Chapter 2

FOURIER TRANSFORMS

2.1 Introduction

The Fourier series expresses any periodic function into a sum of sinusoids. The Fourier transform is the extension of this idea to non-periodic functions by taking the limiting form of Fourier series when the fundamental period is made very large (infinite). Fourier transform finds its applications in astronomy, signal processing, linear time invariant (LTI) systems etc.

Some useful results in computation of the Fourier transforms:

1. $\int_0^\infty e^{-ax} \sin \lambda x \, dx = \frac{\lambda}{a^2+\lambda^2}$
2. $\int_0^\infty e^{-ax} \cos \lambda x \, dx = \frac{a}{a^2+\lambda^2}$
3. $\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}, \lambda > 0$

   When $\lambda = 1$, $\int_0^\infty \frac{sinx}{x} dx = \frac{\pi}{2}$
4. $\sin ax = \frac{e^{iax} - e^{-iax}}{2i}$
5. $\cos ax = \frac{e^{iax} + e^{-iax}}{2}$
6. $\int_0^\infty e^{-a^2x^2} dx = \frac{\sqrt{\pi}}{2a}$

   When $a = 1$, $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$
7. Heaviside Step Function or Unit step function $H(t)$or $U(t) = \begin{cases} 0, & \text{when } t < 0 \\ 1, & \text{when } t \geq 0 \end{cases}$

   At $t = 0$, $H(t)$ is sometimes taken as 0.5 or it may not have any specific value.

   Shifting at $t = a$

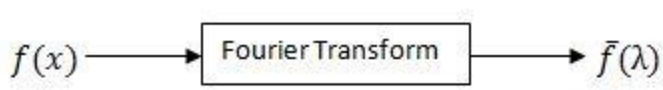
   $H(t-a)$ or $U(t-a) = \begin{cases} 0, & \text{when } t < a \\ 1, & \text{when } t \geq a \end{cases}$
8. Dirac Delta Function or Unit Impulse Function is defined as $\delta(t-a) = 0$, $t \neq a$ *such* that $\int_0^\infty \delta(t-a)dt = 1$, $a \geq 0$. It is zero everywhere except one point 'a'. Delta function in sometimes thought of having infinite value at $t = a$. The delta function can be viewed as the derivative of the Heaviside step function

Dirichlet's Conditions for Existence of Fourier Transform

Fourier transform can be applied to any function $f(x)$ if it satisfies the following conditions:

1. $f(x)$ is absolutely integrable i.e. $\int_{-\infty}^{\infty}|f(x)|dx$ is convergent.
2. The function $f(x)$ has a finite number of maxima and minima.
3. $f(x)$ has only a finite number of discontinuities in any finite

2.2 Fourier Transform, Inverse Fourier Transform and Fourier Integral



The Fourier transform of $f(x), -\infty < x < \infty$, denoted by $\bar{f}(\lambda)$ where $\lambda \in N$, is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx \quad \ldots①$$

Also inverse Fourier transform of $\bar{f}(\lambda)$ gives $f(x)$ as:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x}\, \bar{f}(\lambda)d\lambda \ldots ②$$

Rewriting ① as $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda t} f(t)dt$ and using in ②, Fourier integral representation of $f(x)$ is given by:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{i\lambda(t-x)}\, f(t)dt\, d\lambda$$

2.2.1 Fourier Sine Transform (F.S.T.)

Fourier Sine transform of $f(x), 0 < x < \infty$, denoted by $\bar{f_s}(\lambda)$, is given by

$$F_s\{f(x)\} \equiv \bar{f_s}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(x) \sin \lambda x\, dx \ldots③$$

Also inverse Fourier Sine transform of $\bar{f_s}(\lambda)$ gives $f(x)$ as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} \bar{f_s}(\lambda) \sin \lambda x\, d\lambda \ldots ④$$

Rewriting ③ as $\bar{f_s}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^{\infty} f(t) \sin \lambda t\, dt$ and using in ④, Fourier sine integral representation of $f(x)$ is given by:

$$f(x) = \frac{2}{\pi} \int_0^{\infty}\int_0^{\infty} f(t) \sin \lambda t \sin \lambda x\, dt d\lambda$$

