Ordinary Differential Equations

2.1 Introduction: Relationship between variables and their rate of changes gives rise to differential equations. Mathematical formulation of most of the physical and engineering problems leads to differential equations. It is very important for scientists and engineers to know the inception and solving of differential equations. These are of two types:

1) Ordinary Differential Equations (ODE)
2) Partial Differential Equations (PDE)

An Ordinary Differential Equation (ODE) involves the derivatives of a dependent variable with respect to a single independent variable whereas a partial differential equation (PDE) contains the derivatives of a dependent variable with respect to two or more independent variables. In this chapter we will confine our studies to ordinary differential equations.

Important Results

- Integration by parts when first function vanishes after a finite number of differentiations: If u and v are both differentiable functions of x, such that u vanishes finitely, then

$$\int \boldsymbol{u.v\,dx = uv_1 - u^{(1)}v_2 + u^{(2)}v_3 - u^{(3)}v_4 + \cdots}$$

Here $\boldsymbol{u^{(n)}}$is derivative of $\boldsymbol{u^{(n-1)}}$ and $\boldsymbol{v_n}$ is integral of $\boldsymbol{v_{n-1}}$

For example

$$\int \boldsymbol{x^2.\sin nx\;dx = (x^2)\left(-\frac{\cos nx}{n}\right) - (2x)\left(-\frac{sinnx}{n^2}\right) + (2)\left(\frac{cosnx}{n^3}\right)}$$

$$\boldsymbol{= -x^2\cos x\;\; + 2x\sin x + 2\cos x}$$

$$\boldsymbol{= -\frac{x^2\cos nx}{n} + \frac{2x\;sinnx}{n^2} + \frac{2cosnx}{n^3}}$$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

2.2 Order and Degree of Ordinary Differential Equations (ODE)

A general ordinary differential equation of n^{th}order can be represented in the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$. Order of an ordinary differential equation is that of the highest derivative occurring in it and the degree is the power of highest derivative after it has been freed from all radical signs.

The differential equation$\left(\frac{d^2y}{dx^2} + 2y\right)^3 + \frac{d^3y}{dx^3} + y = 0$ is having order 3 and degree 1, whereas $\left(\frac{d^3y}{dx^3} + 2y\right)^3 + \frac{d^2y}{dx^2} + y = 0$ is of order 3 and degree 3.

The differential equation $\sqrt{\frac{d^2y}{dx^2}} = \frac{d^3y}{dx^3} + y$ is having order 3 and degree 2.

2.3 Geometric Meaning of First and Second Order Differential Equations

The order of a differential equation depends upon the number of arbitrary constants present in the original equation. For instance, the equation $y = mx$ has only one arbitrary constant, therefore the corresponding differential equation will be of first order; while the equation $y = mx + c$ has two arbitrary constants, hence it will lead to a second order differential equation. Now since for any first order differential equation, m can take infinite values, hence the locus of the equation is made up of single infinity of curves. Also, for a second order differential equation, m can take infinite values and

c can take infinite values at the same time, therefore the general solution can be said to have double infinity of curves. Hence, we can conclude that any n^{th} order differential equation has n^{th} infinity of curves as its general solution.

2.4 Approximate Solutions of Differential Equations: In some cases, where analytical methods are tedious to apply, we can find approximate solutions of first order differential equations using graphical or numerical methods.

2.4.1 Approximating the Curve Using Directional Fields or Slope Fields

With the help of direction fields (slope fields), we can approximate the general solution of a first-order differential equation of the type $\frac{dy}{dx} = f(x, y)$ by drawing isoclines (lines having same slopes). Here we use the fact that $\frac{dy}{dx}$ at any given point (x, y) on the curve gives the slope of the tangent (or gradient) to the curve at (x, y).

Algorithm to plot the curve using slope fields:

Step1: Arrange the given first order differential equation in the form $\frac{dy}{dx} = f(x, y)$, where $\frac{dy}{dx} = m$ is the slope if the tangent to the curve at any point (x, y) on the curve.

Step2: Draw the isoclines corresponding to different values of m like -1, -2, 0, 1, 2 etc. Here the isoclines corresponding to $m = 0$, known as null clines (the tangents parallel to x-axis), provide the positioning of the curve about the x-axis.

Step3: Plot the family of the curves by estimating the direction with the help of isoclines plotted on the direction field.

Example1 Find the family of curves for the equation $\frac{dy}{dx} = x$ using slope fields.

Solution: Let $\frac{dy}{dx} = x = m = \tan\theta$

The isoclines corresponding to different values of m are as computed as given below:

$m = \tan\theta$	θ	x
-2	116.57^0	-2
-1	135^0	-1
0	0^0	0
1	45^0	1
2	63.43^0	2

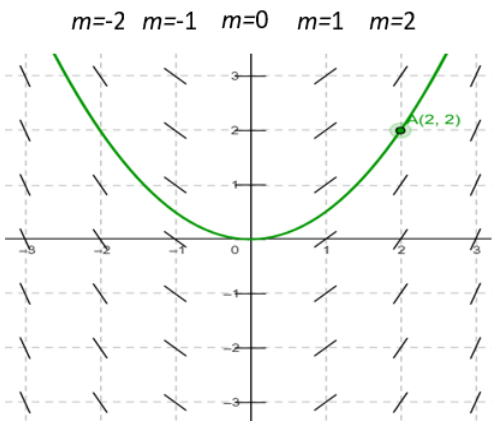


Figure 1

Here θ is the angle made by the tangent to the curve at the point (x, y) with positive direction of x-axis. The table shows that the null clines are placed about the line $x = 0$, i.e. y-axis. Also slopes of all the tangents to the family of curves about the line $x = 1$ are one, i.e. tangents to the curves make an angle $\frac{\pi}{4}$, when the curve passes through the line $x = 1$, and similar interpretations for remaining values of m in the table.

It is evident that infinite number of curves can be drawn in the given direction field. Figure 1 and Figure 2 show the two curves in the family of curves.

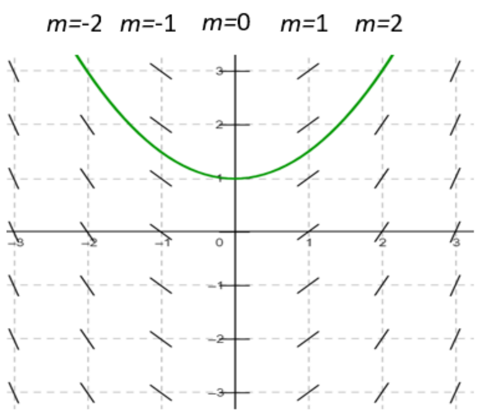


Figure 2

Note: Analytic solution of the given differential equation using variable separable method may be computed as $y = \frac{x^2}{2} + c$. Clearly Figure1 shows a particular solution for $c = 0$ and Figure2 shows the solution corresponding to $c = 1$.

Example2 Find the family of curves for the equation $\frac{dy}{dx} = y$ using slope fields.

Solution: Let $\frac{dy}{dx} = y = m = \tan\theta$

The isoclines corresponding to different values of m are as computed as given below:

$m = \tan\theta$	θ	y
-2	116.57^0	-2
-1	135^0	-1
0	0^0	0
1	45^0	1
2	63.43^0	2

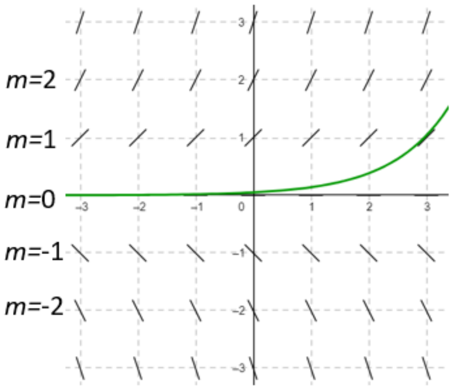


Figure 3

Here θ is the angle made by the tangent to the curve at the point (x, y) with positive direction of x-axis. The table shows that the null clines are placed about the line $y = 0$, i.e. x-axis. Also slopes of all the tangents to the family of curves about the line $y = 1$ are one, i.e. tangents to the curves make an angle $\frac{\pi}{4}$, when the curve passes through the line $y = 1$, and similar interpretations for remaining values of m in the table. It is evident that infinite number of curves can be drawn in the given direction field. Figures 3 and 4 show the two curves in the family of curves.

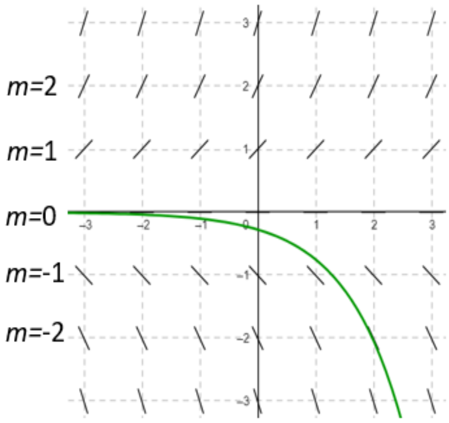


Figure 4

Note: Analytic solution of the given differential equation using variable separable method may be computed as $y = ce^x$. Clearly Figure 3 depicts a particular solution for a positive value of arbitrary constant c and Figure 4 shows a solution corresponding to a negative value of c.

2.4.2 Numerical Methods to Find an Approximate solution

The first-order differential equation and a given initial value constitute a first-order initial value problem given as: $\frac{dy}{dx} = f(x, y)$; $y(x_0) = y_0$. An approximate solution can be found using numerical methods; Euler's method is one of them.

2.4.2.1 Euler's Method

Euler's Method provides us with a numerical solution of the initial value problem

$\frac{dy}{dx} = f(x, y);\ \ y(x_0) = y_0 \cdots ①$, by joining multiple small line segments A_0A_1 , A_1A_2, A_2A_3,⋯, and making an approximation of the actual curve, as shown in the adjoining figure.

Thus if $[x_0, x_1]$ is the small interval, where $x_1 = x_0 + h$, we approximate the curve by the tangent drawn to curve at the point A_0, having coordinates (x_0, y_0), whose equation is given by $y - y_0 = m(x - x_0)$, where m is slope of tangent at the point (x_0, y_0)

Also $m = \left.\frac{dy}{dx}\right]_{(x_0,y_0)} = f(x_0, y_0)$ from 1

$\Rightarrow \quad y = y_0 + f(x_0, y_0)\,(x - x_0)$

$\Rightarrow \; y_1 = y_0 + f(x_0, y_0)\,(x_1 - x_0) \quad \because y(x_1) = y_1$

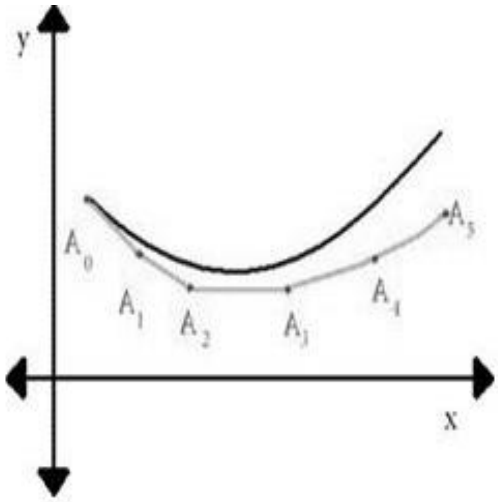
$\Rightarrow \; y_1 = y_0 + hf(x_0, y_0) \qquad \because x_1 - x_0 = h$

Similarly for range $[x_1, x_2]$

$y_2 = y_1 + hf(x_1, y_1)$

$\vdots$

$y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$



It is evident from the given figure that h has to be kept small to avoid the approximations diverging away from the curve. As a result, this method is very slow and needs to be improved.

Example3 Using Euler's method, Compute $y(0.12)$ for the initial value problem:

$\frac{dy}{dx} = x^3 + y$; $y(0) = 1$, taking $h = 0.02$.

Solution: Given $f(x,y) = x^3 + y$, $x_0 = 0$, $y_0 = 1$, $x_n = x_{n-1} + h$, $h = 0.02$

$\therefore x_1 = 0.02$, $x_2 = 0.04$, $x_3 = 0.06$, $x_4 = 0.08$, $x_5 = 0.1$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$\Rightarrow \; y_n = y_{n-1} + h({x_{n-1}}^3 + y_{n-1}) \qquad \cdots 1$

Putting $n = 1$ in 1, $y_1 = y(0.02) = y_0 + h\,({x_0}^3 + y_0)$

$\therefore \; y_1 = 1 + 0.02(\,0 + 1) = 1.02$

Putting $n = 2$ in 1, $y_2 = y(0.04) = y_1 + h\,({x_1}^3 + y_1)$

$\therefore \; y_2 = 1.02 + 0.02(\,(0.02)^3 + 1.02) = 1.04040016$

Putting $n = 3$ in 1, $y_3 = y(0.06) = y_2 + h\,({x_2}^3 + y_2)$

$\therefore \; y_3 = 1.04040016 + 0.02(\,(0.04)^3 + 1.04040016) = 1.061209443$

Putting $n = 4$ in 1, $y_4 = y(0.08) = y_3 + h\,({x_3}^3 + y_3)$

$\therefore \; y_4 = 1.061209443 + 0.02(\,(0.06)^3 + 1.061209443) = 1.082437952$

Putting $n = 5$ in 1, $y_5 = y(0.1) = y_4 + h\,({x_4}^3 + y_4)$

$\therefore \; y_5 = 1.082437952 + 0.02(\,(0.08)^3 + 1.082437952) = 1.104096951$

Putting $n = 6$ in 1, $y_6 = y(0.12) = y_5 + h\,({x_5}^3 + y_5)$

$\therefore \; y_6 = 1.104096951 + 0.02(\,(0.1)^3 + 1.104096951) = 1.126198890$

Thus at $x = 0.12$, $y = 1.126198890 \;\Rightarrow y(0.12) = 1.126198890$

Example4 Using Euler's method, solve $\frac{dy}{dx} = \frac{x-y}{2}$; $y(0) = 1$, over the interval $[0,2]$, taking the step size 0.5.

Solution: Given $f(x,y) = \frac{x-y}{2}$, $x_0 = 0$, $y_0 = 1$, $x_n = x_{n-1} + h$, $h = 0.5$

$\therefore x_1 = \frac{1}{2} = 0.5$, $x_2 = 1$, $x_3 = \frac{3}{2} = 1.5$, $x_4 = 2$

Using Euler's method $y_n = y_{n-1} + hf(x_{n-1}, y_{n-1})$

$\Rightarrow \; y_n = y_{n-1} + \frac{h}{2}(x_{n-1} - y_{n-1})$

or $y_n = y_{n-1} + 0.25(x_{n-1} - y_{n-1})$ $\cdots ①$

Putting $n = 1$ in ①, $y_1 = y\left(\frac{1}{2}\right) = y_0 + 0.25(x_0 - y_0)$

$\therefore\ y_1 = 1 + 0.25(0 - 1) = 0.75$

Putting $n = 2$ in ①, $y_2 = y(1) = y_1 + 0.25(x_1 - y_1)$

$\therefore\ y_2 = 0.75 + 0.25(0.5 - 0.75) = 0.6875$

Putting $n = 3$ in ①, $y_3 = y\left(\frac{3}{2}\right) = y_2 + 0.25(x_2 - y_2)$

$\therefore\ y_3 = 0.6875 + 0.25(1 - 0.6875) = 0.765625$

Putting $n = 4$ in ①, $y_4 = y(2) = y_3 + 0.25(x_3 - y_3)$

$\therefore\ y_4 = 0.765625 + 0.25(1.5 - 0.765625) = 0.94921875$

2.5 Separable Ordinary Differential Equations

Any Separable differential equation can be arranged in the form $N(y)\frac{dy}{dx} = M(x)$, and can be solved by integrating both sides with respect to x as shown:

$$\int N(y)\frac{dy}{dx}dx = \int M(x)dx \Rightarrow \int N(y)dy = \int M(x)dx$$

Example5 Solve the differential equation $\frac{dy}{dx} = x$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow dy = xdx$

Integrating both sides, we have

$\int dy = \int xdx$

$\Rightarrow y = \frac{x^2}{2} + c$ is the required solution of the given differential equation.

Example6 Solve the differential equation $\frac{dy}{dx} = y$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow \frac{dy}{y} = dx$

Integrating both sides, we have

$\int \frac{dy}{y} = \int dx$

$\Rightarrow \log y = x + \log c$

$\Rightarrow \log\frac{y}{c} = x \Rightarrow e^{\log\frac{y}{c}} = e^x \Rightarrow \frac{y}{c} = e^x$

$\Rightarrow y = ce^x$ is the required solution of the given differential equation.

Example7 Solve the differential equation $\frac{dy}{dx} = e^{x+y}\sec y$

Solution: Arranging the given differential equation in variable separable form

$\Rightarrow e^{-y}\cos y\, dy = e^x dx$

Integrating both sides, we have

$\int e^{-y}\cos y\, dy = \int e^x dx$

$\Rightarrow e^{-y}\sin y + \int e^{-y}\sin y\, dy = e^x + c$

$\Rightarrow e^{-y}\sin y + [-e^{-y}\cos y - \int e^{-y}\cos y\, dy] = e^x + c$

$\Rightarrow\ e^{-y}\sin y - e^{-y}\cos y - I = e^x + c$, if $\int e^{-y}\cos y\, dy = I$ say

$\Rightarrow \frac{e^{-y}}{2}(\sin y - \cos y) = e^x + c$ is the required solution

Example8 Solve the differential equation $y - x\frac{dy}{dx} = y^2 + \frac{dy}{dx}$

Solution: Arranging the given differential equation in variable separable form

$$\Rightarrow (1+x)\frac{dy}{dx} = y - y^2$$

$$\Rightarrow \frac{dy}{y(1-y)} = \frac{dx}{(1+x)} \Rightarrow \frac{dy}{y} + \frac{dy}{1-y} = \frac{dx}{(1+x)}$$

Integrating both sides, we have

$$\log|y| - \log|1-y| = \log|1+x| + \log c$$

$$\Rightarrow \log\frac{y}{1-y} = \log c(1+x)$$

$$\Rightarrow \frac{y}{(1-y)(1+x)} = c$$

$\Rightarrow (1+x)(1-y) = Ay$, where $A = \frac{1}{c}$ is an arbitrary constant

2.6 Differential Equations Reducible to Separable Form

In some cases, a differential equation can be reduced to separable form by the substitution $f(x,y) = t$ and can be easily solved thereafter.

Example9 Solve the differential equation $(x+y)^2\frac{dy}{dx} = 1$

Solution: Putting $x + y = t\ \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Given differential equation can be rewritten as $t^2\left(\frac{dt}{dx} - 1\right) = 1$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{t^2} + 1 = \frac{1+t^2}{t^2}$$

$$\Rightarrow \frac{t^2}{1+t^2}dt = dx$$

Integrating both sides, we have

$$\int\frac{t^2}{1+t^2}dt = \int dx$$

$$\Rightarrow \int\frac{1+t^2-1}{1+t^2}dt = \int dx$$

$$\Rightarrow \int\left(1 - \frac{1}{1+t^2}\right)dt = \int dx$$

$$\Rightarrow\ t - \tan^{-1}t = x + c$$

$$\Rightarrow x + y - \tan^{-1}(x+y) = x + c$$

$\Rightarrow y - \tan^{-1}(x+y) = c$, where c is an arbitrary constant

Example10 Solve the differential equation $\cos(x+y)\frac{dy}{dx} = 1$

Solution: Putting $x + y = t\ \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\therefore$ Given differential equation can be rewritten as $\cos t\left(\frac{dt}{dx} - 1\right) = 1$

$$\Rightarrow \frac{dt}{dx} = \frac{1}{\cos t} + 1 = \frac{1+\cos t}{\cos t}$$

$$\Rightarrow \frac{\cos t}{1+\cos t}dt = dx$$

Integrating both sides, we have

$$\int \frac{\cos t}{1+\cos t} dt = \int dx$$

$$\Rightarrow \int \frac{1+\cos t - 1}{1+\cos t} dt = \int dx$$

$$\Rightarrow \int \left(1 - \frac{1}{1+\cos t}\right) dt = \int dx$$

$$\Rightarrow \int \left(1 - \frac{1}{2} sec^2 \frac{t}{2}\right) dt = \int dx$$

$$\Rightarrow\ t - \tan\frac{t}{2} = x + c$$

$$\Rightarrow x + y - \tan\frac{(x+y)}{2} = x + c$$

$\Rightarrow y - \tan\frac{(x+y)}{2} = c$, where c is an arbitrary constant

2.7 Exact Differential Equations of First Order

A differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is said to be exact if it can be directly obtained from its primitive by differentiation.

Figure 1

Theorem: The necessary and sufficient condition for the equation $M(x,y)dx + N(x,y)dy = 0$ to be exact is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Working Rule to Solve an Exact Differential Equation:

1. For the equation $M(x,y)dx + N(x,y)dy = 0$, check the condition for exactness i.e., $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$
2. Solution of the given differential equation is given by $I_1 + I_2 = c$
   Where $I_1 = \int M\, dx$, taking y as constant
   $I_2 = \int N_y\, dy$, Here N_y denotes terms in $N(x,y)$ which do not contain x

or $\underset{y\ constant}{\int M dx} + \int N_y dy = C$

Example11 Solve the differential equation:

$(e^y + 1)\cos x\, dx + e^y \sin x\, dy = 0$ 1

Solution: $M = (e^y + 1)\cos x$, $N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x$, $\frac{\partial N}{\partial x} = e^y \cos x$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of the differential equation 1 is given by:

$$\underset{y\ constant}{\int (e^y + 1)\cos x\ dx} + \int 0\, dy = C$$

$$\Rightarrow (e^y + 1)\sin x = C$$

Example12 Solve the differential equation:

$(\sec x \tan x\ \tan y - e^x)dx + (\sec x\, sec^2\, y)dy = 0$ 1

Solution: $M = \sec x \tan x\ \tan y - e^x$, $N = \sec x\, sec^2\, y$

$\frac{\partial M}{\partial y} = \sec x \tan x\, sec^2\, y$, $\frac{\partial N}{\partial x} = \sec x \tan x\, sec^2\, y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\underset{y\ constant}{\int (\sec x \tan x\ \tan y - e^x)\ dx} + \int 0\ dy = C$$

$$\Rightarrow \sec x \tan y - e^x = C$$

Example13 Solve the differential equation:

$$\left[y\ \left(1+\frac{1}{x}\right) + \cos y\right] dx + [x + \log x - x \sin y]dy = 0\ \ldots\ldots ①$$

Solution: $M = y\ \left(1+\frac{1}{x}\right) + \cos y$, $N = x + \log x - x \sin y$

$\frac{\partial M}{\partial y} = \left(1+\frac{1}{x}\right) - \sin y$, $\frac{\partial N}{\partial x} = \left(1+\frac{1}{x}\right) - \sin y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\int \left[\underset{y\ constant}{y\ \left(1+\frac{1}{x}\right) + \cos y)}\right] dx + \int 0\ dy = C$$

$$\Rightarrow y\ (x + \log x) + x \cos y = C$$

Example14 Solve the differential equation:

$$x\ dx + y\ dy = \frac{a^2(x\ dy - y\ dx)}{x^2+y^2}\ \ldots\ldots ①$$

Solution: ① $\Rightarrow \left(x + \frac{a^2\ y}{x^2+y^2}\right) dx + \left(y - \frac{a^2\ x}{x^2+y^2}\right) dy = 0$

$$M = x + \frac{a^2\ y}{x^2+y^2},\ N = y - \frac{a^2\ x}{x^2+y^2}$$

$$\frac{\partial M}{\partial y} = \frac{a^2\ (x^2-y^2)}{(x^2+y^2)^2},\ \frac{\partial N}{\partial x} = \frac{a^2\ (x^2-y^2)}{(x^2+y^2)^2}$$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution of ① is given by:

$$\underset{y\ constant}{\int \left(x + \frac{a^2\ y}{x^2+y^2}\right) dx} + \int y\ dy = C$$

$$\Rightarrow \frac{x^2}{2} + a^2 tan^{-1}\frac{x}{y} + \frac{y^2}{2} = \mathrm{C}$$

$$\Rightarrow x^2 + 2a^2 tan^{-1}\frac{x}{y} + y^2 = \mathrm{D}\ ,\ D = 2C$$

2.8 Equations Reducible to Exact Differential Equations: Sometimes a differential equation of the form $M(x,y)dx + N(x,y)dy = 0$ is not exact i.e., $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$. It can be made exact by multiplying the equation by some function of x and y known as integrating factor (IF).

