Chapter 3: Series Solutions and Special Functions

3.1 Introduction

Consider the linear differential equation $\frac{d^2y}{dx^2} - 8\frac{dy}{dx} + 15y = 0$, whose solution is given by $y = c_1e^{3x} + c_2e^{5x} = c_1\sum_{n=0}^{\infty}\frac{(3x)^n}{n!} + c_2\sum_{n=0}^{\infty}\frac{(5x)^n}{n!}$, which is a series solution.

Some differential equations with variable coefficients cannot be solved by usual methods, and we need to employ series solution method to find their solutions in terms of infinite convergent series.

3.1.1 Power Series

An infinite series of the form $\sum_{n=0}^{\infty} a_n(x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \cdots$, is called a power series about the point x_0; a_0, a_1, a_2 ... are arbitrary constants. The point $x = x_0$ is called center of the power series. The power series about the origin $(x_0 = 0)$, is called standard power series and is given as: $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1x + a_2x^2 + \cdots a_nx^n + \cdots$

3.1.3 Series Solutions

Consider a second order linear differential equation: $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$ ... ①

where $P(x)$, $Q(x)$ and $R(x)$ are functions of x or constants.

Ordinary and Singular Points

I. The point $x = x_0$ is called an ordinary point of equation ①, if $P(x_0) \neq 0$

II. The point $x = x_0$ is called a singular point of equation ①, if $P(x_0) = 0$

a) If both $\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}$ and $\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$ are finite, then the point $x = x_0$ is called a regular singular point of equation ①.

b) If either or both of $\lim_{x\to x_0}(x-x_0)\frac{Q(x)}{P(x)}$ and $\lim_{x\to x_0}(x-x_0)^2\frac{R(x)}{P(x)}$ are non-finite, then the point $x = x_0$ is called an irregular singular point of equation ①.

Note: Series solution does not exist if $x = x_0$ is an irregular singular point of a differential equation.

Example1 Find the ordinary points, regular singular, and irregular singular points of the differential equation $x^2(x-1)(x-2)\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + 2xy = 0$

Solution: Given $x^2(x-1)(x-2)\frac{d^2y}{dx^2} + (x-1)\frac{dy}{dx} + 2xy = 0$ ... ①

Comparing with the differential equation $P(x)\frac{d^2y}{dx^2} + Q(x)\frac{dy}{dx} + R(x)y = 0$

$P(x) = x^2(x-1)(x-2)$, $Q(x) = (x-1)$ and $R(x) = 2x$,

Now for x_0 to be an ordinary point, $P(x_0) \neq 0$

$\Rightarrow x^2(x-1)(x-2) \neq 0 \Rightarrow x_0 \in R - \{0,1,2\}$

∴ all real numbers except 0, 1 and 2 are ordinary points of differential equation ①

Again, for singular points $P(x_0) = 0$

i.e., $x^2(x-1)(x-2) = 0 \Rightarrow x_0 \in \{0,1,2\}$

$\therefore$ 0, 1 and 2 are singular points of differential equation ①

Now $\lim_{x\to 0}(x-0)\frac{(x-1)}{x^2(x-1)(x-2)} = \lim_{x\to 0}\frac{1}{x(x-2)} = \infty$

$\Rightarrow x = 0$ is an irregular singular point of the differential equation ①

Again, $\lim_{x\to 1}(x-1)\frac{(x-1)}{x^2(x-1)(x-2)} = \lim_{x\to 1}\frac{(x-1)}{x^2(x-2)} = 0$ i.e., finite

And $\lim_{x\to 1}(x-1)^2\frac{2x}{x^2(x-1)(x-2)} = \lim_{x\to 1}\frac{2(x-1)}{x(x-2)} = 0$, i.e., finite

$\Rightarrow x = 1$ is a regular singular point of the differential equation ①

Also, $\lim_{x\to 2}(x-2)\frac{(x-1)}{x^2(x-1)(x-2)} = \lim_{x\to 2}\frac{1}{x^2} = \frac{1}{4}$ i.e., finite