2.2.2 Fourier Cosine Transform (F.C.T.)

Fourier Cosine transform of $f(x), 0 < x < \infty$, denoted by $\bar{f}_c(\lambda)$, is given by

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x \, dx \ldots ⑤$$

Also inverse Fourier Cosine transform of $\bar{f}_c(\lambda)$ gives $f(x)$ as:

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda x \, d\lambda \ldots ⑥$$

Rewriting ⑤ as $\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t \, dt$ and using in ⑥, Fourier cosine integral representation of $f(x)$ is given by:

$$f(x) = \frac{2}{\pi} \int_0^\infty \int_0^\infty f(t) \cos \lambda t \cos \lambda x \, dt d\lambda$$

Remark:

- Parameter λ may be taken as *p*, *s* or ω as per usual notations.
- Fourier transform of $f(x)$ may be given by $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\lambda x} f(x) dx$,

  then Inverse Fourier transform of $\bar{f}(\lambda)$is given by $f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} \bar{f}(\lambda) d\lambda$
- Sometimes Fourier transform of $f(x)$ is taken as $\bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x) dx$,

  thereby Inverse Fourier transform is given by $f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\lambda x} \bar{f}(\lambda) d\lambda$

  Similarly if Fourier Sine transform is taken as $\bar{f}_s(\lambda) = \int_0^\infty f(x) \sin \lambda x \, dx$,

  then Inverse Sine transform is given by $f(x) = \frac{2}{\pi} \int_0^\infty \bar{f}_s(\lambda) \sin \lambda x \, d\lambda$

  Similar is the case with Fourier Cosine transform.

Example 1 State giving reasons whether the Fourier transforms of the following functions exist: i. $\sin \frac{1}{x}$ ii. e^x iii. $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$

Solution: i. The graph of $\sin \frac{1}{x}$ oscillates infinite number of times at $x = n\pi, n \in Z$

$\therefore f(x) \sin \frac{1}{x}$ is having infinite number of maxima and minima in the interval $(-\infty, \infty)$. Hence Fourier transform of $f(x) = \sin \frac{1}{x}$ does not exist.

ii. For $f(x) = e^x$, $\int_{-\infty}^{\infty} |e^x| dx$ is not convergent. Hence Fourier transform of e^x does not exist.

iii. $f(x) = \begin{cases} 1, & \text{if } x \text{ is rational} \\ 0, & \text{if } x \text{ is irrational} \end{cases}$ is having infinite number of maxima and minima in the interval $(-\infty, \infty)$. Hence Fourier transform of $f(x)$ does not exist.

Example 2 Find Fourier Sine transform of

i. $\frac{1}{x}$ ii. $2e^{-3x} + 3e^{-2x}$

Solution: i. By definition, we have $F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{1}{x} \sin \lambda x \, dx = \sqrt{\frac{2}{\pi}} . \frac{\pi}{2} = \sqrt{\frac{\pi}{2}}$$

ii. By definition, $F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x \, dx$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty (2e^{-3x} + 3e^{-2x}) \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \int_0^\infty 2e^{-3x} \sin \lambda x \, dx + \sqrt{\frac{2}{\pi}} \int_0^\infty 3e^{-2x} \sin \lambda x \, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{2e^{-3x}}{9+\lambda^2} (-3 \sin \lambda x - \lambda \cos \lambda x \right]_0^\infty + \sqrt{\frac{2}{\pi}} \left[\frac{3e^{-2x}}{4+\lambda^2} (-2 \sin \lambda x - \lambda \cos \lambda x \right]_0^\infty$$

$$= \sqrt{\frac{2}{\pi}} \left[0 + \frac{2\lambda}{9+\lambda^2} \right] + \sqrt{\frac{2}{\pi}} \left[0 + \frac{3\lambda}{4+\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\lambda}{9+\lambda^2} + \frac{3\lambda}{4+\lambda^2} \right] = \sqrt{\frac{2}{\pi}} \left[\frac{5\lambda^3+35\lambda}{(9+\lambda^2)(4+\lambda^2)} \right]$$

Example 3 Find Fourier transform of Delta function $\delta(x-a)$

Solution: $F\{\delta(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\lambda x} . \delta(x-a) \, dx$

$$= \frac{1}{\sqrt{2\pi}} e^{i\lambda a}$$

$\because \int_{-\infty}^{\infty} f(t) \; \delta(t-a) dt = f(a)$ by virtue of fundamental property of Delta function where $f(t)$ is any differentiable function.