2.8.1 Integrating Factor (IF) Found by Inspection

Some non-exact differential equations can be grouped or rearranged and solved directly by integration, after multiplying by an integrating factor (IF) which can be found just by inspection as shown below:

Term	IF	Result

$xdy + ydx$	1. $\frac{1}{xy}$ 2. $\frac{1}{(xy)^n}, n \neq 1$ 3. $\cos(xy)$ 4. $\sin(xy)$	$\frac{xdy + ydx}{xy} = \frac{1}{y}\mathrm{dy} + \frac{1}{x}\,\mathrm{dx} = d[\log(xy)]$ $\frac{xdy + ydx}{(xy)^n} = \frac{\mathrm{d}(xy)}{(xy)^n} = -d\left[\frac{1}{(n-1)(xy)^{n-1}}\right]$ $\cos(xy)\,(x\,dy + y\,dx) = d\,[\sin(xy)]$ $\sin(xy)\,(x\,dy + y\,dx) = -d\,[\cos(xy)]$
$xdy - ydx$	1. $\frac{1}{x^2}$ 2. $\frac{1}{y^2}$ 3. $\frac{1}{xy}$ 4. $\frac{1}{x^2+y^2}$ 5. $\frac{1}{x\sqrt{x^2-y^2}}$	$\frac{xdy - ydx}{x^2} = d\left[\frac{y}{x}\right]$ $\frac{xdy - ydx}{y^2} = -d\left[\frac{x}{y}\right]$ $\frac{xdy - ydx}{xy} = d\left[log\frac{y}{x}\right]$ $\frac{xdy - ydx}{x^2 + y^2} = d\left[tan^{-1}\frac{y}{x}\right]$ $\frac{xdy - ydx}{x\sqrt{x^2 - y^2}} = d\left[sin^{-1}\frac{y}{x}\right]$
$xdx + ydy$	1. $\frac{1}{x^2+y^2}$ 2. $\frac{1}{(x^2+y^2)^n}, n \neq 1$	$\frac{xdx + ydy}{x^2 + y^2} = \frac{1}{2}d[log(x^2 + y^2)]$ $\frac{xdx + ydy}{(x^2 + y^2)^n} = \frac{1}{2}d\left[\frac{(x^2 + y^2)^{-n+1}}{-n+1}\right]$

Example15 Solve the differential equation:

$x\,dy - y\,dx + 2x^3dx = 0$ .....①

Solution: ① $\Rightarrow (-y + 2x^3)dx + xdy = 0$

$M = -y + 2x^3\,,\; N = x$

$\frac{\partial M}{\partial y} = -1\,,\; \frac{\partial N}{\partial x} = 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(x\,dy - y\,dx)$

① may be rewritten as : $\frac{xdy - ydx}{x^2} + 2x\,dx = 0$

$\Rightarrow d\left[\frac{y}{x}\right] + 2x\,dx = 0$ ......②

Integrating ②, solution is given by : $\frac{y}{x} + x^2 = C$

$\Rightarrow y + x^3 = Cx$

Example16 Solve the differential equation:

$y\,dx - x\,dy + (1 + x^2)dx + x^2 \cos y\,dy = 0$ .....①

Solution: $\Rightarrow (y + 1 + x^2)dx + (x^2 \cos y - x)dy = 0$

$M = y + 1 + x^2,\ N = x^2 \cos y - x$

$\frac{\partial M}{\partial y} = 1\,,\ \frac{\partial N}{\partial x} = 2x \cos y - 1$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2}$ as integrating factor due to presence of the term $(y\,dx - x\,dy)$

① may be rewritten as : $\frac{ydy - xdx}{x^2} + \left(\frac{1}{x^2} + 1\right)dx + \cos y\,dy = 0$

$\Rightarrow -d\left[\frac{y}{x}\right] + \left(\frac{1}{x^2} + 1\right)dx + \cos y\,dy = 0$ ......②

Integrating ②, solution is given by : $-\frac{y}{x} + \left(-\frac{1}{x} + x\right) + \sin y = C$

$\Rightarrow x^2 - y - 1 + x \sin y = Cx$

Example17 Solve the differential equation:

$x\,dx + y\,dy = a(x^2 + y^2)dy$ .....①

Solution: $\Rightarrow xdx + (y - a(x^2 + y^2))dy = 0$

$M = x\,,\ N = y - a(x^2 + y^2)$

$\frac{\partial M}{\partial y} = 0\,,\ \frac{\partial N}{\partial x} = -2ax$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Taking $\frac{1}{x^2 + y^2}$ as integrating factor due to presence of the term $(x\,dx + y\,dy)$

① may be rewritten as: $\frac{xdx + ydy}{x^2 + y^2} - a\,dy = 0$

$\Rightarrow \frac{1}{2}d[log(x^2 + y^2)] - a\,dy = 0$

$\Rightarrow d[log(x^2 + y^2)] - 2\,a\,dy = 0$ ......②

Integrating ②, solution is given by: $(x^2 + y^2) - 2ay = C$, C is an arbitrary constant

Example18 Solve the differential equation:

$a(x\,dy + 2y\,dx) = xy\,dy$ .....①

Solution: ① $\Rightarrow 2aydx + (ax - xy)dy = 0$

$M = 2ay\,,\ N = ax - xy$

$\frac{\partial M}{\partial y} = 2a\,,\ \frac{\partial N}{\partial x} = a - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as $a(x\,dy + y\,dx) + ay\,dx = xy\,dy$ ......②

Taking $\frac{1}{xy}$ as integrating factor due to presence of the term $(x\,dy + y\,dx)$

② may be rewritten as : $a\frac{xdy + ydx}{xy} + \frac{a}{x}\,dx - dy = 0$

$\Rightarrow\ ad[\log(xy)] + \ \frac{a}{x}\, dx - dy = 0$ ……③

Integrating ③ solution is given by: $a\log(xy) + \ a\log x \ \ - y = C$

$\Rightarrow a\log(x^2 y) \ - \ y = C$, C is an arbitrary constant

Example19 Solve the differential equation:

$$x^4 \frac{dy}{dx} + x^3 y + \sec(xy) = 0 \quad .....①$$

Solution: ① $\Rightarrow (x^3 y + \sec(xy))dx + \ x^4 dy = 0$

$M = \ x^3 y + \ \sec(xy)\,,\ N = x^4$

$\frac{\partial M}{\partial y} = \ x^3 + \ x\sec(xy)\tan(xy)\,,\ \frac{\partial N}{\partial x} = 4x^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting ① as: $x^3(x\,dy + \ y\,dx\,) \ + \ \sec(xy)\ dx = 0$

$\Rightarrow \ \frac{(x\,dy + y\,dx\,)}{\sec(xy)} - \ x^{-3}dx = 0$

$\Rightarrow \cos(xy)\,(x\,dy + \ y\,dx\,) - \ x^{-3}dx = 0$

$\Rightarrow d\,[\sin(xy)] - \frac{1}{2}d(x^{-2})dx = 0$ ……②

Integrating ②, we get the required solution as:

$\sin(xy) - \frac{x^{-2}}{2} \ = C$

$\Rightarrow 2\,x^2\sin(xy) - 1 \ = Cx^2$

2.8.2 Integrating Factor (IF) of a Non-Exact Homogeneous Equation

If the equation $Mdx + Ndy = 0$ is a homogeneous equation, then the integrating factor (IF) will be $\frac{1}{Mx+Ny}$, provided $Mx + Ny \neq 0$

Example20 Solve the differential equation:

$(x^3 + \ y^3)dx \ - \ xy^2\,dy = 0$ …..①

Solution: $M = x^3 + \ y^3$, $N = -\,xy^2$

$\frac{\partial M}{\partial y} = \ 3y^2$, $\frac{\partial N}{\partial x} = \ -y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Since ① is a homogeneous equation, ∴ IF = $\frac{1}{Mx+Ny} = \ \frac{1}{x^4 + xy^3 - xy^3} = \frac{1}{x^4}$

∴ ① may be rewritten as : $\left(\frac{1}{x} + \ \frac{y^3}{x^4}\right)dx - \ \frac{y^2}{x^3}dy = 0$ ……②

New $M = \ \frac{1}{x} + \ \frac{y^3}{x^4}$, New $N = -\ \frac{y^2}{x^3}$

$\frac{\partial M}{\partial y} = \ \frac{3y^2}{x^4}$, $\frac{\partial N}{\partial x} = \ \frac{3y^2}{x^4}$

$\frac{\partial M}{\partial y} = \ \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{1}{x} + \ \frac{y^3}{x^4}\right)\ dx \ + \ \int 0\ dy = C$$

$\Rightarrow \log x - \frac{1}{3}\left(\frac{y}{x}\right)^3 = \mathrm{C}$

Example21 Solve the differential equation:

$(3y^4 + 3x^2y^2)dx + (x^3y - 3xy^3)\, dy = 0$ .....①

Solution: $M = 3y^4 + 3x^2y^2$, $N = x^3y - 3xy^3$

$\frac{\partial M}{\partial y} = 12y^3 + 6x^2y$, $\frac{\partial N}{\partial x} = 3x^2y - 3y^3$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Since ① is a homogeneous equation

∴ IF$= \frac{1}{Mx+Ny} = \frac{1}{3xy^4+3x^3y^2+x^3y^2-3xy^4} = \frac{1}{4x^3y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{3y^2}{4x^3} + \frac{3}{4x}\right) dx + \left(\frac{1}{4y} - \frac{3y}{4x^2}\right) dy = 0$ ......②

New $M = \frac{3y^2}{4x^3} + \frac{3}{4x}$, New $N = \frac{1}{4y} - \frac{3y}{4x^2}$

$\frac{\partial M}{\partial y} = \frac{6y}{4x^3} = \frac{3y}{2x^3}$, $\frac{\partial N}{\partial x} = \frac{3y}{2x^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(\frac{3y^2}{4x^3} + \frac{3}{4x}\right) dx + \int \frac{1}{4y}\, dy = C$$

$\Rightarrow \frac{-3y^2}{8x^2} + \frac{3}{4}\log x + \frac{1}{4}\log y = \mathrm{C}$

$\Rightarrow \log x^3y - \frac{3y^2}{2x^2} = \mathrm{D}$, $\mathrm{D} = 4\mathrm{C}$

2.8.3 Integrating Factor of a Non-Exact Differential Equation of the Form $\boldsymbol{yf_1(xy)dx + xf_2(xy)\, dy = 0}$: If the equation $Mdx + Ndy = 0$ is of the given form**,** then the integrating factor (IF) will be $\frac{1}{Mx-Ny}$ provided $Mx - Ny \neq 0$

Example22 Solve the differential equation:

$y(1 + xy)dx + x(1 - xy)\, dy = 0$ .....①

Solution: $M = y + xy^2$, $N = x - x^2y$

$\frac{\partial M}{\partial y} = 1 + 2xy$, $\frac{\partial N}{\partial x} = 1 - 2xy$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ IF $= \frac{1}{Mx-Ny} = \frac{1}{xy+x^2y^2-xy+x^2y^2} = \frac{1}{2x^2y^2}$

∴ ① may be rewritten after multiplying by IF as:

$\left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx + \left(\frac{1}{2xy^2} - \frac{1}{2y}\right) dy = 0$ ......②

New $M = \frac{1}{2x^2y} + \frac{1}{2x}$, New $N = \frac{1}{2xy^2} - \frac{1}{2y}$

$\frac{\partial M}{\partial y} = \frac{-1}{2x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{2x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ 2 is an exact differential equation.

Solution of 2 is given by:

$$\underset{y\ constant}{\int\left(\frac{1}{2x^2y} + \frac{1}{2x}\right) dx} + \int -\frac{1}{2y} dy = 0$$

$\Rightarrow \frac{-1}{2xy} + \frac{1}{2}\log x - \frac{1}{2}\log y = \mathrm{C}$

$\Rightarrow \log\frac{x}{y} - \frac{1}{xy} = \mathrm{D},\ \mathrm{D} = 2\mathrm{C}$

Example23 Solve the differential equation:

$y(xy + 2x^2y^2)dx + x(xy - x^2y^2)\,dy = 0$ 1

Solution: $M = xy^2 + 2x^2y^3,\ N = x^2y - x^3y^2$

$\frac{\partial M}{\partial y} = 2xy + 6x^2y^2$, $\frac{\partial N}{\partial x} = 2xy - 3x^2y^2$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As equation 1 is of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ IF $= \frac{1}{Mx-Ny} = \frac{1}{x^2y^2+2x^3y^3-x^2y^2+x^3y^3} = \frac{1}{3x^3y^3}$

∴ 1 may be rewritten after multiplying by IF as:

$\left(\frac{1}{x^2y} + \frac{2}{x}\right)dx + \left(\frac{1}{xy^2} - \frac{1}{y}\right)dy = 0$ 2

New $M = \frac{1}{x^2y} + \frac{2}{x}$, New $N = \frac{1}{xy^2} - \frac{1}{y}$

$\frac{\partial M}{\partial y} = \frac{-1}{x^2y^2}$, $\frac{\partial N}{\partial x} = \frac{-1}{x^2y^2}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ 2 is an exact differential equation.

Solution of 2 is given by:

$$\underset{y\ constant}{\int\left(\frac{1}{x^2y} + \frac{2}{x}\right) dx} + \int -\frac{1}{y} dy = C$$

$\Rightarrow \frac{-1}{xy} + 2\log x - \log y = \mathrm{C}$

$\Rightarrow \log\frac{x^2}{y} - \frac{1}{xy} = \mathrm{C}$

2.8.4 Integrating Factor (IF) of a Non-Exact Differential Equation

$Mdx + Ndx = 0$ in which $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are connected in a specific way as shown:

i. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = f(x)$, a function of x alone, then IF = $e^{\int f(x)dx}$

ii. If $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = g(y)$, a function of y alone, then IF = $e^{\int -g(y)dy}$

Example24 Solve the differential equation:

$(x^3 + y^2 + x)dx + xy\,dy = 0$ 1

Solution: $M = x^3 + y^2 + x,\ N = xy$

$\frac{\partial M}{\partial y} = 2y$, $\frac{\partial N}{\partial x} = y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As (1) is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{y}{xy} = \frac{1}{x} = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int \frac{1}{x}dx} = e^{\log x} = x$

∴ (1) may be rewritten after multiplying by IF as:

$(x^4 + xy^2 + x^2)dx + x^2y\, dy = 0$ ....(2)

New $M = x^4 + xy^2 + x^2$, New $N = x^2y$

$\frac{\partial M}{\partial y} = 2xy$, $\frac{\partial N}{\partial x} = 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ (2) is an exact differential equation.

Solution of (2) is given by:

$$\int_{y\ constant} (x^4 + xy^2 + x^2)\, dx + \int 0\, dy = C$$

$$\Rightarrow \frac{x^5}{5} + \frac{x^2y^2}{2} + \frac{x^3}{3} = C$$

Example25 Solve the differential equation:

$(y^4 + 2y)dx + (xy^3 + 2y^4 - 4x)\, dy = 0$ .....(1)

Solution: $M = y^4 + 2y$, $N = xy^3 + 2y^4 - 4x$

$\frac{\partial M}{\partial y} = 4y^3 + 2$, $\frac{\partial N}{\partial x} = y^3 - 4$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As (1) is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 3y^3 + 6$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{M} = \frac{3y^3+6}{y^4+2y} = \frac{3}{y} = g(y)$ say

∴ IF = $e^{\int -g(y)dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log y} = \frac{1}{y^3}$

∴ (1) may be rewritten after multiplying by IF as:

$\left(y + \frac{2}{y^2}\right)dx + \left(x + 2y - \frac{4x}{y^3}\right)dy = 0$ ......(2)

New $M = y + \frac{2}{y^2}$, New $N = x + 2y - \frac{4x}{y^3}$

$\frac{\partial M}{\partial y} = 1 - \frac{4}{y^3}$, $\frac{\partial N}{\partial x} = 1 - \frac{4}{y^3}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ (2) is an exact differential equation.

Solution of (2) is given by:

$$\underset{y\ constant}{\int \left(y + \frac{2}{y^2}\right) dx} + \int 2y\, dy = C$$

$$\Rightarrow \left(y + \frac{2}{y^2}\right)x + y^2 = \mathrm{C}$$

Example26 Solve the differential equation:

$(x^2 - y^2 + 2x)dx - 2y\, dy = 0$ 1

Solution: $M = x^2 - y^2 + 2x,\ N = -2y$

$\frac{\partial M}{\partial y} = -2y\,,\ \frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As 1 is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2y}{-2y} = 1 = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int 1 dx} = e^x$

∴ 1 may be rewritten after multiplying by IF as:

$e^x(x^2 - y^2 + 2x)dx - 2e^x y\, dy = 0$.......2

New $M = e^x(x^2 - y^2 + 2x)$, New $N = -2e^x y$

$\frac{\partial M}{\partial y} = -2e^x y\ ,\ \frac{\partial N}{\partial x} = -2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ 2 is an exact differential equation.

Solution of 2 is given by:

$$\int e^x(x^2 - y^2 + 2x)\, dx + \int 0\, dy = C$$

y constant

$\Rightarrow (x^2 - y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = \mathrm{C}$

$\Rightarrow (x^2 - y^2)e^x = \mathrm{C}$, C is an arbitrary constant

Example27 Solve the differential equation:

$2ydx + (2x\log x - xy)\, dy = 0$ 1

Solution: $M = 2y,\ N = 2x\log x - xy$

$\frac{\partial M}{\partial y} = 2\,,\ \frac{\partial N}{\partial x} = 2(1 + \log x) - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As 1 is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = -2\log x + y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{-2\log x + y}{x(2\log x - y)} = -\frac{1}{x} = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int \frac{-1}{x}dx} = e^{\log x^{-1}} = \frac{1}{x}$

∴ 1 may be rewritten after multiplying by IF as:

$\frac{2y}{x}dx + (2\log x - y)\,dy = 0$ …….(2)

New $M = \frac{2y}{x}$, New $N = 2\log x - y$

$\frac{\partial M}{\partial y} = \frac{2}{x} = \quad \frac{\partial N}{\partial x} = \frac{2}{x}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ (2) is an exact differential equation.

Solution of (2) is given by:

$$\int_{y\ constant} \left(\frac{2y}{x}\right) dx + \int -y\,dy = C \Rightarrow 2y\log x - \frac{y^2}{2} = \text{C}$$

2.8.5 Integrating Factor (IF) of a Non-Exact Differential Equation

$\boldsymbol{x^a y^b(m_1 ydx + n_1 xdy) + x^c y^d(m_2 ydx + n_2 xdy) = 0}$, where $a, b, c, d,$ m_1, n_1, m_2, n_2 are constants, is given by $x^{\alpha} y^{\beta}$, where α and β are connected by the relation $\frac{a+\alpha+1}{m_1} = \frac{b+\beta+1}{n_1}$ and $\frac{c+\alpha+1}{m_2} = \frac{d+\beta+1}{n_2}$

Example28 Solve the differential equation:

$(y^2 + 2x^2y)dx + (2x^3 - xy)dy = 0$.….(1)

Solution: $M = y^2 + 2x^2y,\ N = 2x^3 - xy$

$\frac{\partial M}{\partial y} = 2y + 2x^2$, $\frac{\partial N}{\partial x} = 6x^2 - y$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rewriting (1) as $x^2y^0(2ydx + 2xdy) + x^0y^1(ydx - xdy) = 0$ …….(2)

Comparing with standard form $a = 2,\ b = 0,\ c = 0,\ d = 1,$

$$m_1 = 2,\ n_1 = 2,\ m_2 = 1,\ n_2 = -1$$

∴ $\frac{2+\alpha+1}{2} = \frac{0+\beta+1}{2}$ and $\frac{0+\alpha+1}{1} = \frac{1+\beta+1}{-1}$

$\Rightarrow \alpha - \beta = -2$ and $\alpha + \beta = -3$

Solving we get $\alpha = \frac{-5}{2}$ and $\beta = \frac{-1}{2}$

∴ IF = $x^{\alpha}y^{\beta} = x^{\frac{-5}{2}}y^{\frac{-1}{2}}$

∴ (1) may be rewritten after multiplying by IF as:

$x^{\frac{-5}{2}}y^{\frac{-1}{2}}(y^2 + 2x^2y)dx + x^{\frac{-5}{2}}y^{\frac{-1}{2}}(2x^3 - xy)dy = 0$…….(2)

$\Rightarrow \left(x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}\right)dx + \left(2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}\right)dy = 0$

New $M = x^{\frac{-5}{2}}y^{\frac{3}{2}} + 2x^{\frac{-1}{2}}y^{\frac{1}{2}}$, New $N = 2x^{\frac{1}{2}}y^{\frac{-1}{2}} - x^{\frac{-3}{2}}y^{\frac{1}{2}}$

$\frac{\partial M}{\partial y} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$, $\frac{\partial N}{\partial x} = \frac{3}{2}x^{\frac{-5}{2}}y^{\frac{1}{2}} + x^{\frac{-1}{2}}y^{\frac{-1}{2}}$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ (2) is an exact differential equation.