And $\lim_{x\to 2}(x-2)^2\frac{2x}{x^2(x-1)(x-2)} = \lim_{x\to 2}\frac{2(x-2)}{x(x-1)} = 0$, i.e., finite

$\Rightarrow x = 2$ is a regular singular point of the differential equation ①

3.1.3.1 Algorithm to find series solution when $x = 0$ is an ordinary point of equation ①, i.e., $P(0) \neq 0$

Step1: Assume the solution of equation ① as $y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$ ... ②

Step2: Differentiate ② with respect to x to find the values of $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Step3: Substitute the values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in the differential equation ①

Step4: As R.H.S. is zero, equate to zero the coefficients of different powers of x, particularly x^r in most cases to find a recurrence relation between the coefficients.

Step5: Substitute the values of a_0, a_1, a_2, a_3 in ② to get the required solution.

Example2 Find the power series solution about $x = 0$ for the differential equation:

$$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$$

Solution: Given $(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + 2y = 0$ ...①

Let the solution of equation ① be given as $y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots$ ...②

Differentiating ② with respect to x, $\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$ ... ③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$ ... ④

Substituting values of $y, \frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in equation ①

$\Rightarrow (1-x^2)[\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}] - 2x[\sum_{r=1}^{\infty} a_r r x^{r-1}] + 2[\sum_{r=0}^{\infty} a_r x^r] = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + 2\sum_{r=0}^{\infty} a_r x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 - r + 2r - 2]\, x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 + r - 2]\, x^r = 0$ ... ⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1) - a_r(r^2 + r - 2) = 0$

$\Rightarrow a_{r+2} = \frac{(r^2+r-2)}{(r+2)(r+1)} a_r = \frac{(r+2)(r-1)}{(r+2)(r+1)} a_r = \frac{(r-1)}{(r+1)} a_r$

$\Rightarrow a_{r+2} = \frac{(r-1)}{(r+1)} a_r$ ... ⑥ is the required recurrence relation

Putting $r = 0$ in ⑥, $a_2 = -a_0$

Putting $r = 1$ in ⑥, $a_3 = 0$

Putting $r = 2$ in ⑥, $a_4 = \frac{1}{3} a_2 = -\frac{1}{3} a_0$ $\qquad \because a_2 = -a_0$

Putting $r = 3$ in ⑥, $a_5 = \frac{1}{2} a_3 = 0$ $\qquad \because a_3 = 0$

Putting $r = 4$ in ⑥, $a_6 = \frac{3}{5} a_4 = \frac{3}{5}\left(-\frac{1}{3} a_0\right) = -\frac{1}{5} a_0$ $\qquad \because a_4 = -\frac{1}{3} a_0$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of a_2, a_3, a_4, a_5, ... in equation ②

$\Rightarrow y = a_0 + a_1 x + (-a_0)x^2 + \left(-\frac{a_0}{3}\right)x^4 + \left(-\frac{a_0}{5}\right)x^6 + \cdots$

$\Rightarrow y = a_1 x + a_0\left(1 - x^2 - \frac{x^4}{3} - -\frac{x^6}{5} - \cdots\right)$ is the required series solution of equation ①.

3.2 Legendre's Equation

Another important differential equation used in problems showing spherical symmetry is Legendre's equation given by $\boldsymbol{(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + n(n+1)y = 0}$ …… ①

Here n is a real number, though in most practical applications only non-negative integral values are required. Solving equation ① about the point $x = 0$, which is an ordinary point

Let the solution of equation ① be given as

$y = \sum_{r=0}^{\infty} a_r x^r = a_0 + a_1 x + a_2 x^2 + \cdots a_r x^r + \cdots$ ... ②

Differentiating ② with respect to x, $\frac{dy}{dx} = \sum_{r=1}^{\infty} a_r r x^{r-1}$ ... ③

Again differentiating ③ with respect to x, $\frac{d^2y}{dx^2} = \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}$ ... ④

Substituting values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from ②, ③ and ④ in equation ①