Example 4 Show that Fourier sine and cosine transforms of x^{n-1} are $\frac{\lceil n}{\lambda^n} \sin \frac{n\pi}{2}$ and $\frac{\lceil n}{\lambda^n} \cos \frac{n\pi}{2}$ respectively.

Solution: By definition**,** $\int_0^\infty e^{-t} \, t^{n-1} dt = \lceil n$

Putting $t = i\lambda x$ so that $dt = i\lambda dx$

$$\Rightarrow \int_0^\infty e^{-i\lambda x} \, (i\lambda x)^{n-1} i\lambda \, dx = \lceil n$$

$$\Rightarrow \int_0^\infty x^{n-1} e^{-i\lambda x}\, dx = \frac{\lceil n\ i^{-n}}{\lambda^n}$$

$$\Rightarrow \int_0^\infty x^{n-1} (\cos \lambda x - i \sin \lambda x)\, dx = \frac{\lceil n}{\lambda^n}\left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right)$$

$$\because i^{-n} = \left(\cos\frac{n\pi}{2} - i \sin\frac{n\pi}{2}\right)$$

$$\Rightarrow \int_0^\infty x^{n-1} \cos \lambda x\, dx - i \int_0^\infty x^{n-1} \sin \lambda x\, dx = \frac{\lceil n}{\lambda^n}\cos\frac{n\pi}{2} - i\ \frac{\lceil n}{\lambda^n}\sin\frac{n\pi}{2}$$

Equating real and imaginary parts, we get

$$\int_0^\infty x^{n-1} \cos \lambda x\, dx = \frac{\lceil n}{\lambda^n}\cos\frac{n\pi}{2} \text{ and } \int_0^\infty x^{n-1} \sin \lambda x\, dx = \frac{\lceil n}{\lambda^n}\sin\frac{n\pi}{2}$$

$$\Rightarrow \bar{f_c}(\lambda) = \frac{\lceil n}{\lambda^n}\cos\frac{n\pi}{2} \text{ and } \bar{f_s}(\lambda) = \frac{\lceil n}{\lambda^n}\sin\frac{n\pi}{2}$$

Example 5 Find Fourier Cosine transform of $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2 - x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

Solution: By definition, we have $F_c\{f(x)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x\, dx$

$$\therefore \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos \lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}} \left[\int_0^1 x \cos \lambda x\, dx + \int_1^2 (2 - x) \cos \lambda x\, dx + \int_2^\infty 0. \cos \lambda x\, dx\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\left[(x)\left(\frac{\sin \lambda x}{\lambda}\right) - (1)\left(-\frac{\cos \lambda x}{\lambda^2}\right)\right]_0^1 + \left[(2 - x)\left(\frac{\sin \lambda x}{\lambda}\right) - (-1)\left(-\frac{\cos \lambda x}{\lambda^2}\right)\right]_1^2\right]$$

$$= \sqrt{\frac{2}{\pi}} \left[\frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2} - \frac{1}{\lambda^2} - \frac{\cos 2\lambda}{\lambda^2} - \frac{\sin \lambda}{\lambda} + \frac{\cos \lambda}{\lambda^2}\right] = \sqrt{\frac{2}{\pi}} \left[\frac{2\cos \lambda - \cos 2\lambda - 1}{\lambda^2}\right]$$

Example 6 Find Fourier Sine and Cosine transform of $f(x) = e^{-x}$ and hence show that

$$\int_0^\infty \frac{\cos mx}{1+x^2}\, dx = \frac{\pi}{2} e^{-m} = \int_0^\infty \frac{x \sin mx}{1+x^2}\, dx$$

