.3907	0.3925	0.3944	0.3962	0.3980	0.3997	0.4015
1.3	0.4032	0.4049	0.4066	0.4082	0.4099	0.4115	0.4131	0.4147	0.4162	0.4177
1.4	0.4192	0.4207	0.4222	0.4236	0.4251	0.4265	0.4279	0.4292	0.4306	0.4319
1.5	0.4332	0.4345	0.4357	0.4370	0.4382	0.4394	0.4406	0.4418	0.4429	0.4441
1.6	0.4452	0.4463	0.4474	0.4484	0.4495	0.4505	0.4515	0.4525	0.4535	0.4545
1.7	0.4554	0.4564	0.4573	0.4582	0.4591	0.4599	0.4608	0.4616	0.4625	0.4633
1.8	0.4641	0.4649	0.4656	0.4664	0.4671	0.4678	0.4686	0.4693	0.4699	0.4706
1.9	0.4713	0.4719	0.4726	0.4732	0.4738	0.4744	0.4750	0.4756	0.4761	0.4767
2.0	0.4772	0.4778	0.4783	0.4788	0.4793	0.4798	0.4803	0.4808	0.4812	0.4817
2.1	0.4821	0.4826	0.4830	0.4834	0.4838	0.4842	0.4846	0.4850	0.4854	0.4857
2.2	0.4861	0.4864	0.4868	0.4871	0.4875	0.4878	0.4881	0.4884	0.4887	0.4890
2.3	0.4893	0.4896	0.4898	0.4901	0.4904	0.4906	0.4909	0.4911	0.4913	0.4916
2.4	0.4918	0.4920	0.4922	0.4925	0.4927	0.4929	0.4931	0.4932	0.4934	0.4936
2.5	0.4938	0.4940	0.4941	0.4943	0.4945	0.4946	0.4948	0.4949	0.4951	0.4952
2.6	0.4953	0.4955	0.4956	0.4957	0.4959	0.4960	0.4961	0.4962	0.4963	0.4964
2.7	0.4965	0.4966	0.4967	0.4968	0.4969	0.4970	0.4971	0.4972	0.4973	0.4974
2.8	0.4974	0.4975	0.4976	0.4977	0.4977	0.4978	0.4979	0.4979	0.4980	0.4981
2.9	0.4981	0.4982	0.4982	0.4983	0.4984	0.4984	0.4985	0.4985	0.4986	0.4986
3.0	0.4987	0.4987	0.4987	0.4988	0.4988	0.4989	0.4989	0.4989	0.4990	0.4990
3.1	0.4990	0.4991	0.4991	0.4991	0.4992	0.4992	0.4992	0.4992	0.4993	0.4993
3.2	0.4993	0.4993	0.4994	0.4994	0.4994	0.4994	0.4994	0.4995	0.4995	0.4995
3.3	0.4995	0.4995	0.4995	0.4996	0.4996	0.4996	0.4996	0.4996	0.4996	0.4997
3.4	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4997	0.4998