Solution of ② is given by:

$$\int_{y\ constant} \left(x^{\frac{-5}{2}} y^{\frac{3}{2}} + 2x^{\frac{-1}{2}} y^{\frac{1}{2}}\right) dx + \int 0\, dy = 0$$

$$\Rightarrow\ 4(xy)^{\frac{1}{2}} - \frac{2}{3}\left(\frac{y}{x}\right)^{\frac{3}{2}} = C \text{ , C is an arbitrary constant}$$

2.9 Linear Differential Equations

A differential equation of the form $F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}\right) = 0$ in which the dependent variable y and its derivatives viz. $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$ etc occur in first degree and are not multiplied together is called a Linear Differential Equation.

2.9.1 First Order Linear Differential Equations (Leibnitz's Linear Equations)

A first order linear differential equation is of the form $\frac{dy}{dx} + Py = Q, \ldots\ldots.$ Ⓐ

where P and Q are functions of x alone or constants. To solve Ⓐ, multiplying throughout by $e^{\int P\,dx}$ (here $e^{\int P\,dx}$ is known as Integrating Factor (IF)), we get

$$\frac{dy}{dx} e^{\int P\,dx} + Py\, e^{\int P\,dx} = Q\, e^{\int P\,dx}$$

$$\Rightarrow d\left(y\, e^{\int P\,dx}\right) = Q\, e^{\int P\,dx}$$

$$\Rightarrow y\, e^{\int P\,dx} = \int Q\, e^{\int P\,dx}\, dx + C$$

Algorithm to solve a first order linear differential equation (Leibnitz's Equation)

1. Write the given equation in standard form i.e., $\frac{dy}{dx} + Py = Q$
2. Find the integrating factor (IF) = $e^{\int P\,dx}$
3. Solution is given by y. IF = $\int Q$ .IF dx + C, C is an arbitrary constant

Note: If the given equation is of the type $\frac{dx}{dy} + Px = Q,$

then IF = $e^{\int P\,dy}$and the solution is given by x. IF = $\int Q$ .IF $dy + C$

Example29 Solve the differential equation: $\frac{dy}{dx} = \frac{x + y\sin x}{1+cosx}$

Solution: The given equation may be written as:

$$\frac{dy}{dx} - \frac{\sin x}{1+cosx} y = \frac{x}{1+cosx} \ldots\ldots \text{①}$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{\sin x}{1+cosx}$ and $Q = \frac{x}{1+cosx}$

IF = $e^{\int P\,dx} = e^{\int -\frac{\sin x}{1+cosx} dx} = e^{log|1+\cos x|} = 1 + \cos x$

∴ Solution of ① is given by

$$y.\,(1 + \cos x) = \int \frac{x}{1+cosx}(1 + \cos x)\, dx + C$$

$$\Rightarrow y\,(1 + \cos x) = \frac{x^2}{2} + C$$

Example30 Solve the differential equation: $\frac{dy}{dx} = (1 + x) + (1 - y)$

Solution: The given equation may be written as:

$\frac{dy}{dx} + y = 2 + x$ ……. ①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = 1$ and $Q = 2 + x$

IF = $e^{\int P\,dx} = e^{\int dx} = e^x$

∴ Solution of ① is given by

$y.\, e^x = \int (2 + x)e^x\ dx + C$

$\Rightarrow y = 1 + x + Ce^{-x}$

Example31 Solve the differential equation: $(x + y + 1)\frac{dy}{dx} = 1$

Solution: The given equation may be written as:

$\frac{dx}{dy} = x + y + 1 \quad \Rightarrow \quad \frac{dx}{dy} - x = y + 1$ ……①

This is a linear differential equation of the form $\frac{dx}{dy} + Px = Q$

Where $P = -1$ and $Q = y + 1$

IF = $e^{\int P\,dy} = e^{\int -dy} = e^{-y}$

∴ Solution of ① is given by

$x.\, e^{-y} = \int (y + 1)e^{-y}\ dy + C$

$\Rightarrow x\, e^{-y} = -(y + 2)e^{-y} + C$

$\Rightarrow x = -(y + 2) + C\, e^{y}$

Example32 Solve the differential equation: $x\log x\ \frac{dy}{dx} + y = 2\log x$

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{1}{x\log x}y = \frac{2}{x}$ ……..①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = \frac{1}{x\log x}$ and $Q = \frac{2}{x}$

IF = $e^{\int P\,dx} = e^{\int \frac{1}{x\log x}dx} = e^{\log(\log x)} = \log x$

∴ Solution of ① is given by

$y.\log x = \int \frac{2}{x}\log x\ dx + C$

$\Rightarrow y\log x = (\log x)^2 + C$, C is an arbitrary constant

Example33 Solve the differential equation: $\frac{dy}{dx} = \frac{e^{2\sqrt{x}+y}}{\sqrt{x}}$

Solution: The given equation may be written as:

$\frac{dy}{dx} - \frac{1}{\sqrt{x}}\, y = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$ ……①

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

Where $P = -\frac{1}{\sqrt{x}}$ and $Q = \frac{e^{2\sqrt{x}}}{\sqrt{x}}$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{\sqrt{x}}dx} = e^{-2\sqrt{x}}$

$\therefore$ Solution of ① is given by

$$y.e^{-2\sqrt{x}} = \int \frac{e^{2\sqrt{x}}}{\sqrt{x}} e^{-2\sqrt{x}}\, dx + C$$

$$\Rightarrow y.e^{-2\sqrt{x}} = \int \frac{1}{\sqrt{x}} dx + C$$

$$\Rightarrow y.e^{-2\sqrt{x}} = 2\sqrt{x} + C$$

$$\Rightarrow y = 2\sqrt{x}\ e^{2\sqrt{x}} + Ce^{2\sqrt{x}}$$

2.8.2 Equations Reducible to Leibnitz's Equations (Bernoulli's Equations)

Differential equation of the form $\frac{dy}{dx} + Pf(y) = Qg(y)$, ……. Ⓑ

where P and Q are functions of x alone or constant, is called Bernoulli's equation. Dividing both sides of Ⓑ by $g(y)$, we get $\frac{1}{g(y)}\frac{dy}{dx} + P\frac{f(y)}{g(y)} = Q$. Now putting $\frac{f(y)}{g(y)} = t$, Ⓑ reduces to Leibnitz's equation.

Example34 Solve the differential equation: $\frac{dy}{dx} + \frac{1}{x} = \frac{e^y}{x^2}$ ……①

Solution: The given equation may be written as:

$$e^{-y}\frac{dy}{dx} + \frac{1}{x}e^{-y} = \frac{1}{x^2} \text{ ……②}$$

Putting $e^{-y} = t$, $-e^{-y}\frac{dy}{dx} = \frac{dt}{dx}$ ……③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x^2}$ ……..④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x^2}$

IF = $e^{\int P\,dx} = e^{\int -\frac{1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

$\therefore$ Solution of ④ is given by

$$t.\frac{1}{x} = \int -\frac{1}{x^2}.\frac{1}{x}\, dx + \mathrm{C}$$

$$\Rightarrow t.\frac{1}{x} = \frac{1}{2x^2} + \mathrm{C}$$

Substituting $t = e^{-y}$

$$\Rightarrow \frac{e^{-y}}{x} = \frac{1}{2x^2} + \mathrm{C}$$

$$\Rightarrow 2x = e^y(2cx^2 + 1)$$

Example35 Solve the differential equation:

$$\tan y\frac{dy}{dx} + \tan x = \cos y\, cos^3 x \text{ ……①}$$

Solution: The given equation may be written as:

$$\frac{\tan y}{\cos y}\frac{dy}{dx} + \frac{\tan x}{\cos y} = cos^3 x$$

$$\Rightarrow \sec y\ \tan y\frac{dy}{dx} + \sec y\ \tan x = cos^3 x \text{ ……②}$$

Putting $\sec y = t$, $\sec y\ \tan y\frac{dy}{dx} = \frac{dt}{dx}$ ……③

Using ③ in ②, we get $\frac{dt}{dx} + (\tan x)\, t = cos^3 x$ ……..④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = \tan x$ and $Q = cos^3 x$

IF = $e^{\int P\,dx} = e^{\int \tan x\,dx} = e^{\log|\sec x|} = \sec x$

∴ Solution of 4 is given by

$t.\sec x = \int cos^3 x.\sec x \; dx + C$

$\Rightarrow t.\sec x = \int cos^2 x \; dx + C$

$\Rightarrow t.\sec x = \int \frac{1+\cos 2x}{2} \; dx + C$

$\Rightarrow t.\sec x = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Substituting $t = \sec y$,

$\Rightarrow \sec x \sec y = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Example36 Solve the differential equation: $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$ ……1

Solution: The given equation may be written as:

$\frac{dx}{dy} = \frac{x+\sqrt{xy}}{y}$

$\Rightarrow \frac{dx}{dy} - \frac{1}{y}x = \sqrt{\frac{x}{y}}$

Dividing throughout by $\sqrt{x}$

$\Rightarrow \frac{1}{\sqrt{x}}\frac{dx}{dy} - \frac{1}{y}\sqrt{x} = \frac{1}{\sqrt{y}}$ ……2

Putting $\sqrt{x} = t$, $\frac{1}{2\sqrt{x}}\frac{dx}{dy} = \frac{dt}{dy}$ ……3

Using 3 in 2, we get $\frac{dt}{dy} - \frac{1}{2y}t = \frac{1}{2\sqrt{y}}$ ……. 4

4 is a linear differential equation of the form $\frac{dt}{dy} + Pt = Q$

Where $P = -\frac{1}{2y}$ and $Q = \frac{1}{2\sqrt{y}}$

IF = $e^{\int P\,dx} = e^{\int\frac{-1}{2y}dy} = e^{\frac{-1}{2}\log y} = e^{\log\frac{1}{\sqrt{y}}} = \frac{1}{\sqrt{y}}$

∴ Solution of 4 is given by

$t.\frac{1}{\sqrt{y}} = \int \frac{1}{2\sqrt{y}}.\frac{1}{\sqrt{y}} \; dx + C$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \int \frac{1}{2y} \; dx + C$

$\Rightarrow t.\frac{1}{\sqrt{y}} = \frac{1}{2} \; log \; y + C$

Substituting $t = \sqrt{x}$

$\sqrt{\frac{x}{y}} = log\sqrt{y} + C$

Example37 Solve the differential equation: $x\frac{dy}{dx} + y = y^2 \log x$ …..1

Solution: The given equation may be written as:

$\frac{dy}{dx} + \frac{y}{x} = \frac{y^2}{x}\log x$

Dividing throughout by y^2

$\frac{1}{y^2}\frac{dy}{dx} + \frac{1}{xy} = \frac{1}{x}\log x$......②

Putting $\frac{1}{y} = t, \ -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$ ......③

Using ③ in ②, we get $\frac{dt}{dx} - \frac{1}{x}t = -\frac{1}{x}\log x$ ....④

④ is a linear differential equation of the form $\frac{dt}{dx} + Pt = Q$

Where $P = -\frac{1}{x}$ and $Q = -\frac{1}{x}\log x$

IF = $e^{\int P\,dx} = e^{\int \frac{-1}{x}dx} = e^{-\log x} = e^{\log x^{-1}} = \frac{1}{x}$

∴ Solution of ④ is given by

$t.\frac{1}{x} = \int -\frac{1}{x}\log x . \frac{1}{x}\ dx + C$

$\Rightarrow t.\frac{1}{x} = \int -\frac{1}{x^2}\log x\, dx + C$

Putting $\log x = u, \frac{1}{x}dx = du$, also $x = e^u$

$\Rightarrow t.\frac{1}{x} = -\int ue^{-u}\ du + C$

$\Rightarrow t.\frac{1}{x} = -[u(-e^{-u}) - 1(e^{-u})] + C$

$\Rightarrow t.\frac{1}{x} = e^{-u}(u+1) + C$

$\Rightarrow t.\frac{1}{x} = \frac{1}{x}(log\, x + 1) + C$

Substituting $t = \frac{1}{y}$

$\Rightarrow \frac{1}{xy} = \frac{1}{x}(\log x + 1) + C$

$\Rightarrow \frac{1}{y} = (\log x + 1) + Cx$, C is an arbitrary constant

2.8.3 Homogeneous and Non- Homogeneous Linear Differential Equations with Constant Coefficients

A general linear differential equation of n^{th} order with constant coefficients is given by:

$k_0\frac{d^ny}{dx^n} + k_1\frac{d^{n-1}y}{dx^{n-1}} + \cdots + k_{n-1}\frac{dy}{dx} + k_n y = F(x)$ .... ①

where $k's$ are constant and $F(x)$ is a function of x alone or constant. $\Rightarrow (k_0 D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n)y = F(x)$

Or $f(D)y = F(x)$, where $D^n \equiv \frac{d^n}{dx^n}$, $D^{n-1} \equiv \frac{d^{n-1}}{dx^{n-1}}$, ..., $D \equiv \frac{d}{dx}$ are called differential operators. If $F(x) = 0$, ① is called a homogeneous linear differential equation with Constant Coefficients.

2.8.3.1 Solving Homogeneous and Non- Homogeneous Linear Differential Equations with Constant Coefficients

Complete solution of equation $f(D)y = F(x)$ is given by $y =$ C.F + P.I.

where C.F. denotes complimentary function and P.I. is the particular integral.

When $F(x) = 0$, it is a homogeneous linear differential equation with constant coefficients and the solution of equation $f(D)y = 0$ is given by $y =$ C.F

Rules for Finding Complimentary Function (C.F.)

Consider the equation $f(D)y = F(x)$

$\Rightarrow (k_0 D^n + k_1 D^{n-1} + \cdots + k_{n-1} D + k_n)y = F(x)$

Step 1: Put $D = m$, auxiliary equation (A.E) is given by $f(m) = 0$

$\Rightarrow k_0 m^n + k_1 m^{n-1} + \cdots + k_{n-1} m + k_n = 0$ ……③

Step 2: Solve the auxiliary equation given by ③

I. If the *n* roots of A.E. are real and distinct say $m_1, m_2, \ldots m_n$

C.F. = $c_1 e^{m_1 x} + c_2 e^{m_2 x} + \cdots + c_n e^{m_n x}$

II. If two or more roots are equal i.e., $m_1 = m_2 = \ldots = m_k$, $k \le n$

C.F. = $(c_1 + c_2 x + c_3 x^2 + \cdots + c_k x^{k-1}) e^{m_1 x} + \cdots + c_n e^{m_n x}$

III. If A.E. has a pair of imaginary roots i.e., $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$

C.F. = $e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) + c_3 e^{m_3 x} + \cdots + c_n e^{m_n x}$

IV. If 2 pairs of imaginary roots are equal i.e., $m_1 = m_2 = \alpha + i\beta$,

$m_3 = m_4 = \alpha - i\beta$

C.F. = $e^{\alpha x}[(c_1 + c_2 x)\cos \beta x + (c_3 + c_4 x)\sin \beta x] + \cdots + c_n e^{m_n x}$

Example38 Solve the differential equation: $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$

Solution: $\Rightarrow (D^2 - 8D + 15)y = 0$

Auxiliary equation is: $m^2 - 8m + 15 = 0$

$\Rightarrow (m-3)(m-5) = 0$

$\Rightarrow m = 3, 5$

C.F. = $c_1 e^{3x} + c_2 e^{5x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{3x} + c_2 e^{5x}$

Example39 Solve the differential equation: $\frac{d^3y}{dx^3} - 6\frac{d^2y}{dx^2} + 11\frac{dy}{dx} - 6y = 0$

Solution: $\Rightarrow (D^3 - 6D^2 + 11D - 6)y = 0$

Auxiliary equation is: $m^3 - 6m^2 + 11m - 6 = 0$ …. ①

By hit and trial $(m-2)$ is a factor of ①

∴① May be rewritten as

$m^3 - 2m^2 - 4m^2 + 8m + 3m - 6 = 0$

$\Rightarrow m^2(m-2) - 4m(m-2) + 3(m-2) = 0$

$\Rightarrow (m^2 - 4m + 3)(m-2) = 0$

$\Rightarrow (m-3)(m-1)(m-2) = 0$

$\Rightarrow m = 1,2,3$

C.F. = $c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^{x} + c_2 e^{2x} + c_3 e^{3x}$

Example40 Solve $(D^4 - 10D^3 + 35D^2 - 50D + 24)y = 0$

Solution: Auxiliary equation is:

$m^4 - 10m^3 + 35m^2 - 50m + 24 = 0$ ....①

By hit and trial $(m-1)$ is a factor of ①

∴① May be rewritten as

$m^4 - m^3 - 9m^3 + 9m^2 + 26m^2 - 26m - 24m + 24 = 0$

$\Rightarrow m^3(m-1) - 9m^2(m-1) + 26m(m-1) - 24(m-1) = 0$

$\Rightarrow (m-1)(m^3 - 9m^2 + 26m - 24) = 0$ ......②

By hit and trial $(m-2)$ is a factor of ②

∴② May be rewritten as

$(m-1)(m^3 - 2m^2 - 7m^2 + 14m + 12m - 24) = 0$

$\Rightarrow (m-1)[m^2(m-2) - 7m(m-2) + 12(m-2)] = 0$

$\Rightarrow (m-1)(m^2 - 7m + 12)(m-2) = 0$

$\Rightarrow (m-1)(m-3)(m-4)(m-2) = 0$

$\Rightarrow \ m = 1,2,3,4$

C.F. = $c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{4x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x} + c_4 e^{4x}$

Example41 Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = 0$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = 0$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)^2 = 0$

$\Rightarrow \ m = 0, -1, -1$

C.F. = $c_1 + (c_2 + c_3 x)e^{-x}$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = c_1 + (c_2 + c_3 x)e^{-x}$

Example42 Solve the differential equation: $\frac{d^4y}{dx^4} - 2\frac{d^2y}{dx^2} + y = 0$

Solution: $\Rightarrow (D^4 - 2D^2 + 1)y = 0$

Auxiliary equation is: $m^4 - 2m^2 + 1 = 0$

$\Rightarrow (m^2 - 1)^2 = 0$

$\Rightarrow (m+1)^2(m-1)^2 = 0$

$\Rightarrow \ m = -1, -1, 1, 1$

C.F. = $(c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^x$

Since $F(x) = 0$, solution is given by $y =$ C.F

$\Rightarrow y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^x$

Example43 Solve the differential equation: $\frac{d^3y}{dx^3} - 2\frac{dy}{dx} + 4y = 0$

Solution: $\Rightarrow (D^3 - 2D + 4)y = 0$

Auxiliary equation is: $m^3 - 2m + 4 = 0$ ....①

By hit and trial $(m+2)$ is a factor of ①

$\therefore$① May be rewritten as

$m^3+2m^2-2m^2-\ 4m+2m+\ 4=0$

$\Rightarrow m^2(m+2)-2m(m+2)+2(m+2)=0$

$\Rightarrow (m+2)(m^2-2m+2)=0$

$\Rightarrow\ m=-2, 1\pm i$

C.F. = $c_1e^{-2x}+\ e^x(c_2\cos x+c_3\sin x)$

Since $F(x)=0$, solution is given by $y=$ C.F

$\Rightarrow y\ =\ \ c_1e^{-2x}+\ e^x(c_2\cos x+c_3\sin x)$

Example44 Solve the differential equation: $(D^2-2D+5)^2y=0$

Solution: Auxiliary equation is: $(m^2-2m+5)^2$…….①

Solving ①, we get

$\Rightarrow\ m=1\pm 2i, 1\pm 2i$

C.F. = $e^x[(c_1+c_2\,x)\cos 2x+\ (c_3+c_4x)\sin 2x$

Since $F(x)=0$, solution is given by $y=$ C.F

$\Rightarrow y\ =\ \ e^x[(c_1+c_2\,x)\cos 2x+\ (c_3+c_4x)\sin 2x]$

Example45 Solve the differential equation: $(D^2+4)^3y=0$

Solution: Auxiliary equation is: $(m^2+4)^3$…….①

Solving ①, we get

$\Rightarrow\ m=\pm 2i, \pm 2i, \pm 2i$

C.F. = $(c_1+c_2x+c_3\,x^2)\cos 2x+\ (c_4+c_5x+c_6x^2)\sin 2x$

Since $F(x)=0$, solution is given by $y=$ C.F

$\Rightarrow y\ =\ \ (c_1+c_2x+c_3\,x^2)\cos 2x+\ (c_4+c_5x+c_6x^2)\sin 2x$

Shortcut Rules for Finding Particular Integral (P.I.)