Solution: To find Fourier Sine transform

$$F_s\{f(x)\} \equiv \bar{f_s}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin \lambda x\, dx$$

$$\Rightarrow \bar{f_s}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-x} \sin \lambda x\, dx = \sqrt{\frac{2}{\pi}} \left(\frac{\lambda}{1+\lambda^2}\right) \ldots\ldots ①$$

Taking inverse Fourier Sine transform of ①

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f_s}(\lambda) \sin \lambda x\, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{\lambda}{1+\lambda^2} sin\lambda x d\lambda \ldots ②$$

Substituting $f(x) = e^{-x}$ in ②

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^{\infty} \frac{\lambda sin\lambda x}{1+\lambda^2} d\lambda$$

Replacing x by m on both sides

$$\Rightarrow e^{-m} = \frac{2}{\pi}\int_0^{\infty} \frac{\lambda sin\lambda m}{1+\lambda^2} d\lambda$$

Now by property of definite integrals $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx \ \ldots ③$$

Similarly taking Fourier Cosine transform of $f(x) = e^{-x}$

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x) \cos \lambda x \, dx$$

$$\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-x} \cos \lambda x \, dx = \sqrt{\frac{2}{\pi}}\left(\frac{1}{1+\lambda^2}\right) \ldots ④$$

Taking inverse Fourier Cosine transform of ④

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} \bar{f}_c(\lambda) \cos \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^{\infty} \frac{1}{1+\lambda^2} \cos \lambda x \, d\lambda \ldots ⑤$$

Substituting $f(x) = e^{-x}$ in ⑤

$$\Rightarrow e^{-x} = \frac{2}{\pi}\int_0^{\infty} \frac{\cos \lambda x}{1+\lambda^2} d\lambda$$

Replacing x by m on both sides

$$\Rightarrow e^{-m} = \frac{2}{\pi}\int_0^{\infty} \frac{\cos \lambda m}{1+\lambda^2} d\lambda$$

Again by property of definite integrals $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{\cos mx}{1+x^2} dx \ \ldots ⑥$$

From ③and ⑥, we get

$$\int_0^{\infty} \frac{\cos mx}{1+x^2} dx = \frac{\pi}{2} e^{-m} = \int_0^{\infty} \frac{x \sin mx}{1+x^2} dx$$

Example 7 Find Fourier transform of $f(x) = \begin{cases} 1 - x^2, & |x| < 1 \\ 0, & |x| > 1 \end{cases}$

and hence evaluate $\int_0^\infty \left(\frac{x\cos x-\sin x}{x^3}\right)\cos\frac{x}{2}\,dx$

Solution: Fourier transform of $f(x)$ is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-x^2)e^{i\lambda x}dx$$

$$= \frac{1}{\sqrt{2\pi}}\left[(1-x^2)\left(\frac{e^{i\lambda x}}{i\lambda}\right) - (-2x)\left(\frac{e^{i\lambda x}}{i^2\lambda^2}\right) + (-2)\left(\frac{e^{i\lambda x}}{i^3\lambda^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\left[\frac{2e^{i\lambda}}{i^2\lambda^2} - \frac{2e^{i\lambda}}{i^3\lambda^3} + \frac{2e^{-i\lambda}}{i^2\lambda^2} + \frac{2e^{-i\lambda}}{i^3\lambda^3}\right]$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{e^{i\lambda}+e^{-i\lambda}}{\lambda^2} + \frac{e^{i\lambda}-e^{-i\lambda}}{i\lambda^3}\right] \qquad \because i^2 = -1 \text{ and } i^3 = -i$$

$$= \sqrt{\frac{2}{\pi}}\left[-\frac{2\cos\lambda}{\lambda^2} + \frac{2\sin\lambda}{\lambda^3}\right]$$

$$\therefore \bar{f}(\lambda) = \frac{2\sqrt{2}}{\sqrt{\pi}}\left(\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right) \;\ldots..①$$

Taking inverse Fourier transform of ①

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}\,\bar{f}(\lambda)d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty} e^{-i\lambda x}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}(\cos\lambda x - i\sin\lambda x)\left(\frac{\lambda\cos\lambda-\sin\lambda}{\lambda^3}\right)d\lambda \;\; \because e^{-i\lambda x} = \cos\lambda x - i\sin\lambda x$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\cos\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right) - i\sin\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)\right]d\lambda \ldots.②$$