$\Rightarrow (1 - x^2)[\sum_{r=2}^{\infty} a_r r(r-1)x^{r-2}] - 2x[\sum_{r=1}^{\infty} a_r r x^{r-1}] + n(n+1)[\sum_{r=0}^{\infty} a_r x^r] = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r r(r-1)x^r - 2\sum_{r=0}^{\infty} a_r r x^r + n(n+1)\sum_{r=0}^{\infty} a_r x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 - r + 2r - n(n+1)]\, x^r = 0$

$\Rightarrow \sum_{r=2}^{\infty} a_r r(r-1)x^{r-2} - \sum_{r=0}^{\infty} a_r[r^2 + r - n(n+1)]\, x^r = 0$ ... ⑤

Equating to zero the coefficient of x^r in equation ⑤

$\Rightarrow a_{r+2}(r+2)(r+1) - a_r(r^2 + r - n(n+1)) = 0$

$\Rightarrow a_{r+2} = \frac{r(r+1)-n(n+1)}{(r+2)(r+1)} a_r$ ... ⑥ is the required recurrence relation

Putting $r = 0$ in ⑥, $a_2 = \frac{-n(n+1)}{2!} a_0$

Putting $r = 1$ in ⑥, $a_3 = \frac{2-n(n+1)}{6} a_1 = \frac{-(n-1)(n+2)}{3!} a_1$

Putting $r = 2$ in ⑥, $a_4 = \frac{6-n(n+1)}{12} a_2 = \frac{-(n-2)(n+3)}{12}\left(\frac{-n(n+1)}{2}\right) a_0 = \frac{(n-2)n(n+1)(n+3)}{4!} a_0$

$\because a_2 = \frac{-n(n+1)}{2} a_0$

Putting $r = 3$ in ⑥, $a_5 = \frac{12-n(n+1)}{20} a_3 = \frac{-(n-3)(n+4)}{20}\left(\frac{-(n-1)(n+2)}{3!}\right) a_1 = \frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1$

$\because a_3 = \frac{-(n-1)(n+2)}{3!} a_1$

Similarly, all the coefficients can be found using the recurrence relation ⑥

Substituting the values of a_2, a_3, a_4, a_5, ... in equation ②

$y = a_0 + a_1 x - \frac{n(n+1)}{2!} a_0 x^2 - \frac{(n-1)(n+2)}{3!} a_1 x^3 + \frac{(n-2)n(n+1)(n+3)}{4!} a_0 x^4 +$
$\frac{(n-3)(n-1)(n+2)(n+4)}{5!} a_1 x^5 + \cdots$

$\Rightarrow y = a_0\left[1 - \frac{n(n+1)}{2!} x^2 + \frac{(n-2)n(n+1)(n+3)}{4!} x^4 - \cdots\right] +$
$a_1\left[x - \frac{(n-1)(n+2)}{3!} x^3 + \frac{(n-3)(n-1)(n+2)(n+4)}{5!} x^5 - \cdots\right]$ is the required series solution of Legendre's equation given in ①

∴ Series solution of ① in terms of Legendre's function $P_n(x)$ and $Q_n(x)$ is given by

$y = a_0 P_n(x) + a_1 Q_n(x)$,

Here $P_n(x)$ is called Legendre polynomial and $Q_n(x)$ is called Legendre function of 2^{nd} kind.

Results: (i) $\boldsymbol{P_n(1) = 1}$ **(ii)** $\boldsymbol{P_n(-1) = (-1)^n}$

3.2.1 Recurrence Relations of Legendre's Function $\boldsymbol{P_n(x)}$

(1) $\boldsymbol{(n+1)P_{n+1}(x) = (2n+1)x\,P_n(x) - nP_{n-1}(x)}$

Proof: From generating function $(1 - 2xz + z^2)^{-\frac{1}{2}} = \sum_{n=0}^{\infty} z^n P_n(x)$......①

Differentiating both sides of ① partially with respect to z, we get

$-\frac{1}{2}(1 - 2xz + z^2)^{-\frac{3}{2}}(-2x + 2z) = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$