Consider the non- homogeneous linear differential equation

$f(D)y=F(x), F(x)\ \neq 0$

$\Rightarrow (k_0\,D^n+\ k_1\,D^{n-1}+\cdots+\ k_{n-1}\,D+\ k_n\,)y=F(x)$

Then P.I = $\frac{1}{f(\,D)}\,F(x)$, Clearly P.I. = 0 if $F(x)=0$

Case I: When $\boldsymbol{F(x)=\ e^{ax}}$

Use the rule P.I $=\frac{1}{f(\,D)}e^{ax}=\frac{1}{f(\,a)}e^{ax}$, $f(\,a)\ \neq 0$

In case of failure i.e., if $\boldsymbol{f(\,a)=0}$

P.I = $x\frac{1}{f'(D)}e^{ax}=\ \ x\frac{1}{f'(a)}e^{ax}$, $f'(a)\neq 0$

If $f'(a)=0$, P.I. = $x^2\frac{1}{f''(a)}e^{ax}$, $f''(a)\neq 0$ and so on

Example46 Solve the differential equation: $\frac{d^2y}{dx^2}-\ 2\frac{dy}{dx}+10y=e^{2x}$

Solution: $\Rightarrow (D^2-\ 2D+10)y=e^{2x}$

Auxiliary equation is: $m^2-2m+10=0$

$\Rightarrow\ m = 1 \pm 3i$

C.F. $= \ e^{x}(c_1 \cos 3x + c_2 \sin 3x)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{f(D)} e^{2x} = \frac{1}{f(2)} e^{2x}$, by putting $D = 2$

$= \frac{1}{2^2 - 2(2) + 10} e^{2x}\ = \frac{1}{10} e^{2x}$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ e^{x}(c_1 \cos 3x + c_2 \sin 3x) + \frac{1}{10} e^{2x}$

Example47 Solve the differential equation: $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 2y = e^x$

Solution: $\Rightarrow (D^2 + D - 2)y = e^x$

Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow\ m = -2,1$

C.F. $=\ c_1 e^{-2x} + c_2 e^{x}$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{f(D)} e^{x}$, putting $D = 1, f(1) = 0$

$\therefore$P.I. $=\ x \frac{1}{f'(D)} e^x$ $\qquad\qquad \because$ P.I. $= x \frac{1}{f'(a)} e^{ax}$ if $f(a) = 0$

$\Rightarrow$ P.I. $= x \frac{1}{2D+1} e^x = \frac{1}{f'(1)} e^x$, $f'(1) \neq 0$

$\Rightarrow$ P.I. $= \frac{xe^x}{3}$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ c_1 e^{-2x} + c_2 e^{x} + \frac{xe^x}{3}$

Example48 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = \sinh(2x+1) + 4^x$

Solution: $\Rightarrow (D^2 - 4)y = \sinh(2x+1) + 4^x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow\ m = \pm 2$

C.F. $=\ c_1 e^{2x} +\ c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)}\ F(x)$

$= \frac{1}{f(D)} (\sinh(2x+1) + 4^x)$

$= \frac{1}{D^2 - 4}\left(\frac{e^{(2x+1)} - e^{-(2x+1)}}{2}\right) + \frac{1}{D^2 - 4}\left(e^{x \log 4}\right)$

$\because \sinh x = \frac{e^x - e^{-x}}{2}$ and $4^x = e^{x \log 4}$

$= \frac{e}{2}\frac{1}{D^2 - 4} e^{2x} - \frac{e^{-1}}{2}\frac{1}{D^2 - 4} e^{-2x} + \frac{1}{D^2 - 4} e^{x \log 4}$

Putting $D = 2, -2$ and $\log 4$ in the three terms respectively

$f(2) = 0$ and $f(-2) = 0$ for first two terms

$\therefore$ P.I. $= \frac{e}{2} x \frac{1}{2D} e^{2x} - \frac{e^{-1}}{2} x \frac{1}{2D} e^{-2x} + \frac{1}{(log 4)^2 - 4} e^{x \log 4}$

$$\because \text{P.I.} = x\frac{1}{f'(a)}e^{ax} \text{ if } f(a) = 0$$

Now putting $D = 2, -2$ in first two terms respectively

$\Rightarrow$P.I. $= \frac{ex}{8}e^{2x} + \frac{e^{-1}x}{8}e^{-2x} + \frac{4^x}{(log4)^2 - 4}$ $\quad \because e^{x\log 4} = 4^x$

$\Rightarrow$P.I. $= \frac{x}{4}\left(\frac{e^{(2x+1)} + e^{-(2x+1)}}{2}\right) + \frac{4^x}{(log4)^2 - 4}$

$\Rightarrow$P.I. $= \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2 - 4}$ $\quad \because \cosh x = \frac{e^x + e^{-x}}{2}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} + \frac{x}{4}\cosh(2x+1) + \frac{4^x}{(log4)^2 - 4}$

Case II: When $F(x) = Sin\ (ax+b) or\ Cos\ (ax+b)$

If $F(x) = Sin\ (ax+b)\ or\ Cos\ (ax+b)$, put $D^2 = -a^2$,

$D^3 = D^2D = -a^2D, D^4 = (D^2)^2 = a^4, \ldots\ldots$

This will form a linear expression in D in the denominator. Now rationalize the denominator to substitute $D^2 = -a^2$. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

In case of failure i.e., if $f(-a^2) = 0$

P.I. $= x\frac{1}{f'(-a^2)}\ Sin\ (ax+b)\ or\ Cos\ (ax+b), f'(-a^2) \neq 0$

If $f'(-a^2) = 0$, P.I.$= x^2\frac{1}{f''(-a^2)}\ Sin\ (ax+b)\ or\ Cos\ (ax+b), f''(-a^2) \neq 0$

Example49 Solve the differential equation: $(D^2 + D - 2)y = \sin x$

Solution: Auxiliary equation is: $m^2 + m - 2 = 0$

$\Rightarrow (m+2)(m-1) = 0$

$\Rightarrow m = -2,1$

C.F. $= c_1e^{-2x} + c_2e^x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{f(D)}\sin x = \frac{1}{D^2+D-2}\sin x$

putting $D^2 = -1^2 = -1$

P.I $= \frac{1}{D-3}\sin x = \frac{D+3}{D^2-9}\sin x$, Rationalizing the denominator

$= \frac{(D+3)\sin x}{-10}$, Putting $D^2 = -1$

$\therefore$ P.I. $= \frac{-1}{10}(D\ sin\ x + 3\ \sin x)$

$= \frac{-1}{10}(\cos x + 3\ \sin x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{-2x} + c_2e^x - \frac{1}{10}(\cos x + 3\ \sin x)$

Example50 Solve the differential equation: $(D^2 + 2D + 1)y = cos^2x$

Solution: Auxiliary equation is: $m^2 + 2m + 1 = 0$

$(m+1)^2 = 0 \Rightarrow m = -1, -1$

C.F. $= e^{-x}(c_1 + c_2x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} cos^2 x = \frac{1}{D^2+2D+1}\left(\frac{1+\cos 2x}{2}\right)$

$= \frac{1}{2}\frac{1}{D^2+2D+1} e^{0x} + \frac{1}{2}\frac{1}{D^2+2D+1}\cos 2x$

Putting $D = 0$ in the 1st term and $D^2 = -2^2 = -4$ in the 2nd term

P.I $= \frac{1}{2} + \frac{1}{2}\frac{1}{2D-3}\cos 2x$

$= \frac{1}{2} + \frac{1}{2}\frac{2D+3}{4D^2-3^2}\cos 2x$, Rationalizing the denominator

$= \frac{1}{2} + \frac{1}{2}\frac{(2D+3)\cos 2x}{-25}$, Putting $D^2 = -4$

$\therefore$ P. I. $= \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Now $y =$ C.F. + P.I

$\Rightarrow y = e^{-x}(c_1 + c_2 x) + \frac{1}{2} - \frac{1}{50}(-4\sin 2x + 3\cos 2x)$

Example51 Solve the differential equation: $(D^2+9)y = \sin 2x \cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m = \pm 3i$

C.F. = $c_1 \cos 3x + c_2 \sin 3x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)}\sin 2x \cos x = \frac{1}{2}\frac{1}{D^2+9}(\sin 3x + \sin x)$

$= \frac{1}{2}\frac{1}{D^2+9}\sin 3x + \frac{1}{2}\frac{1}{D^2+9}\sin x$

Putting $D^2 = -9$ in the 1st term and $D^2 = -1$ in the 2nd term

We see that $f(D^2 = -9) = 0$ for the 1st term

$\therefore$ P.I. $= \frac{1}{2} x \frac{1}{2D}\sin 3x + \frac{1}{2}\frac{1}{8}\sin x$

$\because$P.I. $= x\frac{1}{f'(-a^2)} Sin\,(ax+b)$, $f'(-a^2) \neq 0$

$\Rightarrow$P.I. $= -\frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x - \frac{x}{12}\cos 3x + \frac{1}{16}\sin x$

Case III: When $F(x) = x^n$, n is a positive integer

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{f(D)} x^n$

1. Take the lowest degree term common from $f(D)$ to get an expression of the form $[1 \pm \phi(D)]$ in the denominator and take it to numerator to become $[1 \pm \phi(D)]^{-1}$
2. Expand $[1 \pm \phi(D)]^{-1}$ using binomial theorem up to nth degree as $(n+1)$th derivative of x^n is zero
3. Operate on the numerator term by term by taking $D \equiv \frac{d}{dx}$

Following expansions will be useful to expand $[1 \pm \phi(D)]^{-1}$

- $(1+x)^{-1} = 1 - x + x^2 - x^3 + \cdots$
- $(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots$

Example52 Solve the differential equation: $\frac{d^2y}{dx^2} - y = 5x - 2$

Solution: $\Rightarrow (D^2 - 1)y = 5x - 2$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow \ m = \pm 1$

C.F. = $c_1 e^{x} + c_2 e^{-x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2-1}(5x - 2)$

$= \frac{1}{-(1-D^2)}(5x - 2)$

$= -(1 - D^2)^{-1}(5x - 2)$

$= -[1 + D^2 + \cdots](5x - 2)$

$= -(5x - 2)$

$\therefore$P.I. $= -5x + 2$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{x} + c_2 e^{-x} - 5x + 2$

Example53 Solve the differential equation: $(D^4 + 4D^2)y = x^2 + 1$

Solution: Auxiliary equation is: $m^4 + 4m^2 = 0$

$\Rightarrow \ m^2(m^2 + 4) = 0$

$\Rightarrow m = 0, 0, \pm 2i$

C.F. = $(c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^4+4D^2}(x^2 + 1)$

$= \frac{1}{D^4+4D^2}(x^2 + 1)$

$= \frac{1}{4D^2\left(1+\frac{D^2}{4}\right)}(x^2 + 1)$

$= \frac{1}{4D^2}\left(1 + \frac{D^2}{4}\right)^{-1}(x^2 + 1)$

$= \frac{1}{4D^2}\left[1 - \frac{D^2}{4} + \cdots\right](x^2 + 1)$

$= \frac{1}{4D^2}\left(x^2 + 1 - \frac{1}{2}\right)$

$= \frac{1}{4D^2}\left(x^2 + \frac{1}{2}\right)$

$= \frac{1}{4D}\int\left(x^2 + \frac{1}{2}\right)dx$

$= \frac{1}{4D}\left(\frac{x^3}{3} + \frac{x}{2}\right)$

$= \frac{1}{4}\int\left(\frac{x^3}{3} + \frac{x}{2}\right)dx$

$\therefore$ P.I $= \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x) + (c_3 \cos 2x + c_4 \sin 2x) + \frac{1}{4}\left(\frac{x^4}{12} + \frac{x^2}{4}\right)$

Example54 Solve the differential equation: $(D^2 - 6D + 9)y = 1 + x + x^2$

Solution: Auxiliary equation is: $m^2 - 6m + 9 = 0$

$\Rightarrow\ (m-3)^2 = 0 \Rightarrow m = 3, 3$

C.F. $= e^{3x}(c_1 + c_2 x)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2-6D+9}(1 + x + x^2)$

$$= \frac{1}{9\left(1-\frac{2D}{3}+\frac{D^2}{9}\right)}(1 + x + x^2)$$

$$= \frac{1}{9}\left(1 - \left(\frac{2D}{3} - \frac{D^2}{9}\right)\right)^{-1}(1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \left(\frac{2D}{3} - \frac{D^2}{9}\right) + \left(\frac{2D}{3} - \frac{D^2}{9}\right)^2 + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \frac{2D}{3} - \frac{D^2}{9} + \frac{4D^2}{9} + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left[1 + \frac{2D}{3} + \frac{D^2}{3} + \cdots\right](1 + x + x^2)$$

$$= \frac{1}{9}\left(1 + x + x^2 + 0 + \frac{2}{3} + \frac{4x}{3} + 0 + 0 + \frac{2}{3}\right)$$

$\therefore$ P.I $= \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ e^{3x}(c_1 + c_2 x) + \frac{1}{9}\left(\frac{7}{3} + \frac{7x}{3} + x^2\right)$

Case IV: When $F(x) = e^{ax}g(x)$, where $g(x)$ is any function of x

Use the rule: $\frac{1}{f(D)}\ e^{ax}g(x) = e^{ax}\left(\frac{1}{f(D+a)}\ g(x)\right)$

Example55 Solve the differential equation: $(D^2 + 2)y = x^2 e^{3x}$

Solution: Auxiliary equation is: $m^2 + 2 = 0$

$\Rightarrow m^2 = -2 \Rightarrow m = \pm\sqrt{2}i$

C.F. $= (c_1 \cos(\sqrt{2}\,x) + c_2 \sin(\sqrt{2}\,x))$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^2+2}x^2 e^{3x}$

$$= e^{3x}\frac{1}{(D+3)^2+2}x^2$$

$$= e^{3x}\frac{1}{D^2+6D+11}x^2$$

$$= \frac{e^{3x}}{11}\frac{1}{\left(1+\frac{6D}{11}+\frac{D^2}{11}\right)}x^2$$

$$= \frac{e^{3x}}{11}\left(1 + \left(\frac{6D}{11} + \frac{D^2}{11}\right)\right)^{-1} x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \left(\frac{6D}{11} + \frac{D^2}{11}\right) + \left(\frac{6D}{11} + \frac{D^2}{11}\right)^2 + \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} - \frac{D^2}{11} + \frac{36D^2}{121} + \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left[1 - \frac{6D}{11} + \frac{25D^2}{121} + \cdots\right]x^2$$

$$= \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$$

$\therefore\ P.I\ = \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Complete solution is: $y\ =\$ C.F. + P.I

$\Rightarrow y\ =\ (c_1\cos(\sqrt{2}\,x) + c_2\sin(\sqrt{2}\,x)) + \frac{e^{3x}}{11}\left(x^2 - \frac{12x}{11} + \frac{50}{121}\right)$

Example56 Solve the differential equation: $(D^3+1)y = e^{2x}\sin x$

Solution: Auxiliary equation is: $m^3 + 1 = 0$

$\Rightarrow m^3 = -1$

$\Rightarrow m = -1, \frac{1\pm\sqrt{3}i}{2}$

C.F. $= c_1e^{-x} +\ e^{\frac{x}{2}}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

P.I. $= \frac{1}{f(D)}\ F(x)\ = \frac{1}{D^3+1}e^{2x}\sin x$

$= e^{2x}\frac{1}{(D+2)^3+1}\sin x$

$= e^{2x}\frac{1}{D^3+6D^2+12D+9}\sin x$

$= e^{2x}\frac{1}{-D-6+12D+9}\sin x$, Putting $D^2 = -1$

$= e^{2x}\frac{1}{11D+3}\sin x$

$= e^{2x}\frac{11D-3}{121D^2-9}\sin x$, Rationalizing the denominator

$= -\frac{e^{2x}}{130}(11D-3)\sin x$, Putting $D^2 = -1$

$\therefore$ P.I $= -\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Complete solution is: $y\ =\$ C.F. + P.I

$\Rightarrow y\ = c_1e^{-x} +\ e^{\frac{x}{2}}\left(c_2\cos\left(\frac{\sqrt{3}}{2}x\right) + c_3\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

$-\frac{e^{2x}}{130}(11\cos x - 3\sin x)$

Example57 Solve the differential equation: $\frac{d^2y}{dx^2} - 4y = x\ \sinh x$

Solution: $\Rightarrow (D^2 - 4)y = x\ \sinh x$

Auxiliary equation is: $m^2 - 4 = 0$

$\Rightarrow\ m = \pm 2$

C.F. $=\ c_1e^{2x} +\ c_2e^{-2x}$

P.I. $= \frac{1}{f(D)}\ F(x)$

$= \frac{1}{f(D)}(x\ \sinh x)$

$= \frac{1}{D^2-4}\left(x\frac{e^x - e^{-x}}{2}\right)$ $\qquad \because \sinh x = \frac{e^x - e^{-x}}{2}$

$= \frac{1}{D^2-4}\left(x\frac{e^x}{2} - x\frac{e^{-x}}{2}\right)$

$= \frac{e^x}{2}\frac{1}{(D+1)^2-4}x - \frac{e^{-x}}{2}\frac{1}{(D-1)^2-4}x$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D-3)}x - \frac{e^{-x}}{2}\frac{1}{D^2-2D-3}x$$

$$= \frac{e^x}{2}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{2D}{3}\right)}x - \frac{e^{-x}}{2}\frac{1}{-3\left(1-\frac{D^2}{3}+\frac{2D}{3}\right)}x$$

$$= -\frac{e^x}{6}\left[1-\left(\frac{D^2}{3}+\frac{2D}{3}\right)\right]^{-1}x + \frac{e^{-x}}{6}\left[1-\left(\frac{D^2}{3}-\frac{2D}{3}\right)\right]^{-1}x$$

$$= -\frac{e^x}{6}\left(1+\frac{2D}{3}\right)x + \frac{e^{-x}}{6}\left(1-\frac{2D}{3}\right)x$$

$$= -\frac{e^x}{6}\left(x+\frac{2}{3}\right) + \frac{e^{-x}}{6}\left(x-\frac{2}{3}\right)$$

$$= -\frac{x}{3}\left(\frac{e^x-e^{-x}}{2}\right) - \frac{2}{9}\left(\frac{e^x+e^{-x}}{2}\right)$$

$\therefore$ P.I. $= -\frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1e^{2x} + c_2e^{-2x} - \frac{x}{3}\sinh x - \frac{2}{9}\cosh x$

Example58 Solve the differential equation: $(D^2+1)y = x^2\sin 2x$

Solution: Auxiliary equation is: $m^2+1=0$

$\Rightarrow m^2 = -1$

$\Rightarrow m = \pm i$

C.F. $= c_1\cos x + c_2\sin x$

P.I. $= \frac{1}{f(D)}F(x) = \frac{1}{D^2+1}x^2\sin 2x$

$=$ Imaginary part of $\frac{1}{D^2+1}x^2e^{i2x}$

Now $\frac{1}{D^2+1}x^2e^{i2x} = e^{i2x}\frac{1}{(D+2i)^2+1}x^2$

$$= e^{i2x}\frac{1}{D^2+4i^2+4iD+1}x^2$$

$$= e^{i2x}\frac{1}{D^2+4iD-3}x^2$$

$$= e^{i2x}\frac{1}{-3\left(1-\frac{D^2}{3}-\frac{4iD}{3}\right)}x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\left(\frac{D^2}{3}+\frac{4iD}{3}\right)\right]^{-1}x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)+\left(\frac{D^2}{3}+\frac{4iD}{3}\right)^2\right]x^2$$

$$= \frac{-e^{i2x}}{3}\left[1+\frac{D^2}{3}+\frac{4iD}{3}+\frac{16i^2D^2}{9}\right]x^2$$

$$= \frac{-e^{i2x}}{3}\left[1-\frac{13D^2}{9}+\frac{4iD}{3}\right]x^2$$

$$= \frac{-e^{i2x}}{3}\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$$= -\frac{1}{3}(\cos 2x + i\sin 2x)\left[x^2-\frac{26}{9}+i\frac{8x}{3}\right]$$

$\therefore$ P.I. $=$ Imaginary part of $\frac{1}{D^2+1}x^2e^{i2x} = -\frac{1}{3}\left(\frac{8x}{3}\cos 2x + \left(x^2-\frac{26}{9}\right)\sin 2x\right)$

$$= -\frac{8x}{9}\cos 2x + \frac{1}{27}(26-9x^2)\sin 2x$$

Complete solution is: y = C.F. + P.I

$$\Rightarrow y = c_1 \cos x + c_2 \sin x - \frac{8x}{9}\cos 2x + \frac{1}{27}(26 - 9x^2)\sin 2x$$

Example59 Solve the differential equation: $(D^2 - 4D + 4)y = x^2 e^{2x} \sin 2x$

Solution: Auxiliary equation is: $m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2$

$\Rightarrow m = 2, 2$

C.F. $= (c_1 + c_2 x)e^{2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 4D + 4} x^2 e^{2x} \sin 2x$

$$= e^{2x} \frac{1}{(D+2)^2 - 4(D+2) + 4} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D^2} x^2 \sin 2x$$

$$= e^{2x} \frac{1}{D} \int x^2 \sin 2x \, dx$$

$$= e^{2x} \frac{1}{D}\left[(x^2)\left(\frac{-\cos 2x}{2}\right) - (2x)\left(\frac{-\sin 2x}{4}\right) + (2)\left(\frac{\cos 2x}{8}\right)\right]$$

$$= e^{2x} \frac{1}{D}\left[-\frac{1}{2}x^2 \cos 2x + \frac{1}{2}x \sin 2x + \frac{1}{4}\cos 2x\right]$$

$$= e^{2x}\left[-\frac{1}{2}\int x^2 \cos 2x + \frac{1}{2}\int x \sin 2x \, dx + \frac{1}{4}\int \cos 2x \, dx\right]$$

$$= e^{2x}\left[-\frac{1}{2}\left[(x^2)\left(\frac{\sin 2x}{2}\right) - (2x)\left(\frac{-\cos 2x}{4}\right) + (2)\left(\frac{-\sin 2x}{8}\right)\right] + \right.$$
$$\left.\frac{1}{2}\left[(x)\left(\frac{-\cos 2x}{2}\right) - (1)\left(\frac{-\sin 2x}{4}\right)\right] + \frac{1}{4}\left(\frac{\sin 2x}{2}\right)\right]$$

$$\therefore \text{P.I.} = e^{2x}\left[\frac{-x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$$

Complete solution is: y = C.F. + P.I

$$\Rightarrow y = (c_1 + c_2 x)e^{2x} + e^{2x}\left[\frac{-x^2}{4}\sin 2x - \frac{x}{2}\cos 2x + \frac{3}{8}\sin 2x\right]$$

Case V: When $F(x) = x\, g(x)$, where g(x) is any function of x

Use the rule: $\frac{1}{f(D)}\left(x\, g(x)\right) = x \frac{1}{f(D)} g(x) + \left(\frac{d}{dD}\frac{1}{f(D)}\right) g(x)$

Example60 Solve the differential equation: $(D^2 + 9)y = x \cos x$

Solution: Auxiliary equation is: $m^2 + 9 = 0$

$\Rightarrow m^2 = -9$

$\Rightarrow m = \pm 3i$

C.F. $= (c_1 \cos 3x + c_2 \sin 3x)$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2 + 9} x \cos x$

$$= x \frac{1}{D^2 + 9}\cos x + \frac{-2D}{(D^2+9)^2}\cos x$$

$$= x \frac{1}{-1+9}\cos x + \frac{-2D}{(-1+9)^2}\cos x, \qquad \text{Putting } D^2 = -1$$

$$= \frac{x \cos x}{8} - \frac{2D \cos x}{64}$$

$$= \frac{x \cos x}{8} - \frac{2D \cos x}{64}$$

$\therefore$ P.I. $= \frac{x\cos x}{8} + \frac{\sin x}{32}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 \cos 3x + c_2 \sin 3x + \frac{x\cos x}{8} + \frac{\sin x}{32}$

Example61 Solve the differential equation: $(D^2-1)y = x\sin x + (1+x^2)e^x$

Solution: Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. $= c_1 e^x + c_2 e^x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2-1}\left(x\sin x + (1+x^2)e^x\right)$

$= \frac{1}{D^2-1} x\sin x + \frac{1}{D^2-1}(1+x^2)e^x$

Now $\frac{1}{D^2-1} x\sin x = x\frac{1}{D^2-1}\sin x + \frac{-2D}{(D^2-1)^2}\sin x$

$= x\frac{1}{-1-1}\sin x + \frac{-2D}{(-1-1)^2}\sin x,$ Putting $D^2 = -1$

$= -\frac{1}{2}(x\sin x + \cos x)$

Also $\frac{1}{D^2-1}(1+x^2)e^x = e^x \frac{1}{(D+1)^2-1}(1+x^2)$

$= e^x \frac{1}{D^2+2D}(1+x^2)$

$= e^x \frac{1}{2D\left(1+\frac{D}{2}\right)}(1+x^2)$

$= e^x \frac{1}{2D}\left(1+\frac{D}{2}\right)^{-1}(1+x^2)$

$= e^x \frac{1}{2D}\left[1 - \frac{D}{2} + \frac{D^2}{4}\right](1+x^2)$

$= e^x \frac{1}{2D}\left[1 + x^2 - x + \frac{1}{2}\right]$

$= e^x \frac{1}{2D}\left[x^2 - x + \frac{3}{2}\right]$

$= \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$

$\therefore$ P.I. $= -\frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^x + c_2 e^x - \frac{1}{2}(x\sin x + \cos x) + \frac{e^x}{2}\left[\frac{x^3}{3} - \frac{x^2}{2} + \frac{3x}{2}\right]$