Substituting $f(x) = \begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases}$ in ②

$$\Rightarrow \begin{cases}1-x^2, & |x|<1\\ 0, & |x|>1\end{cases} = \frac{2}{\pi}\int_{-\infty}^{\infty}\left[\cos\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right) - i\sin\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)\right]d\lambda$$

Equating real parts on both sides, we get

$$\int_{-\infty}^{\infty}\cos\lambda x\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda = \begin{cases}\frac{\pi}{2}(1-x^2), & |x|<1\\ 0, & |x|>1\end{cases}$$

Putting $x = \frac{1}{2}$ on both sides

$$\int_{-\infty}^{\infty}\cos\frac{\lambda}{2}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda = \frac{\pi}{2}\left(1-\frac{1}{4}\right)$$

$$\Rightarrow 2\int_0^{\infty}\cos\frac{\lambda}{2}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right)d\lambda = \frac{3\pi}{8} \qquad \because \cos\frac{\lambda}{2}\left(\frac{\sin\lambda-\lambda\cos\lambda}{\lambda^3}\right) \text{ is an even function of } \lambda$$

Now by property of definite integrals $\int_a^b f(x)dx = \int_a^b f(y)dy$

$$\therefore \int_0^\infty \left(\frac{x\cos x - \sin x}{x^3}\right)\cos\frac{x}{2}\,dx = -\frac{3\pi}{16}$$

Example 8 Find the Fourier cosine transform of $f(x) = \frac{1}{1+x^2}$

Solution: To find Fourier cosine transform

$$F_c\{f(x)\} \equiv \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\,dx$$

$$\Rightarrow \bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx \quad \ldots\ldots ①$$

To evaluate the integral given by ①

Let $g(x) = e^{-x}$ …… ②

$$F_c\{g(x)\} \equiv \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty g(x)\cos\lambda x\,dx$$

$$\Rightarrow \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\cos\lambda x\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\frac{e^{-x}}{1+\lambda^2}(-\cos\lambda x + \lambda\sin\lambda x)\right]_0^\infty$$

$$\Rightarrow \bar{g}_c(\lambda) = \sqrt{\frac{2}{\pi}}\frac{1}{1+\lambda^2}$$

Again taking Inverse Fourier cosine transform

$$g(x) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+\lambda^2}\cos\lambda x\,d\lambda$$

$$\Rightarrow g(\lambda) = \frac{2}{\pi}\int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx$$

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx = \frac{\pi}{2}\,g(\lambda) \quad \ldots\ldots ③$$

Using ② in ③, we get

$$\Rightarrow \int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx = \frac{\pi}{2}e^{-\lambda} \quad \ldots\ldots ④$$

Using ④ in ①, we get

$$\bar{f}_c(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{1+x^2}\cos\lambda x\,dx = \sqrt{\frac{2}{\pi}}.\frac{\pi}{2}\,e^{-\lambda} = \sqrt{\frac{\pi}{2}}\,e^{-\lambda}$$

Example 9 Find the Fourier sine transform of $f(x)=\frac{e^{-ax}}{x}$ and use it to evaluate $\int_0^{\infty} tan^{-1}\left(\frac{x}{a}\right) sinxdx$

Solution: To find Fourier sine transform

$$F_s\{f(x)\} \equiv \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin\lambda x\, dx$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}\sin\lambda x\, dx$$

To evaluate the integral, differentiating both sides with respect to λ

$$\frac{d}{d\lambda}\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}\frac{e^{-ax}}{x}(\cos\lambda x)xdx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^{\infty} e^{-ax}cos\lambda xdx = \sqrt{\frac{2}{\pi}}\frac{a}{a^2+\lambda^2}$$

Now integrating both sides with respect to λ

$$\bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}\int\frac{a}{a^2+\lambda^2}d\lambda$$

$$\Rightarrow \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{\lambda}{a}\right)+c$$

when $\lambda=0$, $\bar{f}_s(\lambda)=0$, $\Rightarrow$ c $=0$

$$\therefore \bar{f}_s(\lambda) = \sqrt{\frac{2}{\pi}}tan^{-1}\left(\frac{\lambda}{a}\right)$$