$\Rightarrow (x - z)(1 - 2xz + z^2)^{-\frac{1}{2}-1} = \sum_{n=0}^{\infty} nz^{n-1} P_n(x)$

$\Rightarrow (x - z)(1 - 2xz + z^2)^{-\frac{1}{2}} = (1 - 2xz + z^2)\sum_{n=0}^{\infty} nz^{n-1} P_n(x)$

$\Rightarrow (x - z)\sum_{n=0}^{\infty} z^n P_n(x) = (1 - 2xz + z^2)\sum_{n=0}^{\infty} nz^{n-1} P_n(x)$ using ①

Equating coefficient of z^n on both sides

$$xP_n(x) - P_{n-1}(x) = (n+1)P_{n+1}(x) - 2xnP_n(x) + (n-1)P_{n-1}(x)$$

$$\Rightarrow (n+1)P_{n+1}(x) = (2n+1)x\,P_n(x) - nP_{n-1}(x)$$

(2) $\boldsymbol{P_n(x) = \mathrm{P}'_{n+1}(x) - 2x\,\mathrm{P}'_n(x) + \mathrm{P}'_{n-1}(x)}$

Differentiating both sides of ① partially with respect to x, we get

$$-\frac{1}{2}\,(1-2xz+z^2)^{-1-\frac{1}{2}}(-2z) = \sum_{n=0}^{\infty} z^n\,\mathrm{P}'_n(x)$$

$$\Rightarrow z(1-2xz+z^2)^{-\frac{1}{2}} = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\,\mathrm{P}'_n(x)$$

$$\Rightarrow z\sum_{n=0}^{\infty} z^n P_n(x) = (1-2xz+z^2)\sum_{n=0}^{\infty} z^n\,\mathrm{P}'_n(x) \text{ using ①}$$

Equating coefficient of z^{n+1} on both sides

$$P_n(x) = \mathrm{P}'_{n+1}(x) - 2x\,\mathrm{P}'_n(x) + \mathrm{P}'_{n-1}(x)$$

(3) $\boldsymbol{nP_n(x) = x\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)}$

Differentiating recurrence relation (1) partially with respect to x, we get

$$(n+1)\mathrm{P}'_{n+1}(x) = (2n+1)\,x\,\mathrm{P}'_n(x) + (2n+1)P_n(x) - n\mathrm{P}'_{n-1}(x) \ldots\ldots ②$$

Also from recurrence relation (2)

$$\mathrm{P}'_{n+1}(x) = P_n(x) + 2x\,\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x) \ \ldots\ldots ③$$

Using ③ in ②, we get

$$(n+1)[P_n(x) + 2x\,\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)] = (2n+1)\,x\,\mathrm{P}'_n(x) + (2n+1)P_n(x) - n\mathrm{P}'_{n-1}(x)$$

$$\Rightarrow nP_n(x) = x\mathrm{P}'_n(x) - \mathrm{P}'_{n-1}(x)$$

(4) $\boldsymbol{(n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - x\mathrm{P}'_n(x)}$

Adding recurrence relations (2) and (3), we get

$$(n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - x\mathrm{P}'_n(x)$$

(5) $\boldsymbol{(2n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - \mathrm{P}'_{n-1}(x)}$

Adding recurrence relations (3) and (4), we get

$$(2n+1)P_n(x) = \mathrm{P}'_{n+1}(x) - \mathrm{P}'_{n-1}(x)$$

(6) $\boldsymbol{(1-x^2)\,\mathrm{P}'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]}$

Replacing n by $(n-1)$ in recurrence relation (4)

$$nP_{n-1}(x) = \mathrm{P}'_n(x) - x\mathrm{P}'_{n-1}(x) \quad \ldots\ldots ④$$

Also multiplying recurrence relation (3) by x

$$n\,xP_n(x) = x^2\mathrm{P}'_n(x) - x\mathrm{P}'_{n-1}(x) \quad \ldots\ldots. ⑤$$