Case VI: When $F(x)$ is any general function of x not covered in shortcut methods I to V above

Resolve $f(D)$ into partial fractions and use the rule:

$\frac{1}{D+a} F(x) = e^{-ax}\int e^{ax} F(x)\,dx$

Example62 Solve the differential equation: $(D^2 + 3D + 2)y = e^{e^x}$

Solution: Auxiliary equation is: $m^2 + 3m + 2 = 0$

$\Rightarrow (m+1)(m+2) = 0$

$\Rightarrow m = -1, -2$

C.F. $= c_1 e^{-x} + c_2 e^{-2x}$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+3D+2} e^{e^x}$

$= \frac{1}{(D+1)(D+2)} e^{e^x}$

$= \left(\frac{1}{(D+1)} - \frac{1}{(D+2)}\right) e^{e^x}$

$= e^{-x} \int e^x e^{e^x} dx - e^{-2x} \int e^{2x} e^{e^x} dx$

$= e^{-x} \int D e^{e^x} dx - e^{-2x} \int e^x D e^{e^x} dx$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int e^x e^{e^x} dx\right]$, Integrating 2nd term by parts

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - \int D e^{e^x} dx\right]$

$= e^{-x} e^{e^x} - e^{-2x}\left[e^x e^{e^x} - e^{e^x}\right]$

$\therefore$ P.I. $= e^{-2x} e^{e^x}$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2 e^{-2x} + e^{-2x} e^{e^x}$

Example63 Solve the differential equation: $(D^2 + 4)y = \tan 2x$

Solution: Auxiliary equation is: $m^2 + 4 = 0$

$\Rightarrow m = \pm 2i$

C.F. $= c_1 \cos 2x + c_2 \sin 2x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2+4} \tan 2x$

$= \frac{1}{(D-2i)(D+2i)} \tan 2x$

$= \frac{1}{4i}\left(\frac{1}{(D-2i)} - \frac{1}{(D+2i)}\right) \tan 2x$

P.I. $= \frac{1}{4i}\left(\frac{1}{D-2i} \tan 2x\right) - \frac{1}{4i}\left(\frac{1}{D+2i} \tan 2x\right)$ …….1

Now $\frac{1}{D-2i} \tan 2x = e^{2ix} \int e^{-2ix} \tan 2x \, dx$

$= e^{2ix} \int (\cos 2x - i \sin 2x) \tan 2x \, dx$

$= e^{2ix} \int (\sin 2x - i \frac{sin^2 2x}{cos 2x}) \, dx$

$= e^{2ix} \int \left(\sin 2x - i \frac{1-cos^2 2x}{cos 2x}\right) dx$

$= e^{2ix} \int (\sin 2x - i \sec 2x + i \cos 2x) \, dx$

$= e^{2ix} \left(-\frac{1}{2} \cos 2x - \frac{i}{2} log|\sec 2x + \tan 2x| + \frac{i}{2} \sin 2x\right)$

$\therefore \frac{1}{D-2i} \tan 2x = e^{2ix}\left(-\frac{1}{2} e^{-2ix} - \frac{i}{2} log|\sec 2x + \tan 2x|\right)$ …2

Replacing i by $-i$

$\frac{1}{D+2i} \tan 2x = e^{-2ix}\left(-\frac{1}{2} e^{2ix} + \frac{i}{2} log|\sec 2x + \tan 2x|\right)$ …3

Using 2 and 3 in 1

P.I. $= \frac{1}{4i}\left[e^{2ix}\left(-\frac{1}{2} e^{-2ix} - \frac{i}{2} log|\sec 2x + \tan 2x|\right)\right]$

$- \frac{1}{4i}\left[e^{-2ix}\left(-\frac{1}{2} e^{2ix} + \frac{i}{2} log|\sec 2x + \tan 2x|\right)\right]$

$$= \frac{1}{4i}\left[-\frac{1}{2} - \frac{i}{2}e^{2ix} log|\sec 2x + \tan 2x| + \frac{1}{2} - \frac{i}{2}e^{-2ix} log|\sec 2x + \tan 2x|\right]$$

$$= \frac{1}{4i}\left[-i\,\frac{e^{2ix} + e^{-2ix}}{2}\, log|\sec 2x + \tan 2x|\right]$$

$\therefore$P.I. $= -\frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$

Complete solution is: $y =$ C.F. + P.I.

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \frac{1}{4}[\cos 2x\, log|\sec 2x + \tan 2x|]$

2.8.4 Differential Equations Reducible to Linear Form with Constant Coefficients

Some special type of homogenous and non-homogeneous linear differential equations with variable coefficients after suitable substitutions can be reduced to linear differential equations with constant coefficients.

2.8.4.1 Euler–Cauchy Differential Equation

The differential equation of the form:

$$k_0\, x^n \frac{d^n y}{dx^n} + k_1\, x^{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + k_{n-1}\, x \frac{dy}{dx} + k_n\, y = F(x)$$

is called Euler–Cauchy Equation and it can be reduced to linear differential equations with constant coefficients by following substitutions:

$x = e^t \quad \Rightarrow \log x = t$

$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\frac{1}{x}$

$\Rightarrow x\frac{dy}{dx} = \frac{dy}{dt} = Dy$, where $D \equiv \frac{d}{dt}$

Similarly, $x^2 \frac{d^2y}{dx^2} = D(D-1)y$, $x^3 \frac{d^3y}{dx^3} = D(D-1)(D-2)y$ and so on.

Example64 Solve the differential equation:

$$x^3 \frac{d^3y}{dx^3} + 3x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + 8y = 13\, cos(log\, x),\ x > 0 \qquad \ldots\ldots 1$$

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2 \frac{\mathrm{d^2y}}{\mathrm{dx^2}} = \mathrm{D(D-1)y}$ and $x^3 \frac{\mathrm{d^3y}}{\mathrm{dx^3}} = \mathrm{D(D-1)(D-2)y}$

$\therefore$1 May be rewritten as

$(\mathrm{D(D-1)(D-2) + 3D(D-1) + D + 8})y = 13\cos t$

$\Rightarrow (D^3 + 8)y = 13\cos t$, $D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^3 + 8 = 0$

$\Rightarrow (m+2)(m^2 - 2m + 2) = 0$

$\Rightarrow m = -2, 1 \pm \sqrt{3}i$

C.F. $= c_1 e^{-2t} + e^t\left(c_2 \cos\sqrt{3}t + c_3 \sin\sqrt{3}t\right)$

$= \frac{c_1}{x^2} + x\left(c_2 \cos(\sqrt{3}\log x) + c_3 \sin(\sqrt{3}\log x\right)$

P.I. $= \frac{1}{f(D)}\, F(x) = 13\frac{1}{D^3+8}\cos t$

$= 13\frac{1}{-D+8}\cos t$, Putting $D^2 = -1$

$$= 13\frac{(8+D)}{64-D^2}\cos t = 13\frac{(8+D)}{65}\cos t \qquad \text{Putting } D^2 = -1$$

$$\therefore \text{P.I.} = \frac{1}{5}(8\cos t + D\cos t)$$

$$= \frac{1}{5}(8\cos t - \sin t)$$

$$= \frac{1}{5}(8\cos(\log x) - \sin(\log x))$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{x^2} + x\left(c_2\cos(\sqrt{3}\log x) + c_3\sin(\sqrt{3}\log x\right) + \frac{1}{5}(8\cos(\log x) - \sin(\log x))$$

Example65 Solve the differential equation: $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \frac{x^3}{1+x^2}$ …….1

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{dx}^2} = \mathrm{D(D-1)y}$$

$\therefore$ 1 May be rewritten as

$$(\mathrm{D(D-1)+D-1})y = \frac{e^{3t}}{1+e^{2t}}$$

$$\Rightarrow (D^2-1)y = \frac{e^{3t}}{1+e^{2t}},\ D \equiv \frac{d}{dt}$$

Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow\ m = \pm 1$

C.F. $= c_1e^{-t} + c_2e^t$

$$= \frac{c_1}{x} + c_2x$$

$$\text{P.I.} = \frac{1}{f(D)}F(x) = \frac{1}{D^2-1}\frac{e^{3t}}{1+e^{2t}}$$

$$= \frac{1}{(D-1)(D+1)}\frac{e^{3t}}{1+e^{2t}} = \frac{1}{2}\left(\frac{1}{(D-1)} - \frac{1}{(D+1)}\right)\frac{e^{3t}}{1+e^{2t}}$$

$$= \frac{1}{2}\left(\frac{1}{(D-1)}\frac{e^{3t}}{1+e^{2t}} - \frac{1}{(D+1)}\frac{e^{3t}}{1+e^{2t}}\right)$$

$$= \frac{1}{2}\left(e^t\int e^{-t}\frac{e^{3t}}{1+e^{2t}}dt - e^{-t}\int e^t\frac{e^{3t}}{1+e^{2t}}dt\right) \quad \because \frac{1}{D+a}F(x) = e^{-ax}\int e^{ax}F(x)\,dx$$

$$= \frac{1}{2}\left(e^t\int\frac{e^{2t}}{1+e^{2t}}dt - e^{-t}\int\frac{e^{4t}}{1+e^{2t}}dt\right)$$

Put $e^{2t} = u \Rightarrow 2e^{2t}dt = du$

$$\therefore \text{P.I} = \frac{1}{4}\left(e^t\int\frac{1}{1+u}du - e^{-t}\int\frac{u}{1+u}du\right)$$

$$= \frac{1}{4}\left(e^t\log(1+u) - e^{-t}\int\frac{1+u-1}{1+u}du\right)$$

$$= \frac{1}{4}\left(e^t\log(1+u) - e^{-t}\int\left(1-\frac{1}{1+u}\right)du\right)$$

$$= \frac{1}{4}(e^t\log(1+u) - e^{-t}(u - \log(1+u)))$$

$$= \frac{1}{4}(e^t\log(1+e^{2t}) - e^{-t}(e^{2t} - \log(1+e^{2t})))$$

$$= \frac{1}{4}\left(x\log(1+x^2) - \frac{1}{x}(x^2 - \log(1+x^2)\right)$$

$$= \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1 + x^2) - \frac{x}{4}$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{x} + c_2 x + \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1 + x^2) - \frac{x}{4}$$

$$\Rightarrow y = \frac{c_1}{x} + c_3 x + \frac{1}{4}\left(x + \frac{1}{x}\right)\log(1 + x^2) \ , c_3 = c_2 - \frac{1}{4}$$

Example66 Solve the differential equation:

$$x^2 D^2 - 2x\mathrm{D} - 4y = x^2 + 2\log x \ , \quad x > 0 \qquad \ldots\ldots (1)$$

Solution: This is a Euler–Cauchy Equation with variable coefficients.

Putting $x = e^t$ $\therefore \log x = t$

$$\Rightarrow xD = \delta y, x^2 D^2 = \delta(\delta - 1)\mathrm{y} \ , \delta \equiv \frac{d}{dt}$$

$\therefore$ (1) May be rewritten as

$$(\delta(\delta - 1) - 2\delta - 4)y = e^{2t} + 2t$$

$$\Rightarrow (\delta^2 - 3\delta - 4)y = e^{2t} + 2t$$

Auxiliary equation is: $m^2 - 3m - 4 = 0$

$$\Rightarrow (m + 1)(m - 4) = 0$$

$$\Rightarrow m = -1, 4$$

$$\text{C.F.} = c_1 e^{-t} + c_2 e^{4t}$$

$$= \frac{c_1}{x} + \frac{c_2}{x^4}$$

$$\text{P.I.} = \frac{1}{f(\delta)} F(x) = \frac{1}{\delta^2 - 3\delta - 4}(e^{2t} + 2t)$$

$$= \frac{1}{\delta^2 - 3\delta - 4} e^{2t} + \frac{1}{\delta^2 - 3\delta - 4} 2t$$

$$= \frac{1}{-6} e^{2t} + 2\frac{1}{-4\left(1 - \frac{\delta^2}{4} + \frac{3\delta}{4}\right)} t \quad \text{Putting } \delta = 2 \text{ in the 1st term}$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2}\left(1 - \left(\frac{\delta^2}{4} - \frac{3\delta}{4}\right)\right)^{-1} t$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2}\left[1 + \frac{\delta^2}{4} - \frac{3\delta}{4} + \cdots\right] t$$

$$= \frac{-e^{2t}}{6} - \frac{1}{2}\left[t - \frac{3}{4}\right]$$

$$\therefore \text{P.I.} = \frac{-x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

Complete solution is: $y =$ C.F. + P.I

$$\Rightarrow y = \frac{c_1}{x} + \frac{c_2}{x^4} \frac{-x^2}{6} - \frac{1}{2}\left[\log x - \frac{3}{4}\right]$$

2.9 Method of Variation of Parameters for Finding Particular Integral

Method of Variation of Parameters enables us to find the solution of 2^{nd} and higher order differential equations with constant coefficients as well as equations with variable coefficients.

Working rule

Consider a 2^{nd} order linear differential equation:

$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy\ = F(x)$ …….①

1. Find complimentary function given as: C.F. $= \ c_1y_1+c_2y_2$, where y_1 and y_2 are two linearly independent solutions of ①
2. Calculate $W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}$, W is called Wronskian of y_1 andy_2
3. Compute $u_1=-\int\frac{y_2F(x)}{W}dx$, $u_2=\int\frac{y_1F(x)}{W}dx$
4. Find P.I. $= u_1y_1+u_2y_2$
5. Complete solution is given by: $y\ =$ C.F. + P.I

Note: Method is commonly used to solve 2^{nd} order differential equations, but it can be extended to solve differential equations of higher orders.

Example67 Solve the differential equation: $\frac{d^2y}{dx^2}+y\ = cosec\ x$

using method of variation of parameters.

Solution: $\Rightarrow (D^2+1)y=\operatorname{cosec} x$

Auxiliary equation is: $(m^2+1)=0$

$\Rightarrow\ m=\pm i$

C.F. $= \ c_1\cos x+c_2\sin x=c_1y_1+c_2y_2$

$\therefore\ y_1=\cos x$ and $y_2=\sin x$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}=1$$

$u_1=-\int\frac{y_2F(x)}{W}dx=-\int\sin x\ \operatorname{cosec} x\ dx=\ -\int 1dx=-x$

$u_2=\int\frac{y_1F(x)}{W}dx=\int\cos x\ \operatorname{cosec} x\ dx=\ \int\cot x\,dx=log|\sin x|$

$\therefore$ P.I. $= u_1y_1+u_2y_2$

$\quad = -x\cos x+\ \sin x\ log|\sin x|$

Complete solution is: $y\ =$ C.F. = P.I

$\Rightarrow y=c_1\cos x+c_2\sin x-x\cos x+\ \sin x\ log|\sin x|$

Example68 Solve the differential equation: $(D^2-2D+1)y=e^x$

using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2-2m+1)=0$

$\Rightarrow\ m=1{,}1$

C.F. $= (c_1+c_2\,x)e^x=c_1e^x+c_2x\ e^x=c_1y_1+c_2y_2$

$\therefore\ y_1=e^x$ and $y_2=x\ e^x$

$$W=\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}=\begin{vmatrix} e^x & x\ e^x \\ e^x & x\ e^x+\ e^x \end{vmatrix}=e^{2x}$$

$u_1=-\int\frac{y_2F(x)}{W}dx=-\int\frac{xe^xe^x}{e^{2x}}\ dx=\ -\int xdx=\ -\frac{x^2}{2}$

$u_2=\int\frac{y_1F(x)}{W}dx=\int\frac{e^xe^x}{e^{2x}}\ dx=\ \int 1dx=\ x$

$\therefore$ P.I. $= u_1y_1+u_2y_2$

$\quad = -\frac{x^2}{2}e^x+x^2e^x=\frac{x^2}{2}e^x$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = (c_1 + c_2\, x)e^x + \frac{x^2}{2}e^x$

Example69 Solve the differential equation: $\frac{d^2y}{dx^2} + 4y = x\, sin\, 2x$ using method of variation of parameters.

Solution: $\Rightarrow (D^2 + 4)y = x \sin 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow\ m = \pm 2i$

C.F. = $c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\frac{1}{2}\int x \sin^2 2x\ dx = -\frac{1}{4}\int x(1 - \cos 4x)\, dx$

$= -\frac{1}{4}\left[\frac{x^2}{2} - \left[(x)\left(\frac{\sin 4x}{4}\right) - (1)\left(-\frac{\cos 4x}{16}\right)\right]\right]$

$= \left[-\frac{x^2}{8} + \frac{x\ \sin 4x}{16} + \frac{\cos 4x}{64}\right]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int x \sin 2x \cos 2x\ dx = \frac{1}{4}\int x \sin 4x\ dx$

$= \frac{1}{4}\left[(x)\left(-\frac{\cos 4x}{4}\right) - (1)\left(-\frac{\sin 4x}{16}\right)\right]$

$= \left[-\frac{x\ \cos 4x}{16} + \frac{\sin 4x}{64}\right]$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= \cos 2x\left[-\frac{x^2}{8} + \frac{x\ \sin 4x}{16} + \frac{\cos 4x}{64}\right] + \sin 2x\left[-\frac{x\ \cos 4x}{16} + \frac{\sin 4x}{64}\right]$

$= \frac{x}{16}(\sin 4x \cos 2x - \cos 4x \sin 2x) + \frac{1}{64}(\cos 4x \cos 2x + \sin 4x \sin 2x)$

$-\frac{x^2}{8}\cos 2x\ \ = \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Complete solution is: $y = $ C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x + \frac{x}{16}\sin 2x + \frac{1}{64}\cos 2x - \frac{x^2}{8}\cos 2x$

Example70 Solve the differential equation: $(D^2 - D - 2)y = e^{(e^x + 3x)}$ using method of variation of parameters.

Solution: Auxiliary equation is: $(m^2 - m - 2) = 0$

$\Rightarrow\ m = -1, 2$

C.F. $= c_1 e^{-x} + c_2\, e^{2x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{-x}$ and $y_2 = e^{2x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{2x} \\ -e^{-x} & 2e^{2x} \end{vmatrix} = 3e^x$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{e^{2x} e^{(e^x + 3x)}}{3e^x}\ dx = -\int \frac{e^{2x} e^{e^x} e^{3x}}{3e^x} dx$

$= -\frac{1}{3}\int e^{4x} e^{e^x} dx$, Putting $e^x = t \Rightarrow e^x dx = tdt$

$u_1 = -\frac{1}{3}\int t^3 e^t dt = -\frac{1}{3}[(t^3)(e^t) - (3t^2)(e^t) + (6t)(e^t) - (6)(e^t)]$

$\Rightarrow u_1 = -\frac{e^{e^x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{-x} e^{(e^x + 3x)}}{3e^x}\ dx = \ \int \frac{e^{-x} e^{e^x} e^{3x}}{3e^x} dx\ \frac{1}{3}\int e^x e^{e^x} dx\ = \frac{e^{e^x}}{3}$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= -\frac{e^{e^x} e^{-x}}{3}[e^{3x} - 3e^{2x} + 6e^x - 6] + \frac{e^{e^x} e^{2x}}{3}$

$= \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Complete solution is: $y\ =\ $C.F. + P.I

$\Rightarrow y = c_1 e^{-x} + c_2\, e^{2x} + \frac{e^{e^x}}{3}[3e^x - 6 + 6e^{-x}]$

Example71 Given that $\therefore\ y_1 = x$ and $y_2 = \ \frac{1}{x}$ are two linearly independent solutions of the differential equation: $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y\ = x,\ x \neq 0$. Find the particular integral and general solution using method of variation of parameters.

Solution: Rewriting the equation as: $\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{1}{x^2}y\ = \frac{1}{x}$

Given that $\therefore\ y_1 = x$ and $y_2 = \ \frac{1}{x}$

$\therefore$ C.F. $=\ c_1 y_1 + c_2 y_2 = c_1 x + \ \frac{c_2}{x}$

$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & \frac{1}{x} \\ 1 & -\frac{1}{x^2} \end{vmatrix} = -\frac{2}{x}$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = \int \frac{1}{x}.\frac{1}{x}.\frac{x}{2}\ dx = \ \frac{1}{2}\int \frac{1}{x} dx = \ \frac{1}{2}\log x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = -\int x.\frac{1}{x}.\frac{x}{2}\ dx = \ -\frac{x^2}{4}$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$= \frac{x}{2}\log x - \frac{x}{4}$

Complete solution is: $y\ =\ $C.F. + P.I

$\Rightarrow y = c_1 x + \ \frac{c_2}{x} + \frac{x}{2}\log x - \frac{x}{4}$

Example72 Solve the differential equation: $x^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} - 4x\frac{\mathrm{dy}}{\mathrm{d}x} + 6y\ = x^2 \log x$ using method of variation of parameters.

Solution: This is a Euler-Cauchy linear differential equation with variable coefficients.

Putting $x = e^t \quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy, x^2\frac{\mathrm{d}^2\mathrm{y}}{\mathrm{d}x^2} = \mathrm{D}(\mathrm{D}-1)\mathrm{y}$

$\therefore$ Given differential equation may be rewritten as

$(\mathrm{D}(\mathrm{D}-1) - 4\mathrm{D} + 6)y = te^{2t}$

$\Rightarrow (D^2 - 5D + 6)y = te^{2t}$

Auxiliary equation is: $(m-2)(m-3) = 0$

$\Rightarrow\ m = 2,3$

C.F. = $c_1 e^{2t} + c_2 e^{3t} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{2t}$ and $y_2 = e^{3t}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{2t} & e^{3t} \\ 2e^{2t} & 3e^{3t} \end{vmatrix} = e^{5t}$$

$$u_1 = -\int \frac{y_2 F(t)}{W} dt = -\int \frac{e^{3t} t e^{2t}}{e^{5t}}\ dt = -\int t dt = -\frac{t^2}{2}$$

$$u_2 = \int \frac{y_1 F(t)}{W} dt = \int \frac{e^{2t} t e^{2t}}{e^{5t}}\ dt = \int t e^{-t} dt = [(t)(-e^{-t}) - (1)(e^{-t})]$$

$$= -te^{-t} - e^{-t}$$

$\therefore$ P.I. $= u_1 y_1 + u_2 y_2$

$$= -\frac{t^2}{2} e^{2t} - (te^{-t} + e^{-t})e^{3t}$$

$$= -\frac{t^2}{2} e^{2t} - te^{2t} - e^{2t} = -e^{2t}\left(\frac{t^2}{2} + t + 1\right)$$

Complete solution is: $y\ =$ C.F. + P.I

$$\Rightarrow y = c_1 e^{2t} + c_2 e^{3t} - e^{2t}\left(\frac{t^2}{2} + t + 1\right)$$

or $y = c_1 x^2 + c_2\, x^3 - x^2\left(\frac{(\log x)^2}{2} + \log x + 1\right)$

$$\Rightarrow y = c_3 x^2 + c_2\, x^3 - \frac{x^2}{2}(\log x)^2 - x^2 \log x,\ c_3 = c_1 - 1$$

2.10 Population Dynamics

A Population is the group of individuals of same species, and population dynamics is the study of population changes over time, which can be estimated by mathematical modelling. Here we confine our study to two most practical growth models viz. Exponential and Logistic patterns.