Again taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty} tan^{-1}\left(\frac{\lambda}{a}\right)\sin\lambda x\, d\lambda$$

Substituting $f(x)=\frac{e^{-ax}}{x}$ on both sides

$$\frac{e^{-ax}}{x} = \frac{2}{\pi}\int_0^{\infty} tan^{-1}\left(\frac{\lambda}{a}\right)\sin\lambda x\, d\lambda$$

Putting x =1 on both sides

$$\frac{\pi}{2}e^{-a} = \int_0^{\infty} tan^{-1}\frac{\lambda}{a}\sin\lambda\, d\lambda$$

$$\Rightarrow \int_0^{\infty} tan^{-1}\frac{\lambda}{a}\sin x\, dx = \frac{\pi}{2}e^{-a}$$

Example 10 If $t>0$ Show that i. $\int_0^{\infty}\frac{\cos\lambda t}{\lambda^2+a^2}d\lambda = \frac{\pi}{2a}e^{-at}, a>0$

ii. $\int_0^\infty \frac{\lambda \sin \lambda t}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2} e^{at}, a \leq 0$

Solution: i. Let $f(t) = \frac{\pi}{2a} e^{-at}, a > 0,\ t > 0$

Taking Fourier cosine transform of $f(t)$, we get

$$F_c\{f(t)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \cos \lambda t \, dt$$

$$= \frac{\pi}{2a} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-at} \cos \lambda t \, dt$$

$$= \frac{1}{a} \sqrt{\frac{\pi}{2}} \frac{a}{a^2 + \lambda^2}$$

Also inverse Fourier cosine transform of $\bar{f_c}(\lambda)$ gives $f(t)$ as:

$$f(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f_c}(\lambda) \cos \lambda t \, d\lambda$$

$$= \frac{1}{a} \sqrt{\frac{\pi}{2}} \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{a}{a^2 + \lambda^2} \cos \lambda t \, d\lambda$$

$$\Rightarrow f(t) = \int_0^\infty \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda$$

$$\therefore \int_0^\infty \frac{\cos \lambda t}{\lambda^2 + a^2} d\lambda = \frac{\pi}{2a} e^{-at}, a > 0$$

ii. Again let $g(t) = \frac{\pi}{2} e^{at}, a \leq 0,\ t > 0$

Taking Fourier sine transform of $g(t)$, we get

$$F_s\{g(t)\} \equiv \bar{g}_s(\lambda) = \sqrt{\frac{2}{\pi}} \int_0^\infty g(t) \sin \lambda t \, dt$$

$$= \frac{\pi}{2} \sqrt{\frac{2}{\pi}} \int_0^\infty e^{at} \sin \lambda t \, dt, a \leq 0$$

$$= \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-at} \sin \lambda t \, dt, a > 0$$

$$= \sqrt{\frac{\pi}{2}} \frac{\lambda}{a^2 + \lambda^2}$$

Also inverse Fourier sine transform of $\bar{g}_s(\lambda)$ gives $g(t)$ as:

$$g(t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{g}_s(\lambda) \sin \lambda t \, d\lambda$$

$$= \sqrt{\frac{\pi}{2}}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{\lambda}{a^2+\lambda^2}\sin\lambda t\, d\lambda$$

$$\Rightarrow g(t) = \int_0^\infty \frac{\lambda \sin\lambda t}{\lambda^2+a^2}d\lambda$$

$$\therefore \int_0^\infty \frac{\lambda \sin\lambda t}{\lambda^2+a^2}d\lambda = \frac{\pi}{2}e^{at}, a \le 0$$

Example 11 Prove that Fourier transform of $e^{\frac{-x^2}{2}}$ is self reciprocal.