Subtracting ⑤ from ④

$$(1-x^2)\,\mathrm{P}'_n(x) = n\,[P_{n-1}(x) - xP_n(x)]$$

(7) $\boldsymbol{(1-x^2)\,\mathrm{P}'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]}$

Replacing n by $(n+1)$ in recurrence relation (3)

$$\Rightarrow (n+1)P_{n+1}(x) = x\mathrm{P}'_{n+1}(x) - \mathrm{P}'_n(x) \ \ldots\ldots ⑥$$

Also multiplying recurrence relation (4) by x

$$(n+1)xP_n(x) = xP'_{n+1}(x) - x^2P'_n(x) \quad .... ⑦$$

Subtracting ⑥ from ⑦, we get

$$(1-x^2)\,P'_n(x) = (n+1)\,[xP_n(x) - P_{n+1}(x)]$$

3.2.2 Rodrigue's Formula

Rodrigue's formula is helpful in producing Legendre's polynomials of various orders and is given by $P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$

Proof: Let $y = (x^2-1)^n$

$$\therefore \frac{dy}{dx} = n(x^2-1)^{n-1}2x = 2nx\frac{(x^2-1)^n}{(x^2-1)}$$

$$\Rightarrow y_1(x^2-1) - 2nxy = 0\,,\; y_1 \equiv \frac{dy}{dx} \quad ....②$$

Differentiating ② $(n+1)$ times using Leibnitz's theorem:

$$\Rightarrow y_{n+2}(x^2-1) + (n+1)y_{n+1}(2x) + \frac{(n+1).n}{2!}y_n(2) - 2n[y_{n+1}(x) + (n+1)y_n(1)] = 0$$

$$\Rightarrow y_{n+2}(x^2-1) + 2xy_{n+1} - (n^2+n)y_n = 0$$

$$\Rightarrow (1-x^2)y_{n+2} - 2xy_{n+1} + n(n+1)y_n = 0 \quad ....③$$

Putting $y_n = V$, so that $y_{n+1} = \frac{dV}{dx}$ and $y_{n+2} = \frac{d^2V}{dx}$

$$③\Rightarrow (1-x^2)\frac{d^2V}{dx^2} - 2x\frac{dV}{dx} + n(n+1)V = 0$$

which is Legendre's equation with the solution $V = AP_n(x) + BQ_n(x)$

But since $V = y_n = \frac{d^n}{dx^n}(x^2-1)^n$ contains only positive powers of x, solution can only be a constant multiple of $P_n(x)$.

$$\therefore P_n(x) = CV = Cy_n$$

$$= C\frac{d^n}{dx^n}(x^2-1)^n \quad .... ④$$

$$= CD^n[(x-1)^n(x+1)^n] \quad , \frac{d^n}{dx^n} \equiv D^n$$

$$= CD^n[(x-1)^n(x+1)^n]$$

$$= C\left[D^n(x-1)^n(x+1)^n + n_{C_1}D^{n-1}(x-1)^n n(x+1)^{n-1} + \cdots + (x-1)^n D^n(x+1)^n\right]$$

$$= C[n!\,(x+1)^n + n.n(n-1)\ldots 3.2.(x-1)n(x+1)^{n-1} + \cdots + (x-1)^n n!]$$

Taking $x = 1$ on both sides

$$\Rightarrow 1 = Cn!\,2^n + 0 \quad \because P_n(1) = 1$$

$$\Rightarrow C = \frac{1}{2^n n!} \quad ....⑤$$

Using ⑤ in ④, we get

$$P_n(x) = \frac{1}{2^n n!}\frac{d^n}{dx^n}(x^2-1)^n$$

Putting $n = 0$, $P_0(x) = 1$

Putting $n = 1$, $P_1(x) = \frac{1}{2}\frac{d}{dx}(x^2-1)^1 = \frac{1}{2}2x = x$

Putting $n = 2$, $P_2(x) = \frac{1}{2^2 2!}\frac{d^2}{dx^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1)$

Putting $n = 3$, $P_3(x) = \frac{1}{2}(5x^3-3x)$

Putting $n = 4$, $P_4(x) = \frac{1}{8}(35x^4-30x^2+3)$

Putting $n = 5$, $P_5(x) = \frac{1}{8}(63x^5-70x^3+15x)$ etc…

Example2 Expand the following functions in series of Legendre's polynomials.