I. Exponential Model

An exponential growth model is possible only if there are unlimited resources and the population can reproduce to its maximum capacity. This is generally not feasible under natural phenomena due to limited available resources. Exponential growth can be achieved if all favorable conditions are provided in a specific environment, for example, bacteria culture in a laboratory. Exponential growth can be represented by a J shaped curve as shown in *Figure4*. If $N(t)$ denotes the size of the population at any time t, then under normal circumstances rate of change of population is directly proportional to population itself i.e., $\frac{dN}{dt} \propto \mathrm{N}$.

$\Rightarrow \frac{dN}{dt} = \text{rN}$, where r is the relative growth rate

$\Rightarrow \frac{dN}{N} = rdt$

Integrating both sides, we have

$\int \frac{dN}{N} = \int rdt$

$\Rightarrow \log N = rt + \log c$

$\Rightarrow \log \frac{N}{c} = rt \Rightarrow e^{\log \frac{N}{c}} = e^{rt} \Rightarrow \frac{N}{c} = e^{rt}$

$\Rightarrow N = ce^{rt}$ is the required solution of the given differential equation.

If N_0 be the initial population at $t = 0$, then population at any time t is given by

$$N(t) = N_0 e^{rt}$$

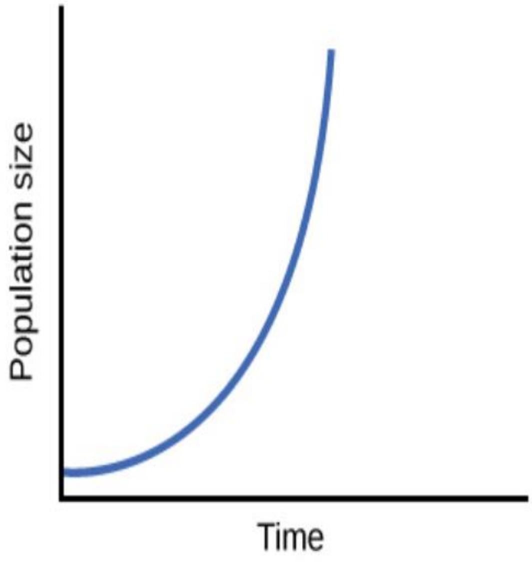


Figure 4

Example73 The population of rabbits in a zoo by the end of year 2010 is 1125 and by the end of the year 2011 is 1242. Assuming population growth to follow exponential model by providing all necessary resources,

(a) Determine the relative growth rate

(b) Write the general equation for population dynamics in exponential model

(c) Calculate the time required for the population to be doubled

(d) Estimate the number of rabbits by the end of the year 2021

Solution: If N denotes the size of the population at any time t, r is the relative growth rate and $N_0 (at\ t = 0)$ be the initial population by the end of year 2010. Then the general equation for population dynamics in exponential model is given by $N(t) = N_0 e^{rt}$

$$\Rightarrow N(t) = 1125e^{rt} \quad \because N_0 = 1125$$

(a) Population by the end of year 2011, i.e., at $t = 1$ is 1242

$\Rightarrow N(1) = 1242 = 1125e^{r}$

$\Rightarrow e^{r} = \frac{1242}{1125} = 1.104$

$\Rightarrow r = \ln 1.104 = 0.0989$

(b) The general equation for population dynamics is given by:

$N(t) = 1125e^{0.0989t}$

(c) For population to be doubled, i.e., $N(t) = 2250$

$\therefore 2250 = 1125e^{0.0989t}$

$\Rightarrow e^{0.0989t} = \frac{2250}{1125} = 2$

$\Rightarrow 0.0989t = \ln 2 = 0.6931$

$\Rightarrow t = \frac{0.6931}{0.0989} = 7.008$ years approximately

(d) We have $N(t) = 1125e^{0.0989t}$

$\therefore$ The population by the end of the year 2021, i.e., at $t = 11$ is given by

$N(11) = 1125e^{1.0879} = 1125\ (2.96803) = 3339$ approx.

Example74 A sample culture has initially P_0 bacteria. After five hours, the number of bacteria is measured to be $5P_0$. Determine the time required for number of bacteria to be ten times as of initial number.

Solution: The general equation for bacteria growth is given by

$$P(t) = P_0 e^{rt}, \text{ where } P(0) = P_0$$

Given $\quad 5P_0 = P_0 e^{5r}$

$$\Rightarrow\ e^{5r} = 5$$

$$\Rightarrow\ 5r = \ln 5 \ \Rightarrow\ r = \frac{1}{5}\ln 5$$

$\therefore$ The general equation for population dynamics is given by:

$$P(t) = P_0 e^{\frac{t}{5}\ln 5}$$

Now for bacteria to grow ten times of initial number

$$10P_0 = P_0 e^{\frac{t}{5}\ln 5}$$

$$\Rightarrow e^{\frac{t}{5}\ln 5} = 10$$

$$\Rightarrow \frac{t}{5}\ln 5 = \ln 10$$

$$\Rightarrow t = \frac{5\ln 10}{\ln 5} = 7.1534 \text{ hours approx.}$$

II. Logistic Growth

Logistic population growth is more practical approach under limited resources. Logistic growth takes place under all-natural phenomenon, when a population becomes almost constant, as it approaches a maximum quantity imposed by limited resources. Logistic growth produces an S-shaped curve (*Figure5*), where L is the carrying capacity of the system. If $N(t)$ denotes the size of the population at any time t and r is the relative growth rate in the logistic growth model, then the rate of growth of population may be defined by the differential equation:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{L}\right) \qquad \ldots Ⓐ$$

$$\Rightarrow \frac{dN}{dt} = rN - \frac{r}{L}N^2 \qquad \ldots Ⓑ$$

Equation Ⓑ implies that if L is very large compared to N, $\frac{r}{L}N^2 \to 0$ and hence $\frac{dN}{dt} \approx rN$

Thus, any population follows exponential growth model for small population number. This is called as Malthus's law.

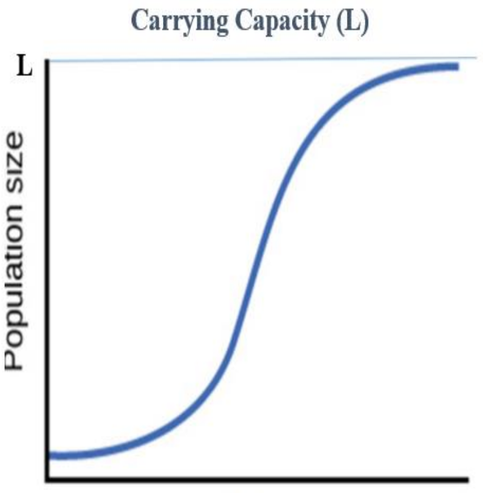


Figure 5

Now Ⓐ $\Rightarrow \frac{dN}{N\left(1-\frac{N}{L}\right)} = rdt$

$$\Rightarrow \frac{LdN}{N(L-N)} = rdt$$

Integrating both sides

$$\int \frac{L}{N(L-N)} dN = \int rdt$$

$$\Rightarrow \int \left(\frac{1}{N} - \frac{1}{L-N}\right) dN = \int rdt$$

$$\Rightarrow \ln N - \ln(L-N) = rt + \ln c\,, \; c \text{ is an arbitrary constant}$$

$$\Rightarrow \ln \frac{N}{c(L-N)} = rt$$

$$\Rightarrow \frac{N}{(L-N)} = ce^{rt}$$

$$\Rightarrow (L-N) = Nbe^{-rt} \quad \text{putting } \; b = \frac{1}{c}$$

$$\Rightarrow L = N + Nbe^{-rt}$$

$$\Rightarrow L = N(1 + be^{-rt})$$

$$\Rightarrow N(t) = \frac{L}{(1+be^{-rt})}$$

Example75 100 fishes of an exotic species were released in a large fish aquarium of a museum having a maximum capacity of 1100 fishes. After seven months, there were 220 fishes in the aquarium. Assuming logistic growth,

(a) Write a general equation that describes the population $N(t)$ at time t.

(b) How many fishes will be there in the aquarium after one year?

(c) In how many months the fish population can reach 500?

Solution: (a) The general equation that describes the population $N(t)$ at time t, assuming logistic growth is $N(t) = \frac{L}{(1+be^{-rt})}$ ...①

Here $N(0) = 100$ and $L = 1100$

Putting $t = 0$, ① $\Rightarrow 100 = \frac{1100}{(1+b)}$

$\Rightarrow (1+b) = 11 \;\; \Rightarrow \text{b} = 10$

$\Rightarrow N(t) = \frac{1000}{(1+10e^{-rt})}$ ...②

Again, given that $N(7) = 220$

Putting $t = 7$, ② $\Rightarrow 220 = \frac{1100}{(1+10e^{-7r})}$

$\Rightarrow (1 + 10e^{-7r}) = 5$

$\Rightarrow e^{-7r} = 0.4 \Rightarrow -7r = \ln 0.4$

$\Rightarrow r = \frac{\ln 0.4}{-7} = \frac{-0.9163}{-7} = 0.1309$

∴ The general equation that describes the population $N(t)$ at time t is given by:

$$N(t) = \frac{1100}{(1+10e^{-0.1309t})}$$

(b) Number of fishes in the aquarium after 12 months is given by

$N(12) = \frac{1100}{\left(1+10e^{-0.1309(12)}\right)} = 357.2834$, i.e., 357 fishes approximately

(c) Here $N(t) = 500\,, \quad t = ?$

We have $N(t) = \frac{1100}{(1+10e^{-0.1309t})}$

$\Rightarrow 500 = \frac{1100}{(1+10e^{-0.1309t})}$

$\Rightarrow (1 + 10e^{-0.1309t}) = 2.2$

$\Rightarrow e^{-0.1309t} = 0.11$

$\Rightarrow -0.1309t = \ln 0.11$

$$\Rightarrow t = \frac{\ln 0.11}{-0.1309} = \frac{-2.2073}{-0.1309} = 16.8625 \text{ months, i.e., nearly 17 months}$$

2.11 Orthogonal Trajectories

Orthogonal trajectories are the curves that are perpendicular to a given family of curves. Let the family of curves $F(x, y, c)$ be the solution of a given differential equation $\frac{dy}{dx} = f(x, y)$; then the family of curves $G(x, y, d)$ represents the orthogonal trajectory of $F(x, y, c)$, if every curve of $G(x, y, d)$ is orthogonal (perpendicular) to each curve of the family $F(x, y, c)$.

Example76 Find the orthogonal trajectories of the families of parabolas $x = ky^2$

Solution: Given family of parabolas is $x = ky^2$ ... ①

Differentiating both sides of ① with respect to x

$$1 = 2ky\frac{dy}{dx} \quad \Rightarrow \frac{dy}{dx} = \frac{1}{2ky} = \frac{y}{2x} = m\,, \quad \because k = \frac{x}{y^2}$$

where m is the slope of the family of parabolas $x = ky^2$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = -\frac{2x}{y}$

Hence the differential equation of orthogonal trajectories is given by

$$\frac{dy}{dx} = -\frac{2x}{y}$$

$$\Rightarrow ydy = -2xdx$$

Integrating both sides, we have

$$\int ydy = \int -2xdx$$

$$\Rightarrow \frac{y^2}{2} = -x^2 + c\,, \quad c \text{ is an arbitrary constant}$$

$$\Rightarrow 2x^2 + y^2 = d \quad , d = 2c \text{ is an arbitrary constant}$$

$$\therefore 2x^2 + y^2 = d$$

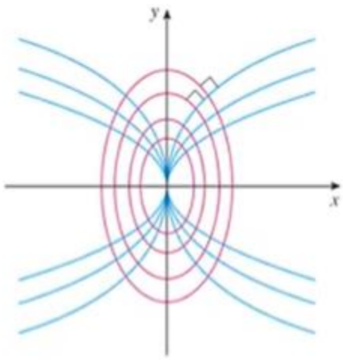


Figure 6

Figure 6 shows the required orthogonal trajectory, which is a family of ellipse.

Example77 Find the orthogonal trajectories of the family of circles $x^2 + y^2 = r^2$

Solution: Given family of circles is: $x^2 + y^2 = r^2$ ... ①

Differentiating both sides of ① with respect to x

$$2x + 2y\frac{dy}{dx} = 0 \quad \Rightarrow \frac{dy}{dx} = -\frac{x}{y} = m\,,$$

where m is the slope of the family of circles $x^2 + y^2 = r^2$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = \frac{y}{x}$

Hence the differential equation of orthogonal trajectories is given by

$$\frac{dy}{dx} = \frac{y}{x} \Rightarrow \frac{dy}{y} = \frac{dx}{x}$$

Integrating both sides, we have

$$\int \frac{dy}{y} = \int \frac{dx}{x}$$

$$\Rightarrow \log y = \log x + \log c\,, \quad c \text{ is an arbitrary constant}$$

$$\Rightarrow \log y = \log cx \quad \Rightarrow y = cx$$

$\therefore y = cx$ is the required orthogonal trajectory.

Example78 Find the orthogonal trajectories of the family of curves $y = e^{ax}$

Solution: Given family of curves is: $y = e^{ax}$ ...①

Differentiating both sides of ① with respect to x

$\frac{dy}{dx} = ae^{ax} = m$ ...②

where m is the slope of the family of curves $y = e^{ax}$

To remove the constant a from m,

Taking natural log on both sides of equation ①

$\Rightarrow \ln y = ax \Rightarrow a = \frac{\ln y}{x}$

Substituting $e^{ax} = y$ and $a = \frac{\ln y}{x}$ in ②

$\Rightarrow m = \frac{y \ln y}{x}$

$\therefore$ slope of orthogonal trajectories $= -\frac{1}{m} = -\frac{x}{y \ln y}$

Hence the differential equation of orthogonal trajectories is given by

$$\frac{dy}{dx} = -\frac{x}{y \ln y}$$

$$\Rightarrow y \ln y \, dy = -x dx$$

Integrating both sides, we have

$\int y \ln y \, dy = -\int x dx$ ...③

Now let $I = \int y \ln y \, dy$

$\Rightarrow I = y(y \ln y - y) - \left(I - \frac{y^2}{2}\right) + c_1$, $\because \int \ln y \, dy = y \ln y - y$

$\Rightarrow 2I = y(y \ln y - y) + \frac{y^2}{2} + c_1 \Rightarrow I = \frac{y}{2}(y \ln y - y) + \frac{y^2}{4} + c_1$

$\Rightarrow I = \frac{y^2 \ln y}{2} - \frac{y^2}{2} + \frac{y^2}{4} + c_1 \quad \Rightarrow I = \frac{y^2 \ln y}{2} - \frac{y^2}{4} + c_1$ ...④

Using ④ in ③

$\Rightarrow \frac{y^2 \ln y}{2} - \frac{y^2}{4} + c_1 = -\frac{x^2}{2} + c_2$, c_1 and c_2 are arbitrary constants

$\therefore 2x^2 - y^2 + 2y^2 \ln y = c$ is the required orthogonal trajectory.

2.12 Modeling of Free Oscillations of a Mass-Spring System

Consider an undamped (unaffected by any external forces like air or friction) mass-spring system as shown in Figure7. Assume that the spring can resist both extension and compression with stiffness constant 'K'. The system is purely theoretical because it neglects damping forces resulting in uninterrupted free oscillations, which is impracticable. Any practical model will always have damping forces resulting in oscillations to stop eventually.

Suppose we have an elastic spring with stiffness constant 'K' hanging from a fixed surface (Figure7a). We attach an object with mass 'm', resulting the string to stretch by a length 'y_0' after the system attains its rest position (Figure7b).

While in rest position, the gravitational force on the system acting in downward direction is mg, and an upwards restoring force 'F' also acts on the system due to initial displacement 'y_0'.

Let's define a reference frame, where the downward direction is the positive y-direction, primarily because gravitational forces pull the spring in the downwards direction.

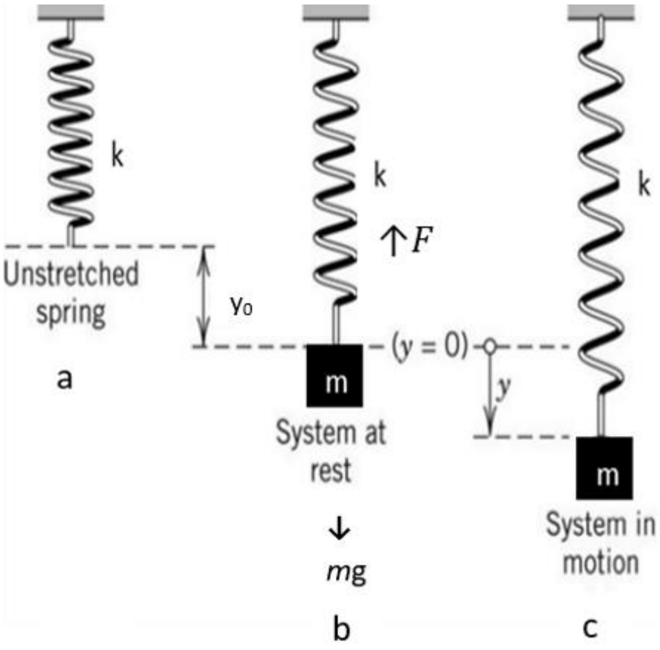


Figure 7

Also, let $y = 0$ be the equilibrium position of the top surface of the suspended mass 'm' and upward is the negative y-direction.

Now, the upward restoring force 'F' on the system caused due to string stiffness is directly proportional to the initial displacement 'y_0', i.e., $F \propto y_0$

$\Rightarrow\ F = -ky_0\ \ \ldots$ 1

Note that we have used negative sign because upward displacement is in negative y-direction as per our reference system.

Again, the downward gravitational force on the system 'mg', and the restoring force 'F' balance each other, so that the system is at rest in its equilibrium position.

$\Rightarrow\ F + mg = 0\quad \ldots$ 2

From 1 and 2, $-ky_0 + mg = 0$

$\Rightarrow k = \frac{mg}{y_0}\ \ \ldots$ 3, k is the spring stiffness measure.

Now, when we pull the mass 'm' in downwards direction (Figure7c) by a distance y,

as per Hooke's Law, the system produces an upward restoring force 'F' to resist the displacement 'y', such that $F \propto y$.

$$\Rightarrow\ F = -ky$$

$$\Rightarrow m\frac{d^2y}{dt^2} = -ky\ ,\ \because Force = mass \times \text{acceleration}$$

$$\Rightarrow\ \frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

$$\Rightarrow\ \frac{d^2y}{dt^2} + \omega^2 y = 0\ ,\ \text{ putting } \frac{k}{m} = \omega^2\quad \ldots 4$$

is the required differential equation of the mass-spring system.

Clearly, 4 is a homogeneous linear equation with constant coefficients.

$$\Rightarrow (D^2 + \omega^2)y = 0\ ,\ D = \frac{d}{dt}$$

Auxiliary equation is: $p^2 + \omega^2 = 0 \Rightarrow p = \pm i\omega$

C.F. $= c_1 \cos \omega\, t + c_2 \sin \omega t$

$\therefore$ Solution of the mass-spring equation $\frac{d^2y}{dx^2} + \frac{k}{m}y = 0$ is given by

$y = c_1 \cos \omega t + c_2 \sin \omega t, \ \omega^2 = \frac{k}{m}$

Clearly ④ is the differential equation of the Simple Harmonic Motion (SHM).

Also, period of oscillation (T) is given by $\frac{2\pi}{\omega}$ and frequency (ρ) is $\frac{1}{T}$

$\therefore T = 2\pi\sqrt{\frac{m}{k}}$ and $\rho = \frac{1}{2\pi} = \sqrt{\frac{k}{m}}$

Example 78 Solve the mass-spring equation $\frac{d^2y}{dt^2} + 64y = 0$, $y(0) = 4$, $y'(0) = 0$.

Also, interpret the initial value problem and find the period and the frequency of the simple harmonic motion.

Solution: Given the mass spring equation $\frac{d^2y}{dt^2} + 64y = 0$ ... ①

Comparing with the equation $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$, $\frac{k}{m} = \omega^2$ ... ②

$\Rightarrow \frac{k}{m} = \omega^2 = 64$

Solution of ② is given by $y = c_1 \cos \omega t + c_2 \sin \omega t, \ \omega^2 = \frac{k}{m}$

$\therefore$ Solution of ① is $y(t) = c_1 \cos 8t + c_2 \sin 8t$ ... ③

Also, given $y(0) = 4$, i.e. ... ④

Initial condition ④ implies that the mass is displaced downwards by 4 units to initiate the simple harmonic motion in the mass-spring system.

Given $y'(0) = 0$ ... ⑤

Also, ⑤ implies that no initial velocity is given to the system.

Using ④ in ③ $\Rightarrow y(0) = 4 = c_1 \cos 0 + c_2 \sin 0 \Rightarrow c_1 = 4$

$\therefore$ ③ $\Rightarrow y(t) = 4 \cos 8t + c_2 \sin 8t$ ... ⑥

Differentiating equation ⑥, $y'(t) = -32 \sin 8t + 8c_2 \cos 8t$ ... ⑦

Using ⑤ in ⑦ $\Rightarrow y'(0) = 0 = -32 \sin 0 + 8c_2 \cos 0 \Rightarrow c_2 = 0$

$\therefore y(t) = 4 \cos 8t$ is the required solution of the given mass -spring equation.

Also, the Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{8} = \frac{\pi}{4}$

Frequency $\rho = \frac{1}{T} = \frac{4}{\pi}$

Example79 An object of weight 4lbs stretches a string by 12 inches. Find the equation of motion if the spring is released from its equilibrium position with an upward velocity of 10 ft/s. Find the frequency and the period of the motion.

Solution: Let the equation of mass-spring system be given by $\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$

Given that $W = mg = 4$ lbs, initial displacement $y_0 = 1$ft

Also, $y(0) = 0$, $y'(0) = -10$ ft/s

Now, $k = \frac{mg}{y_0} = \frac{4}{1} = 4$

Also, $W = mg$

$\Rightarrow 4 = 32m \qquad \because \quad g = 32.14$ ft/s^2 in British system

$\Rightarrow \ m = 1/8$

The differential equation of mass spring- system be given is given by

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0 \ \Rightarrow \frac{d^2y}{dt^2} + 32y = 0 \quad \ldots (1)$$

$$\because k = 4 \text{ and } m = 1/8$$

Solution of (1) is $y = c_1 \cos \omega\, t + c_2 \sin \omega t,\ \omega^2 = \frac{k}{m} = 32$

$$\Rightarrow y = c_1 \cos \sqrt{32}\, t + c_2 \sin \sqrt{32} t \ \ldots (2)$$

Given $y(0) = 0,\ y'(0) = -10$, substituting in (2), we get $c_1 = 0$ and $c_2 = -\frac{10}{\sqrt{32}}$

$\Rightarrow y = -\frac{10}{\sqrt{32}} c_2 \sin \sqrt{32} t$ is the required equation of motion.