(i) $(1+2x-x^2)$

(ii) (x^3-5x^2+x+1)

Solution: $1 = P_0(x)$, $x = P_1(x)$,

$P_2(x) = \frac{1}{2}(3x^2-1) \Rightarrow x^2 = \frac{1}{3}(2P_2(x)+1) = \frac{1}{3}\left(2P_2(x)+P_0(x)\right)$

$P_3(x) = \frac{1}{2}(5x^3-3x) \Rightarrow x^3 = \frac{1}{5}(2P_3(x)+3x) = \frac{1}{5}\left(2P_3(x)+3P_1(x)\right)$

(i) Let $E = (1+2x-x^2)$

Substituting values of 1, x and x^2 in terms of Legendre's polynomials, we get

$E = \left(P_0(x)+2P_1(x)-\frac{1}{3}\left(2P_2(x)+P_0(x)\right)\right)$

$= \frac{1}{3}\left(3P_0(x)+6P_1(x)-2P_2(x)-P_0(x)\right)$

$= \frac{2}{3}\left(P_0(x)+3P_1(x)-P_2(x)\right)$

(ii) Let $F = (x^3-5x^2+x+1)$

Substituting values of 1, x, x^2 and x^3in terms of Legendre's polynomials, we get

$F = \left[\frac{1}{5}\left(2P_3(x)+3P_1(x)\right)-\frac{5}{3}\left(2P_2(x)+P_0(x)\right)+P_1(x)+P_0(x)\right]$

$= \frac{2}{5}P_3(x)-\frac{10}{3}P_2(x)+\frac{8}{5}P_1(x)-\frac{2}{3}P_0(x)$

Example3 Prove that

(i) $P_n'(1) = \frac{n(n+1)}{2}$

(ii) $P_n'(-1) = (-1)^{(n+1)}\frac{n(n+1)}{2}$

Solution: $P_n(x)$ is the solution of Legendre's equation given by:

$$(1-x^2)\frac{d^2y}{dx^2}-2x\frac{dy}{dx}+n(n+1)y = 0 \quad \text{.......①}$$

$\therefore y = P_n(x)$ will satisfy equation ①

$\Rightarrow (1-x^2)P_n''(x)-2x\,P_n'(x)+n(n+1)P_n(x) = 0$ ....②

Putting $x = 1$ in ② we get

$-2P_n'(1)+n(n+1)P_n(1) = 0$

$\Rightarrow P_n'(1) = \frac{n(n+1)}{2} \because P_n(1) = 1$

Putting $x = -1$ in ② we get

$2P_n'(-1)+n(n+1)P_n(-1) = 0$

$$\Rightarrow P_n'(-1) = -\frac{n(n+1)}{2} P_n(-1)$$

$$= (-1)^{(n+1)} \frac{n(n+1)}{2} \because P_n(-1) = (-1)^n$$

3.3 Bessel's Equation

The differential equation $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - n^2)y = 0$ ……①

is known as Bessel's equation of order n and its solutions are called Bessel's functions.

Note that $x = 0$ is a regular singular point of Bessel's equation.