Also, the Period $T = \frac{2\pi}{\omega} = \frac{2\pi}{\sqrt{32}} = \frac{2\pi}{2\sqrt{8}} = \frac{\pi}{\sqrt{8}}$

Frequency $\rho = \frac{1}{T} = \frac{\sqrt{8}}{\pi}$

2.13 Series Solutions and Special Functions

Consider the linear differential equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$, whose solution is given by

$y = c_1 e^{3x} + c_2 e^{5x} = c_1 \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} + c_2 \sum_{n=0}^{\infty} \frac{(5x)^n}{n!}$, which is a series solution.

Some differential equations with variable coefficients cannot be solved by usual methods, and we need to employ series solution method to find their solutions in terms of infinite convergent series.

2.13.1 Power Series

An infinite series of the form $\sum_{n=0}^{\infty} a_n (x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots$, is called a power series about the point x_0; a_0, a_1, a_2 ... are arbitrary constants. The point $x = x_0$ is called center of the power series. The power series about the origin $(x_0 = 0)$, is called standard power series and is given as: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \cdots a_n x^n + \cdots$

2.13.2Series Solutions

Consider a second order linear differential equation:

$$P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0 \ldots (1)$$

where $P(x)$, $Q(x)$ and $R(x)$ are functions of x or constants.

Ordinary and Singular Points

I. The point $x = x_0$ is called an ordinary point of equation (1), if $P(x_0) \neq 0$

II. The point $x = x_0$ is called a singular point of equation (1), if $P(x_0) = 0$

a) If both $\lim\limits_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}$ and $\lim\limits_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$ are finite, then the point $x = x_0$ is called a regular singular point of equation (1).

b) If either or both of $\lim\limits_{x \to x_0} (x - x_0)\frac{Q(x)}{P(x)}$ and $\lim\limits_{x \to x_0} (x - x_0)^2 \frac{R(x)}{P(x)}$ are non-finite, then the point $x = x_0$ is called an irregular singular point of equation (1).

Note: Series solution does not exist if $x = x_0$ is an irregular singular point of a differential equation.

Example80 Find the ordinary points, regular singular, and irregular singular points of the differential equation $x^2(x-1)(x-2)\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+2xy\ =0$

Solution: Given $x^2(x-1)(x-2)\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+2xy\ =0$ ... ①

Comparing with the differential equation $P(x)\frac{d^2y}{dx^2}+Q(x)\frac{dy}{dx}+R(x)y\ =0$

$P(x)=x^2(x-1)(x-2)$, $Q(x)=(x-1)$ and $R(x)=2x$,

Now for x_0 to be an ordinary point, $P(x_0)\neq 0$

$\Rightarrow x^2(x-1)(x-2)\neq 0\ \Rightarrow x_0\in R-\{0,1,2\}$

∴ all real numbers except 0, 1 and 2 are ordinary points of differential equation ①

Again, for singular points $P(x_0)=0$

i.e., $x^2(x-1)(x-2)=0\ \Rightarrow x_0\in\{0,1,2\}$

∴ 0, 1 and 2 are singular points of differential equation ①

Now $\lim\limits_{x\to 0}(x-0)\frac{(x-1)}{x^2(x-1)(x-2)}=\lim\limits_{x\to 0}\frac{1}{x(x-2)}=\infty$

$\Rightarrow x=0$ is an irregular singular point of the differential equation ①

Again, $\lim\limits_{x\to 1}(x-1)\frac{(x-1)}{x^2(x-1)(x-2)}=\lim\limits_{x\to 1}\frac{(x-1)}{x^2(x-2)}=0$ i.e., finite

And $\lim\limits_{x\to 1}(x-1)^2\frac{2x}{x^2(x-1)(x-2)}=\lim\limits_{x\to 1}\frac{2(x-1)}{x(x-2)}=0$, i.e., finite

$\Rightarrow x=1$ is a regular singular point of the differential equation ①

Also, $\lim\limits_{x\to 2}(x-2)\frac{(x-1)}{x^2(x-1)(x-2)}=\lim\limits_{x\to 2}\frac{1}{x^2}=\frac{1}{4}$ i.e., finite

And $\lim\limits_{x\to 2}(x-2)^2\frac{2x}{x^2(x-1)(x-2)}=\lim\limits_{x\to 2}\frac{2(x-2)}{x(x-1)}=0$, i.e., finite

$\Rightarrow x=2$ is a regular singular point of the differential equation ①

2.13.3 Algorithm to find series solution when $x=0$ is an ordinary point of equation ①, i.e., $P(0)\neq 0$

Step1: Assume the solution of equation ① as $y=\sum_{r=0}^{\infty}a_rx^r\ =a_0+a_1x+a_2x^2+\cdots$

... ②

Step2: Differentiate ② with respect to x to find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Step3: Substitute the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the differential equation ①

Step4: As R.H.S. is zero, equate to zero the coefficients of different powers of x, particularly x^r in most cases to find a recurrence relation between the coefficients.

Step5: Substitute the values of a_0, a_1, a_2, a_3 in ② to get the required solution.

Example81 Find the power series solution about $x=0$ for the differential equation:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$$

Solution: Given $(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+2y=0$ ... ①

Let the solution of equation ① be given as $y=\sum_{r=0}^{\infty}a_rx^r=a_0+a_1x+a_2x^2+\cdots$

... ②

Differentiating ② with respect to x, $\frac{dy}{dx}=\sum_{r=1}^{\infty}a_rrx^{r-1}$ ... ③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2}=\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}$ ... ④

Substituting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in equation ①

$\Rightarrow(1-x^2)[\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}]-2x[\sum_{r=1}^{\infty}a_rrx^{r-1}]+2[\sum_{r=0}^{\infty}a_rx^r]=0$

$\Rightarrow\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}-\sum_{r=0}^{\infty}a_rr(r-1)x^r-2\sum_{r=0}^{\infty}a_rrx^r+2\sum_{r=0}^{\infty}a_rx^r=0$

$\Rightarrow\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}-\sum_{r=0}^{\infty}a_r[r^2-r+2r-2]\,x^r=0$

$\Rightarrow\sum_{r=2}^{\infty}a_rr(r-1)x^{r-2}-\sum_{r=0}^{\infty}a_r[r^2+r-2]\,x^r=0$ ... ⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1)-a_r(r^2+r-2)=0$

$\Rightarrow a_{r+2}=\frac{(r^2+r-2)}{(r+2)(r+1)}a_r=\frac{(r+2)(r-1)}{(r+2)(r+1)}a_r=\frac{(r-1)}{(r+1)}a_r$

$\Rightarrow a_{r+2}=\frac{(r-1)}{(r+1)}a_r$ ... ⑥ is the required recurrence relation

Putting $r=0$ in ⑥, $a_2=-a_0$

Putting $r=1$ in ⑥, $a_3=0$

Putting $r=2$ in ⑥, $a_4=\frac{1}{3}a_2=-\frac{1}{3}a_0$ $\qquad\because a_2=-a_0$

Putting $r=3$ in ⑥, $a_5=\frac{1}{2}a_3=0$ $\qquad\because a_3=0$

Putting $r=4$ in ⑥, $a_6=\frac{3}{5}a_4=\frac{3}{5}\left(-\frac{1}{3}a_0\right)=-\frac{1}{5}a_0$ $\qquad\because a_4=-\frac{1}{3}a_0$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of a_2, a_3, a_4, a_5, ... in equation ②

$\Rightarrow y=a_0+a_1x+(-a_0)x^2+\left(-\frac{a_0}{3}\right)x^4+\left(-\frac{a_0}{5}\right)x^6+\cdots$

$\Rightarrow y=a_1x+a_0\left(1-x^2-\frac{x^4}{3}-\frac{x^6}{5}-\cdots\right)$ is the required series solution of equation ①.

2.14 Legendre's Equation

Another important differential equation used in problems showing spherical symmetry is Legendre's equation given by $\boldsymbol{(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y=0}$ ...... ①

Here n is a real number, though in most practical applications only non-negative integral values are required. Solving equation ① about the point $x=0$, which is an ordinary point

Let the solution of equation ① be given as

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots a_r x^r + \cdots$ ... ②

Differentiating ② with respect to x, $\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$ ... ③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$ ... ④

Substituting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in equation ①

$\Rightarrow (1-x^2)[\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}] - 2x[\sum_{r=1}^{\infty} a_r r x^{r-1}] + n(n+1)[\sum_{r=0}^{\infty} a_r x^r] = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r [r^2 - r + 2r - n(n+1)]\, x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r [r^2 + r - n(n+1)]\, x^r = 0$ ... ⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1) - a_r(r^2 + r - n(n+1)) = 0$

$\Rightarrow a_{r+2} = \frac{r(r+1)-n(n+1)}{(r+2)(r+1)} a_r$ ... ⑥ is the required recurrence relation

Putting $r = 0$ in ⑥, $a_2 = \frac{-n(n+1)}{2!} a_0$

Putting $r = 1$ in ⑥, $a_3 = \frac{2-n(n+1)}{6} a_1 = \frac{-(n-1)(n+2)}{3!} a_1$

Putting $r = 2$ in ⑥, $a_4 = \frac{6-n(n+1)}{12} a_2 = \frac{-(n-2)(n+3)}{12}\left(\frac{-n(n+1)}{2}\right) a_0 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0$

$\because a_2 = \frac{-n(n+1)}{2} a_0$

Putting $r = 3$ in ⑥, $a_5 = \frac{12-n(n+1)}{20} a_3 = \frac{-(n-3)(n+4)}{20}\left(\frac{-(n-1)(n+2)}{3!}\right) a_1 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$

$\because a_3 = \frac{-(n-1)(n+2)}{3!} a_1$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of $a_2, a_3, a_4, a_5, \ldots$ in equation ②

$$y = a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} a_1 x^3 + \frac{(n-2)n(n+1)(n+3)}{4!} a_0 x^4 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 x^5 + \cdots$$

$\Rightarrow y = a_0\left[1 - \frac{n(n+1)}{2!}x^2 + \frac{(n-2)n(n+1)(n+3)}{4!}x^4 - \cdots\right] +$

$a_1\left[x - \frac{(n-1)(n+2)}{3!}x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!}x^5 - \cdots\right]$ is the required series solution of Legendre's equation given in ①

$\therefore$ Series solution of ① in terms of Legendre's function $P_n(x)$ and $Q_n(x)$ is given by

$y = a_0 P_n(x) + a_1 Q_n(x)$,

Here $P_n(x)$ is called Legendre polynomial and $Q_n(x)$ is called Legendre function of 2^{nd} kind.

Results: (i) $\boldsymbol{P_n(1) = 1}$ (ii) $\boldsymbol{P_n(-1) = (-1)^n}$

2.14. 1 Recurrence Relations of Legendre's Function $\boldsymbol{P_n(x)}$

(1) $\boldsymbol{(n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)}$

Proof: From generating function $(1-2xz+z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$……①

Differentiating both sides of ① partially with respect to z, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-\frac{3}{2}}(-2x+2z) = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)\,(1-2xz+z^2)^{-\frac{1}{2}-1} = \sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x)$$

$$\Rightarrow (x-z)\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} nz^{n-1}P_n(x) \text{ using ①}$$

Equating coefficient of z^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)x\, P_n(x) - nP_{n-1}(x)$$

(2) $\boldsymbol{P_n(x) = P'_{n+1}(x) - 2x\, P'_n(x) + P'_{n-1}(x)}$

Differentiating both sides of ① partially with respect to x, we get

$$-\frac{1}{2}(1-2xz+z^2)^{-1-\frac{1}{2}}(-2z) = \sum_{n=0}^{\infty} z^n\, P'_n(x)$$

$$\Rightarrow z(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\, P'_n(x)$$

$$\Rightarrow z\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\, P'_n(x) \text{ using ①}$$

Equating coefficient of z^{n+1} on both sides

$$P_n(x) = P'_{n+1}(x) - 2x\, P'_n(x) + P'_{n-1}(x)$$

(3) $\boldsymbol{nP_n(x) = xP'_n(x) - P'_{n-1}(x)}$

Differentiating recurrence relation (1) partially with respect to x, we get

$$(n+1)P'_{n+1}(x) = (2n+1)\, x\, P'_n(x) + (2n+1)P_n(x) - nP'_{n-1}(x)\text{…..②}$$

Also from recurrence relation (2)

$$P'_{n+1}(x) = P_n(x) + 2x\, P'_n(x) - P'_{n-1}(x) \;\; \text{…… ③}$$

Using ③ in ②, we get

$$(n+1)[P_n(x) + 2x\, P'_n(x) - P'_{n-1}(x)] = (2n+1)\, x\, P'_n(x) + (2n+1)P_n(x) - nP'_{n-1}(x)$$

$$\Rightarrow nP_n(x) = xP'_n(x) - P'_{n-1}(x)$$

(4) $\boldsymbol{(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)}$

Adding recurrence relations (2) and (3), we get

$$(n+1)P_n(x) = P'_{n+1}(x) - xP'_n(x)$$

(5) $\boldsymbol{(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)}$

Adding recurrence relations (3) and (4), we get

$$(2n+1)P_n(x) = P'_{n+1}(x) - P'_{n-1}(x)$$

(6) $\boldsymbol{(1-x^2)\, P'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]}$

Replacing n by $(n-1)$ in recurrence relation (4)

$$nP_{n-1}(x) = P'_n(x) - xP'_{n-1}(x) \quad \text{…… ④}$$

Also multiplying recurrence relation (3) by x

$n\, xP_n(x) = x^2 P_n'(x) - xP_{n-1}'(x)$ .... ⑤

Subtracting ⑤ from ④

$(1 - x^2)\, P_n'(x) = n\, [P_{n-1}(x) - xP_n(x)]$

(7) $\boldsymbol{(1 - x^2)\, P_n'(x) = (n + 1)\, [xP_n(x) - P_{n+1}(x)]}$

Replacing n by $(n + 1)$ in recurrence relation (3)

$\Rightarrow (n + 1)P_{n+1}(x) = xP_{n+1}'(x) - P_n'(x)$ ......⑥

Also multiplying recurrence relation (4) by x

$(n + 1)xP_n(x) = xP_{n+1}'(x) - x^2 P_n'(x)$ .... ⑦

Subtracting ⑥ from ⑦, we get

$(1 - x^2)\, P_n'(x) = (n + 1)\, [xP_n(x) - P_{n+1}(x)]$

2.14.2 Rodrigue's Formula

Rodrigue's formula is helpful in producing Legendre's polynomials of various orders and is given by

$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2 - 1)^n$

Proof: Let $y = (x^2 - 1)^n$

$\therefore \frac{dy}{dx} = n(x^2 - 1)^{n-1} 2x = 2nx \frac{(x^2-1)^n}{(x^2-1)}$

$\Rightarrow y_1(x^2 - 1) - 2nxy = 0\,,\ y_1 \equiv \frac{dy}{dx}$ ....②

Differentiating ② $(n + 1)$ times using Leibnitz's theorem:

$\Rightarrow y_{n+2}(x^2 - 1) + (n + 1)y_{n+1}(2x) + \frac{(n+1).n}{2!} y_n(2) - 2n[y_{n+1}(x) + (n + 1)y_n(1)] = 0$

$\Rightarrow y_{n+2}(x^2 - 1) + 2xy_{n+1} - (n^2 + n)y_n = 0$

$\Rightarrow (1 - x^2)y_{n+2} - 2xy_{n+1} + n(n + 1)y_n = 0$ ....③

Putting $y_n = V$, so that $y_{n+1} = \frac{dV}{dx}$ and $y_{n+2} = \frac{d^2V}{dx}$

③$\Rightarrow (1 - x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + n(n + 1)V = 0$

which is Legendre's equation with the solution $V = AP_n(x) + BQ_n(x)$

But since $V = y_n = \frac{d^n}{dx^n}(x^2 - 1)^n$ contains only positive powers of x, solution can only be a constant multiple of $P_n(x)$.

$\therefore P_n(x) = CV = Cy_n$

$= C\frac{d^n}{dx^n}(x^2 - 1)^n$ .... ④

$= CD^n[(x - 1)^n(x + 1)^n]$, $\frac{d^n}{dx^n} \equiv D^n$

$= CD^n[(x - 1)^n(x + 1)^n]$

$= C\big[D^n(x - 1)^n(x + 1)^n + n_{C_1}D^{n-1}(x - 1)^n n(x + 1)^{n-1} + \cdots + (x - 1)^n D^n(x + 1)^n\big]$

$= C[n!\,(x + 1)^n + n.n(n - 1) \ldots 3.2.\,(x - 1)n(x + 1)^{n-1} + \cdots + (x - 1)^n n!]$

Taking $x = 1$ on both sides

$\Rightarrow 1 = Cn!\,2^n + 0 \quad \because P_n(1) = 1$

$\Rightarrow C = \frac{1}{2^n n!}$ ....⑤

Using ⑤ in ④, we get

$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}(x^2-1)^n$

Putting $n = 0$, $P_0(x) = 1$

Putting $n = 1$, $P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2}2x = x$

Putting $n = 2$, $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1)$

Putting $n = 3$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

Putting $n = 4$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Putting $n = 5$, $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$ etc…

Example82 Expand the following functions in series of Legendre's polynomials.

(i) $(1+2x-x^2)$

(ii) (x^3-5x^2+x+1)

Solution: $1 = P_0(x)$, $x = P_1(x)$,

$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{1}{3}(2P_2(x)+1) = \frac{1}{3}\big(2P_2(x)+P_0(x)\big)$

$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow x^3 = \frac{1}{5}(2P_3(x)+3x) = \frac{1}{5}\big(2P_3(x)+3P_1(x)\big)$

(i) Let $E = (1+2x-x^2)$

Substituting values of 1, x and x^2 in terms of Legendre's polynomials, we get

$E = \left(P_0(x)+2P_1(x)-\frac{1}{3}\big(2P_2(x)+P_0(x)\big)\right)$

$= \frac{1}{3}\big(3P_0(x)+6P_1(x)-2P_2(x)-P_0(x)\big)$

$= \frac{2}{3}\big(P_0(x)+3P_1(x)-P_2(x)\big)$

(ii) Let $F = (x^3-5x^2+x+1)$

Substituting values of 1, x, x^2 and x^3in terms of Legendre's polynomials, we get

$F = \left[\frac{1}{5}\big(2P_3(x)+3P_1(x)\big)-\frac{5}{3}\big(2P_2(x)+P_0(x)\big)+P_1(x)+P_0(x)\right]$

$= \frac{2}{5}P_3(x)-\frac{10}{3}P_2(x)+\frac{8}{5}P_1(x)-\frac{2}{3}P_0(x)$

Example83 Prove that

(i) $P_n'(1) = \frac{n(n+1)}{2}$

(ii) $P_n'(-1) = (-1)^{(n+1)}\frac{n(n+1)}{2}$

Solution: $P_n(x)$ is the solution of Legendre's equation given by:

$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0$ …….①

$\therefore y = P_n(x)$ will satisfy equation ①

$\Rightarrow (1-x^2)P_n''(x)-2x\,P_n'(x)+n(n+1)P_n(x) = 0$ …….②

Putting $x = 1$ in ② we get

$-2P_n'(1)+n(n+1)P_n(1) = 0$

$\Rightarrow P_n'(1) = \frac{n(n+1)}{2} \because P_n(1) = 1$

Putting $x = -1$ in ② we get

$$2P_n'(-1) + n(n+1)P_n(-1) = 0$$
$$\Rightarrow P_n'(-1) = -\frac{n(n+1)}{2}P_n(-1)$$
$$= (-1)^{(n+1)}\frac{n(n+1)}{2} \quad \because P_n(-1) = (-1)^n$$

2.15 Bessel's Equation

The differential equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ ……①

is known as Bessel's equation of order n and its solutions are called Bessel's functions.

Note that $x = 0$ is a regular singular point of Bessel's equation.