Series solution of ① in terms of Bessel's functions $J_n(x)$ and $J_{-n}(x)$ is given by

$$y = AJ_n(x) + BJ_{-n}(x)$$

where $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(n+r+1)}\left(\frac{x}{2}\right)^{n+2r}$

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Proposition If n is any integer then $J_{-n}(x) = (-1)^n J_n(x)$

Proof: Case I: n is a positive integer

$$J_{-n}(x) = \sum_{r=0}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

If n is a positive integer, values of r from 0 to $(n-1)$ will give gamma function of $-ve$ integers in the denominator, which being infinite all such terms will vanish.

$$\therefore J_{-n}(x) = \sum_{r=n}^{\infty}(-1)^r \frac{1}{r!\,\Gamma(-n+r+1)}\left(\frac{x}{2}\right)^{-n+2r}$$

Putting $r = n + k$, we get

$$J_{-n}(x) = \sum_{k=0}^{\infty}(-1)^{n+k} \frac{1}{(n+k)!\,\Gamma(k+1)}\left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n \sum_{k=0}^{\infty}(-1)^k \frac{1}{k!\,\Gamma(n+k+1}\left(\frac{x}{2}\right)^{n+2k}$$

$$= (-1)^n J_n(x)$$

Case II: $n = 0$

$$J_{-0}(x) = (-1)^0 J_0(x)$$

or $J_0(x) = J_0(x)$, which is true

Case III: n is a negative integer

Let $= -p$, where p is a positive integer

From case I $J_p(x) = (-1)^{-p} J_{-p}(x) \Rightarrow J_{-n}(x) = (-1)^n J_n(x)$

3.3.1 Expansions of $J_0(x)$, $J_1(x)$, $J_{\frac{1}{2}}(x)$ and $J_{-\frac{1}{2}}(x)$

We have $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

1. $J_0(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{r!\,\Gamma(r+1)} = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{(r!)^2}$

$\because \Gamma(r + 1 = r!$ when r is a positive integer

$$\Rightarrow J_0(x) = 1 - \frac{1}{(1!)^2}\left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 + \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots$$

2. $J_1(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{1+2r} \frac{1}{r!\,\Gamma(r+2)} = \frac{x}{2}\sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{2r} \frac{1}{r!\,(r+1)!}$

$\because \Gamma(r+2 = (r+1)!$ when r is a positive integer

$$\Rightarrow J_1(x) = \frac{x}{2}\left[1 - \frac{1}{1!\,2!}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,3!}\left(\frac{x}{2}\right)^4 + \frac{1}{3!\,4!}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

3. $J_{\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{\frac{1}{2}+2r} \frac{1}{r!\,\left\lceil\left(r+\frac{3}{2}\right)\right.}$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\left\lceil\frac{3}{2}\right.} - \frac{1}{1!\left\lceil\frac{5}{2}\right.}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\left\lceil\frac{7}{2}\right.}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\left\lceil\frac{9}{2}\right.}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{x}{2}}\left[\frac{1}{\frac{1}{2}\left\lceil\frac{1}{2}\right.} - \frac{1}{1!\,\frac{3}{2}\frac{1}{2}\left\lceil\frac{1}{2}\right.}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}\left\lceil\frac{1}{2}\right.}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}\left\lceil\frac{1}{2}\right.}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(n+1 = n\Gamma n$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{1}{\frac{1}{2}} - \frac{1}{1!\,\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{7}{2}\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right] \quad \because \left\lceil\frac{1}{2}\right. = \sqrt{\pi}$$

$$= \sqrt{\frac{x}{2\pi}}\left[\frac{2}{1!} - \frac{2x^2}{3!} + \frac{2x^4}{5!} - \frac{2x^6}{7!} + \cdots\right]$$

$$= \sqrt{\frac{x}{2\pi}}\,\frac{2}{x}\left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\sin x$$

4. $J_{-\frac{1}{2}}(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{-\frac{1}{2}+2r} \frac{1}{r!\,\left\lceil\left(r+\frac{1}{2}\right)\right.}$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\left\lceil\frac{1}{2}\right.} - \frac{1}{1!\left\lceil\frac{3}{2}\right.}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\left\lceil\frac{5}{2}\right.}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\left\lceil\frac{7}{2}\right.}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$$= \sqrt{\frac{2}{x}}\left[\frac{1}{\left\lceil\frac{1}{2}\right.} - \frac{1}{1!\,\frac{1}{2}\left\lceil\frac{1}{2}\right.}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{3}{2}\frac{1}{2}\left\lceil\frac{1}{2}\right.}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}\left\lceil\frac{1}{2}\right.}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \Gamma(n+1 = n\Gamma n$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{1}{1!\,\frac{1}{2}}\left(\frac{x}{2}\right)^2 + \frac{1}{2!\,\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^4 - \frac{1}{3!\,\frac{5}{2}\frac{3}{2}\frac{1}{2}}\left(\frac{x}{2}\right)^6 + \cdots\right]$$

$\because \left\lceil\frac{1}{2}\right. = \sqrt{\pi}$

$$= \sqrt{\frac{2}{\pi x}}\left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right] = \sqrt{\frac{2}{\pi x}}\cos x$$

3.3.2 Recurrence Relations of Bessel's Function

(1) $\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$ **or** $\int x^n J_{n-1}(x)dx = x^n J_n(x)$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow x^n J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2n+2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow \frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty}(-1)^r \frac{2(n+r)x^{2n+2r-1}}{2^{n+2r}} \frac{1}{r!\,(n+r)\Gamma(n+r)}$$

$\because \Gamma(n+r+1) = (n+r)\Gamma(n+r)$

$$= x^n \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{(n-1)+2r} \frac{1}{r!\,\Gamma((n-1)+r+1)}$$

$$= x^n J_{n-1}(x)$$

(2) $\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$ **or** $\int x^{-n}J_{n+1}(x)\,dx = -\frac{d}{dx}[x^{-n}J_n(x)]$

Proof: $J_n(x) = \sum_{r=0}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{r!\,\Gamma(n+r+1)}$

$$\Rightarrow\ x^{-n}J_n(x) = \sum_{r=0}^{\infty}(-1)^r \frac{x^{2r}}{2^{n+2r}} \frac{1}{r!\,\Gamma(n+r+1)}$$

$$\Rightarrow\ \frac{d}{dx}[x^{-n}J_n(x)] = \sum_{r=1}^{\infty}(-1)^r \frac{2r\,x^{2r-1}}{2^{n+2r}} \frac{1}{(r-1)!\ r\,\Gamma(n+r+1)}$$

$$= x^{-n}\sum_{r=1}^{\infty}(-1)^r \left(\frac{x}{2}\right)^{n+2r-1} \frac{1}{(r-1)!\ \Gamma(n+r+1)}$$

$$= -x^{-n}\sum_{k=0}^{\infty}(-1)^k \left(\frac{x}{2}\right)^{(n+1)+2k} \frac{1}{k!\ \Gamma((n+1)+k+1)}$$

Putting $r = k+1$

$$= -x^{-n}J_{n+1}(x)$$

(3) $J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$

Proof: From recurrence relation (1)

$$\frac{d}{dx}[x^n J_n(x)] = x^n J_{n-1}(x)$$

$$\Rightarrow x^n J_n{}'(x) + n x^{n-1} J_n(x) = x^n J_{n-1}(x)$$

Dividing by x^n, we get

$$J_n'(x) + \frac{n}{x}J_n(x) = J_{n-1}(x)$$

$$\Rightarrow J_n{}'(x) = J_{n-1}(x) - \frac{n}{x}J_n(x)$$

(4) $J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$

Proof: From recurrence relation (2)

$$\frac{d}{dx}[x^{-n}J_n(x)] = -x^{-n}J_{n+1}(x)$$

$$\Rightarrow x^{-n}J_n{}'(x) - n x^{-n-1}J_n(x) = -x^{-n}J_{n+1}(x)$$

Dividing by x^{-n}, we get

$$J_n'(x) = -J_{n+1}(x) + \frac{n}{x}J_n(x)$$

(5) $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

Proof: Adding recurrence relations (3) and (4), we get

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$$

(6) $2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$

Proof: Subtracting recurrence relations (3) from (4), we get

$$2\frac{n}{x}J_n(x) = J_{n-1}(x) + J_{n+1}(x)$$

$$\Rightarrow\ 2nJ_n(x) = x[J_{n-1}(x) + J_{n+1}(x)]$$