Series solution of ① in terms of Bessel's functions $J_n(x)$ and $J_{-n}(x)$ is given by

$$y = AJ_n(x) + BJ_{-n}(x)$$

where $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Proposition If n is any integer then $J_{-n}(x) = (-1)^n J_n(x)$

Proof: Case I: n is a positive integer

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

If n is a positive integer, values of r from 0 to $(n-1)$ will give gamma function of $-ve$ integers in the denominator, which being infinite all such terms will vanish.

$$\therefore J_{-n}(x) = \sum_{r=n}^{\infty}(-1)^r\frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Putting $r = n + k$, we get

$$J_{-n}(x) = \sum_{k=0}^{\infty}(-1)^{n+k}\frac{1}{(n+k)!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{n+2k}$$
$$= (-1)^n\sum_{k=0}^{\infty}(-1)^k\frac{1}{k!\,\Gamma(n+k+1}\left(\frac{x}{2}\right)^{n+2k}$$
$$= (-1)^n J_n(x)$$

Case II: $n = 0$

$$J_{-0}(x) = (-1)^0 J_0(x)$$

or $J_0(x) = J_0(x)$, which is true

Case III: n is a negative integer

Let $= -p$, where p is a positive integer

From case I $J_p(x) = (-1)^{-p}J_{-p}(x) \Rightarrow J_{-n}(x) = (-1)^n J_n(x)$

2.15.1 Expansions of $J_0(x)$, $J_1(x)$, $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$

We have $J_n(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{n+2r}\frac{1}{r!\,\Gamma(n+r+1)}$

1. $J_0(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{r!\,\Gamma(r+1)} = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{(r!)^2}$

$\because \Gamma(r+1 = r!$ when r is a positive integer

$$\Rightarrow J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

2. $J_1(x) = \sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{1+2r}\frac{1}{r!\,\Gamma(r+2)} = \frac{x}{2}\sum_{r=0}^{\infty}(-1)^r\left(\frac{x}{2}\right)^{2r}\frac{1}{r!\,(r+1)!}$

$\because \Gamma(r+2 = (r+1)!$ when r is a positive integer

$$\Rightarrow J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 + \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

3. $J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \frac{1}{r!\,\Gamma\left(r+\frac{3}{2}\right)}$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\Gamma\frac{3}{2}} - \frac{1}{1!\,\Gamma\frac{5}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\frac{7}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma\frac{9}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\frac{1}{2}\Gamma\frac{1}{2}} - \frac{1}{1!\,\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(\mathrm{n}+1 = \mathrm{n}\Gamma\mathrm{n}$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{1}{\frac{1}{2}} - \frac{1}{1!\,\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right] \quad \because \Gamma\frac{1}{2} = \sqrt{\pi}$$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \frac{2x^6}{7!} + \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\,\frac{2}{x}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

4. $J_{-\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{-\frac{1}{2}+2r} \frac{1}{r!\,\Gamma\left(r+\frac{1}{2}\right)}$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma\frac{1}{2}} - \frac{1}{1!\,\Gamma\frac{3}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\Gamma\frac{5}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\Gamma\frac{7}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\Gamma\frac{1}{2}} - \frac{1}{1!\,\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}\Gamma\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(\mathrm{n}+1 = \mathrm{n}\Gamma\mathrm{n}$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{1}{1!\,\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma\frac{1}{2} = \sqrt{\pi}$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\cos x$$

2.15.2 Recurrence Relations of Bessel's Function

(1) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ **or** $\int x^n J_{n-1}(x)dx = x^n J_n(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow x^n J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty}(-1)^r \frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}} \frac{1}{r!\,(n+r)\Gamma(n+r)}$$

$\because \Gamma(\mathrm{n}+\mathrm{r}+1) = (\mathrm{n}+\mathrm{r})\Gamma(\mathrm{n}+\mathrm{r})$

$$= x^n \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{r!\,\Gamma((n-1)+r+1)}$$

$$= x^n J_{n-1}(x)$$

(2) $\frac{d}{dx}[x^{-n} J_n(x)] = -x^{-n} J_{n+1}(x)$ **or** $\int x^{-n} J_{n+1}(x)\,dx = -\frac{d}{dx}[x^{-n} J_n(x)]$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow x^{-n} J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty}(-1)^r \frac{2r\,x^{2r-1}}{2^{n+2r}} \frac{1}{(r-1)!\ r\,\Gamma(n+r+1)}$$

$$= x^{-n}\sum_{r=1}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{(r-1)!\ \Gamma(n+r+1)}$$

$$= -x^{-n}\sum_{k=0}^{\infty}(-1)^k \left(\frac{x}{2}\right)^{(n+1)+2k} \frac{1}{k!\ \Gamma((n+1)+k+1)}$$

Putting $r = k+1$

$$= -x^{-n}J_{n+1}(x)$$

(3) $\boldsymbol{J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)}$

Proof: From recurrence relation (1)

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n{}'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we get

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$$

$$\Rightarrow J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

(4) $\boldsymbol{J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)}$

Proof: From recurrence relation (2)

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow x^{-n}J_n{}'(x) - n x^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

Dividing by x^{-n}, we get

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$$

(5) $\boldsymbol{J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]}$

Proof: Adding recurrence relations (3) and (4), we get

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

(6) $\boldsymbol{2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]}$

Proof: Subtracting recurrence relations (3) from (4), we get

$$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$

2.13 Previous Years Solved Questions

Q1. Solve $y(2xy + e^x)dx - e^x dy = 0$

$\langle Q1(g), GGSIPU, December\ 2012\rangle$

Solution: $M = y(2xy + e^x)$, $N = -e^x$

$\frac{\partial M}{\partial y} = 4xy + e^x$, $\frac{\partial N}{\partial x} = -e^x$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

Rearranging the equation as $(ye^x dx - e^x dy) + 2xy^2 dx = 0$ ….①

Taking $\frac{1}{y^2}$ as integrating factor, ① may be rewritten as:

$$\frac{ye^x dx - e^x dy}{y^2} + 2x\,dx = 0$$

$$\Rightarrow d\left[\frac{e^x}{y}\right] + 2x\,dx = 0 \ldots\ldots ②$$

Integrating ②, solution is given by : $\frac{e^x}{y} + x^2 = C$

$\Rightarrow e^x + yx^2 = Cy$

Q2. Solve the differential equation: $\langle Q8(a),\ GGSIPU,\ December\ 2012\rangle$

$(x^2 + y^2 + 2x)dx + 2y\,dy = 0$ .....①

Solution: $M = x^2 + y^2 + 2x,\ N = 2y$

$\frac{\partial M}{\partial y} = 2y\ ,\ \frac{\partial N}{\partial x} = 0$

$\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$, ∴ given differential equation is not exact.

As ① is neither homogeneous nor of the form $yf_1(xy)dx + xf_2(xy)dy = 0$,

∴ Computing $\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x} = 2y$

Clearly $\frac{\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}}{N} = \frac{2y}{2y} = 1 = f(x)$ say

∴ IF = $e^{\int f(x)dx} = e^{\int 1dx} = e^x$

∴ ① may be rewritten after multiplying by IF as:

$e^x(x^2 + y^2 + 2x)dx + 2e^x y\,dy = 0$.......②

New $M = e^x(x^2 + y^2 + 2x)$, New $N = 2e^x y$

$\frac{\partial M}{\partial y} = 2e^x y\ ,\ \frac{\partial N}{\partial x} = 2e^x y$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ ② is an exact differential equation.

Solution of ② is given by:

$\int e^x(x^2 + y^2 + 2x)\,dx + \int 0\,dy$

y constant

$\Rightarrow (x^2 + y^2 + 2x)e^x - (2x + 2)e^x + (2)e^x = \mathrm{C}$

$\Rightarrow (x^2 + y^2)e^x = \mathrm{C}$, C is an arbitrary constant.

Q3. Solve $(xy^2 + x)dx + (yx^2 + y)dy$

$\langle Q1(f),\ GGSIPU,\ December\ 2013\rangle$

Solution: $M = xy^2 + x\ ,\ N = yx^2 + y$

$\frac{\partial M}{\partial y} = 2xy,\ \frac{\partial N}{\partial x} = 2xy$

$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, ∴ given differential equation is exact.

Solution is given by:

$\int (xy^2 + x)\,dx + \int y\,dy = C$

y constant

$\Rightarrow \frac{x^2y^2}{2} + \frac{x^2}{2} + \frac{y^2}{2} = \mathrm{C}_1$

$\Rightarrow x^2y^2 + x^2 + y^2 = \mathrm{C}$

Q.4 Solve $(D^2 + D + 1)^2(D - 2)y = 0$ $\langle \boldsymbol{Q1(h), GGSIPU,\ December\ 2012}\rangle$

Solution: Auxiliary equation is: $(m^2 + m + 1)^2(m - 2)y = 0$.......①

Solving ①, we get

$\Rightarrow m = 2, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i, -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$

C.F. = $c_1e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3\,x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5x)\sin\frac{\sqrt{3}}{2}x$

Since $F(x) = 0$, solution is given by $y = \text{C.F}$

$$\Rightarrow y = c_1 e^{2x} + e^{\frac{-x}{2}}[(c_2 + c_3 x)\cos\frac{\sqrt{3}}{2}x + (c_4 + c_5 x)\sin\frac{\sqrt{3}}{2}x$$

Q5. Solve $(D^2 - 1)y = \cosh x \cos x$

$$\langle \boldsymbol{Q8(b)},\ \boldsymbol{GGSIPU},\ \boldsymbol{December\ 2012} \rangle$$

Solution: Auxiliary equation is: $m^2 - 1 = 0$

$\Rightarrow m = \pm 1$

C.F. = $c_1 e^x + c_2 e^{-x}$

P.I. = $\frac{1}{f(D)} F(x)$

$$= \frac{1}{D^2-1}\left(\frac{e^x + e^{-x}}{2}\cos x\right) \qquad \because \cosh x = \frac{e^x + e^{-x}}{2}$$

$$= \frac{1}{D^2-1}\left(\frac{e^x}{2}\cos x + \frac{e^{-x}}{2}\cos x\right)$$

$$= \frac{e^x}{2}\frac{1}{(D+1)^2-1}\cos x + \frac{e^{-x}}{2}\frac{1}{(D-1)^2-1}\cos x$$

$$= \frac{e^x}{2}\frac{1}{(D^2+2D)}\cos x + \frac{e^{-x}}{2}\frac{1}{D^2-2D}\cos x$$

$$= \frac{e^x}{2}\frac{1}{2D-1}\cos x + \frac{e^{-x}}{2}\frac{1}{-2D-1}\cos x \qquad \text{Putting } D^2 = -1$$

$$= \frac{e^x}{2}\frac{2D+1}{4D^2-1}\cos x - \frac{e^{-x}}{2}\frac{2D-1}{4D^2-1}\cos x$$

$$= -\frac{e^x}{10}(2D+1)\cos x + \frac{e^{-x}}{10}(2D-1)\cos x \qquad \text{Putting } D^2 = -1$$

$$= -\frac{e^x}{10}(-2\sin x + \cos x) + \frac{e^{-x}}{10}(-2\sin x - \cos x)$$

$$\therefore \text{P.I.} = \frac{e^x}{10}(2\sin x - \cos x) - \frac{e^{-x}}{10}(2\sin x + \cos x)$$

Complete solution is: $y = \text{C.F.} + \text{P.I}$

$$\Rightarrow y = c_1 e^x + c_2 e^{-x} + \frac{e^x}{10}(2\sin x - \cos x) - \frac{e^{-x}}{10}(2\sin x + \cos x)$$

Q6. Solve $\frac{d^2y}{dx^2} + 4y = 4\tan 2x$ by the method of variation of parameters.

$$\langle \boldsymbol{Q9(a)},\ \boldsymbol{GGSIPU},\ \boldsymbol{December\ 2012} \rangle$$

Solution: $\Rightarrow (D^2 + 4)y = 4\tan 2x$

Auxiliary equation is: $(m^2 + 4) = 0$

$\Rightarrow m = \pm 2i$

C.F. = $c_1 \cos 2x + c_2 \sin 2x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = \cos 2x$ and $y_2 = \sin 2x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} \cos 2x & \sin 2x \\ -2\sin 2x & 2\cos 2x \end{vmatrix} = 2$$

$$u_1 = -\int \frac{y_2 F(x)}{W}dx = -\frac{4}{2}\int \sin 2x \tan 2x\ dx = -2\int \frac{\sin^2 2x}{\cos 2x}\ dx$$

$$-2\int \frac{1-\cos^2 2x}{\cos 2x}\ dx = -2\int(\sec 2x - \cos 2x)\,dx$$

$$= -2\left[\frac{1}{2}\log|\sec 2x + \tan 2x| - \frac{1}{2}\sin 2x\right]$$

$= [\sin 2x - \log|\sec 2x + \tan 2x|]$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \frac{1}{2}\int 4 \tan 2x \cos 2x \ dx = \ 2\int \sin 2x \ dx$

$= -\cos 2x$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= \cos 2x\, [\sin 2x - \log|\sec 2x + \tan 2x|] - \sin 2x \cos 2x$

$= -\cos 2x \log|\sec 2x + \tan 2x|$

Complete solution is: $y \ = \ $C.F. + P.I

$\Rightarrow y = c_1 \cos 2x + c_2 \sin 2x - \cos 2x \log|\sec 2x + \tan 2x|$

Q7. Solve the system of equations: $\frac{\mathrm{d}x}{\mathrm{dt}} + x \ = y + e^t, \ \ \frac{\mathrm{d}y}{\mathrm{dt}} + y \ = x + e^t$

$\langle \boldsymbol{Q9(b)}, \ \boldsymbol{GGSIPU}, \ \ \boldsymbol{December\ 2012} \rangle$

Solution: Rewriting given system of differential equations as:

$(D+1)x - y = e^t$……①

$(D+1)y - x = e^t$ …..②, $D \equiv \frac{d}{dt}$

Multiplying ① by $(D+1)$

$\Rightarrow (D+1)^2 x - (D+1)y = (D+1)e^t$

$(D^2 + 2D + 1)x - (D+1)y = 2e^t$…..③

Adding ② and ③, we get

$(D^2 + 2D)x = 3e^t$ …….④

which is a linear differential equation in x with constant coefficients.

To solve ④ for x, Auxiliary equation is $m^2 + 2m = 0$

$\Rightarrow m = 0, -2$

C.F. = $c_1 + c_2 e^{-2t}$

P.I. = $\frac{1}{f(D)} F(t) \ = 3 \frac{1}{D^2+2D} e^t$

$= e^t$ Putting $D = 1$

$\therefore x = c_1 + c_2 e^{-2t} + e^t$…….⑤

Using ⑤in ① $\Rightarrow D[c_1 + c_2 e^{-2t} + e^t] + c_1 + c_2 e^{-2t} + e^t - y = e^t$

$\Rightarrow -2c_2 e^{-2t} + e^t + c_1 + c_2 e^{-2t} - y = 0$

$\Rightarrow y = c_1 - c_2 e^{-2t} + e^t$ ……⑥

⑤ and ⑥ give the required solution.

Q8. Solve by method of variation of parameters $y'' - 6y' + 9y \ = \frac{e^{3x}}{x^2}$

$\langle \boldsymbol{Q8(a)}, \boldsymbol{GGSIPU}, \boldsymbol{December\ 2013} \rangle$, $\langle \boldsymbol{Q3(b)}, \boldsymbol{GGSIPU}, \boldsymbol{2^{nd} term\ 2014} \rangle$

Solution: Auxiliary equation is: $\mathrm{m}^2 - 6\mathrm{m} + 9 = 0$

$(\mathrm{m} - 3)^2 = 0$

$\Rightarrow \ m = 3,3$

C.F. = $(c_1 + c_2\,x)e^{3x} = c_1 e^{3x} + c_2 x\, e^{3x} = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^{3x}$ and $y_2 = xe^{3x}$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^{3x} & xe^{3x} \\ 3e^{3x} & 3xe^{3x} + e^{3x} \end{vmatrix} = e^{6x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{xe^{3x}e^{3x}}{x^2 e^{6x}}\, dx = -\int \frac{1}{x} dx = -\log x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^{3x}e^{3x}}{x^2 e^{6x}}\, dx = \int \frac{1}{x^2} dx = -\frac{1}{x}$

$\therefore$ P.I$= u_1 y_1 + u_2 y_2$

$= -e^{3x}\log x - e^{3x} = -e^{3x}(1 + \log x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^{3x} - e^{3x}(1 + \log x)$

Q9. Solve the differential equation: $\frac{d^3y}{dx^3} + 2\frac{d^2y}{dx^2} + \frac{dy}{dx} = e^{2x} + \sin 2x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2013} \rangle$

Solution: $\Rightarrow (D^3 + 2D^2 + D)y = e^{2x} + \sin 2x$

Auxiliary equation is: $m^3 + 2m^2 + m = 0$

$\Rightarrow m(m^2 + 2m + 1) = 0$

$\Rightarrow m(m+1)^2 = 0$

$\Rightarrow m = 0, -1, -1$

C.F. = $c_1 + e^{-x}(c_2 + c_3 x)$

P.I. = $\frac{1}{f(D)} F(x) = \frac{1}{D^3 + 2D^2 + D}(e^{2x} + \sin 2x)$

$= \frac{1}{D^3 + 2D^2 + D} e^{2x} + \frac{1}{D^3 + 2D^2 + D} \sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{-4D - 8 + D} \sin 2x$, putting $D = 2$ in 1st term, $D^2 = -4$ in the 2nd term

$= \frac{1}{18} e^{2x} - \frac{3D-8}{(3D+8)(3D-8)} \sin 2x = \frac{1}{18} e^{2x} - \frac{3D-8}{(9D^2-64)} \sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{100}(3\text{D} - 8)\sin 2x$

$= \frac{1}{18} e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = c_1 + e^{-x}(c_2 + c_3 x) + \frac{1}{18} e^{2x} + \frac{1}{100}(6\cos 2x - 8\sin 2x)$

Q10. Solve $(D^2 - 2D + 1)y = xe^x \cos x$

$\langle \boldsymbol{Q8(a), GGSIPU, December\ 2014} \rangle$

Solution: Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2$

$\Rightarrow m = 1,1$

C.F. = $(c_1 + c_2 x)e^x$

P.I. $= \frac{1}{f(D)} F(x) = \frac{1}{D^2 - 2D + 1} xe^x \cos x$

$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} x \cos x$

$$= e^x \frac{1}{D^2} x\cos x$$
$$= e^x \frac{1}{D}\int x\cos x\, dx$$
$$= e^x \frac{1}{D}[(x)(\sin x) - (1)(-\cos x)]$$
$$= e^x \frac{1}{D}[x\sin x + \cos x]$$
$$= e^x[\int x\sin x\, dx + \int \cos x\, dx]$$
$$= e^x\big[[(x)(-\cos x) - (1)(-\sin x)] + \sin x\big]$$
$\therefore$ P.I. $= e^x[-x\cos x + 2\sin x]$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^x + e^x[-x\cos x + 2\sin x]$

Q11. Solve by M.O.V.P. $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = e^x \log x$

$\langle \boldsymbol{Q8(b), GGSIPU, December\ 2014} \rangle$

Solution: Given differential equation may be rewritten as

$(D^2 - 2D + 1)y = e^x \log x$

: Auxiliary equation is: $m^2 - 2m + 1 = 0$

$\Rightarrow (m-1)^2$

$\Rightarrow m = 1,1$

C.F. $= (c_1 + c_2 x)e^x = c_1 y_1 + c_2 y_2$

$\therefore y_1 = e^x$ and $y_2 = xe^x$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1{}' & y_2{}' \end{vmatrix} = \begin{vmatrix} e^x & x\,e^x \\ e^x & x\,e^x + e^x \end{vmatrix} = e^{2x}$$

$u_1 = -\int \frac{y_2 F(x)}{W} dx = -\int \frac{x\,e^x\,e^x \log x}{e^{2x}}\, dx = -\int x\log x\, dx$

$\int x\log x\, dx = I = \left[(x)(x\log x - x) - (1)\left(I - \frac{x^2}{2}\right)\right]$

$\because \int \log x\, dx = x\log x - x$

$\Rightarrow 2I = x^2\log x - x^2 + \frac{x^2}{2}$

$\Rightarrow I = \int x\log x\, dx = \frac{x^2}{2}\log x - \frac{x^2}{4}$

$\therefore u_1 = \frac{x^2}{4} - \frac{x^2}{2}\log x$

$u_2 = \int \frac{y_1 F(x)}{W} dx = \int \frac{e^x e^x \log x}{e^{2x}}\, dx = \int \log x\, dx = x\log x - x$

$\therefore$ P.I$= \left(\frac{x^2}{4} - \frac{x^2}{2}\log x\right)e^x + (x\log x - x)xe^x$

$= e^x\left(\frac{x^2}{4} - \frac{x^2}{2}\log x + x^2\log x - x^2\right)$

$= \frac{x^2}{2}e^x\left(\log x - \frac{3}{2}\right)$

Complete solution is: $y =$ C.F. + P.I

$\Rightarrow y = (c_1 + c_2 x)e^x + \frac{x^2}{2}e^x\left(\log x - \frac{3}{2}\right)$

Q12. Solve $(D-1)^2(D+1)^2 = \sin^2\frac{x}{2} + e^x + x$

$\langle \boldsymbol{Q1(a), GGSIPU, December\ 2015} \rangle$

Solution: Auxiliary equation is: $(m-1)^2(m+1)^2 = 0$

$\Rightarrow\ m = 1,1,-1,-1$

C.F. $= (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x}$

P.I. $= \frac{1}{f(D)}\ F(x) = \frac{1}{((D-1)(D+1))^2}(\sin^2\frac{x}{2} + e^x + x)$

$= \frac{1}{2}\frac{1}{D^4-2D^2+1}(1-\cos x) + \frac{1}{D^4-2D^2+1}\ e^x + \frac{1}{D^4-2D^2+1}x$

$=\frac{1}{2}\frac{1}{D^4-2D^2+1}\ e^{0x} - \frac{1}{2}\frac{1}{D^4-2D^2+1}\cos x + \frac{1}{D^4-2D^2+1}\ e^x + \frac{1}{D^4-2D^2+1}x$

Now $\frac{1}{2}\frac{1}{D^4-2D^2+1}\ e^{0x} = \frac{1}{2}$, putting $D = 0$

Also $\frac{1}{2}\frac{1}{D^4-2D^2+1}\cos x = \frac{1}{8}\cos x$ putting $D^2 = -1$

Again $\frac{1}{D^4-2D^2+1}\ e^x = x\frac{1}{4D^3-4D}\ e^x$ as $f(1) = 0$, a case of failure 2 times

$= x^2\frac{1}{12D^2-4}\ e^x = \frac{x^2}{8}\ e^x$, putting $D = 1$

And $\frac{1}{D^4-2D^2+1}x = \frac{1}{1+(D^4-2D^2)}x = [1+(D^4-2D^2)]^{-1}x = x$

$\therefore$ P.I.$= \frac{1}{2} - \frac{1}{8}\cos x + \frac{x^2}{8}\ e^x + x$

Complete solution is: y = C.F. + P.I

$\Rightarrow y = (c_1 + c_2x)e^x + (c_3 + c_4x)e^{-x} - \frac{1}{8}\cos x + \frac{x^2}{8}\ e^x + x + \frac{1}{2}$

Q.13 Solve $x^2\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + 6y = x^4 \sin x$

$\langle \boldsymbol{Q3(b), GGSIPU, December\ 2015} \rangle$

Solution: This is a Cauchy's linear equation with variable coefficients.

Putting $x = e^t\quad \therefore \log x = t$

$\Rightarrow x\frac{dy}{dx} = Dy,\ x^2\frac{d^2y}{dx^2} = D(D-1)y$

$\therefore$ Equation may be rewritten as

$(D(D-1) - 4D + 6)y = e^{4t}\sin e^t$

$\Rightarrow (D^2 - 5D + 6)y = e^{4t}\sin e^t\ ,\ D \equiv \frac{d}{dt}$

Auxiliary equation is: $m^2 - 5m + 6 = 0$

$\Rightarrow (m-2)(m-3) = 0$

$\Rightarrow\ m = 2,3$

C.F. $= c_1e^{2t} + c_2e^{3t} = c_1x^2 + c_2x^3\quad \because e^t = x$

P.I. $= \frac{1}{f(D)}\ F(x) = \frac{1}{D^2-5D+6}e^{4t}\sin e^t$

$= e^{4t}\frac{1}{(D+4)^2-5(D+4)+6}\sin e^t$

$= e^{4t}\frac{1}{D^2+3D+2}\sin e^t = e^{4t}\frac{1}{(D+1)(D+2)}\sin e^t$

$= e^{4t}\left[\frac{1}{(D+1)} - \frac{1}{(D+2)}\right]\sin e^t = e^{4t}\left[\frac{1}{(D+1)}\sin e^t - \frac{1}{(D+2)}\sin e^t\right]$

$= e^{4t}[e^{-t}\int e^t \sin e^t\, dt - e^{-2t}\int e^{2t}\sin e^t\, dt]$

$\because \frac{1}{(D+a)}F(t) = e^{-at}\int e^{at}F(t)dt$

$= e^{4t}[e^{-t}(-cose^t) - e^{-2t}(-e^t\cos e^t + \sin e^t]$

Solving the two integrals by putting $e^t = u, \therefore e^t dt = du$

$\therefore$ P.I = $-e^{2t}\sin e^t = -x^2\sin x$

Complete solution is: $y\ =$ C.F. + P.I

$\Rightarrow y\ =\ c_1x^2 + c_2x^3 - x^2\sin x$