Solution: Fourier transform of $f(x)$ is given by

$$F\{f(x)\} \equiv \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx$$

$$\therefore F\left\{e^{\frac{-x^2}{2}}\right\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}} e^{i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-x^2}{2}+i\lambda x}\, dx = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2\,i\lambda x)}$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x^2-2i\lambda x+(i\lambda)^2-(i\lambda)^2)}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-i\lambda)^2+\frac{i^2\lambda^2}{2}}\, dx$$

$$= \frac{e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-1}{2}(x-i\lambda)^2}\, dx$$

$$= \frac{e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\frac{-z^2}{2}}\, dz \qquad \text{By putting } z = (x-i\lambda)$$

$$= \frac{2e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_0^{\infty} e^{\frac{-z^2}{2}}\, dz \qquad e^{\frac{-z^2}{2}} \text{ being even function of } z$$

$$= \frac{2e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_0^{\infty} e^{-\left(\frac{z}{\sqrt{2}}\right)^2}\, dz$$

Put $\frac{z}{\sqrt{2}} = t \;\Rightarrow dz = \sqrt{2}\; dt$

$$\therefore \bar{f}(\lambda) = \frac{2\sqrt{2}e^{\frac{-\lambda^2}{2}}}{\sqrt{2\pi}}\int_0^\infty e^{-t^2}\, dt$$

$$= \frac{2e^{\frac{-\lambda^2}{2}}}{\sqrt{\pi}}.\frac{\sqrt{\pi}}{2} = e^{\frac{-\lambda^2}{2}} \qquad \because \int_0^\infty e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}$$

$$\therefore F\left\{e^{\frac{-x^2}{2}}\right\} = e^{\frac{-\lambda^2}{2}}$$

Hence we see that Fourier transform of $e^{\frac{-x^2}{2}}$ is given by $e^{\frac{-\lambda^2}{2}}$. Variable x is transformed to λ. ∴ We can say that Fourier transform of $e^{\frac{-x^2}{2}}$ is self reciprocal.

Example 12 Find Fourier Cosine transform of e^{-x^2}.

Solution: By definition, $F_c\{f(x)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos\lambda x\, dx$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\left(\frac{e^{i\lambda x}+e^{-i\lambda x}}{2}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{-x^2}e^{i\lambda x} + e^{-x^2}e^{-i\lambda x}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty (e^{-x^2+i\lambda x} + e^{-x^2-i\lambda x})\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{-\left(x^2-2\left(\frac{i\lambda}{2}\right)x+\left(\frac{i\lambda}{2}\right)^2-\left(\frac{i\lambda}{2}\right)^2\right)} + e^{-\left(x^2+2\left(\frac{i\lambda}{2}\right)x+\left(\frac{i\lambda}{2}\right)^2-\left(\frac{i\lambda}{2}\right)^2\right)}\right)dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_0^\infty \left(e^{-\left(x-\frac{i\lambda}{2}\right)^2+\frac{i^2\lambda^2}{4}} + e^{-\left(x+\frac{i\lambda}{2}\right)^2+\frac{i^2\lambda^2}{4}}\right)dx$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}}\left[\int_0^\infty e^{-\left(x-\frac{i\lambda}{2}\right)^2}dx + \int_0^\infty e^{-\left(x+\frac{i\lambda}{2}\right)^2}dx\right]$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}}\left[\frac{\sqrt{\pi}}{2}+\frac{\sqrt{\pi}}{2}\right] = \frac{\sqrt{\pi}e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}}$$

$$\Rightarrow \bar{f_c}(\lambda) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\lambda^2}{4}}$$

Or

Fourier Cosine transform of e^{-x^2} can also be found using the method given below:

$$\bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos\lambda x\, dx \quad \ldots. ①$$

Differentiating both sides with respect to λ

$$\Rightarrow \frac{d}{d\lambda}\bar{f_c}(\lambda) = -\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-x^2}\sin\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[\sin \lambda x.\frac{e^{-x^2}}{2}\right]_0^\infty - \sqrt{\frac{2}{\pi}}\int_0^\infty \lambda\cos \lambda x.\frac{e^{-x^2}}{2}\,dx$$

$$= 0 - \frac{\lambda}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos \lambda x\,dx \quad ....②$$

$$\Rightarrow \frac{d}{d\lambda}\bar{f}_c(\lambda) = -\frac{\lambda}{2}\bar{f}_c(\lambda) \qquad \text{using ① in ②}$$

$$\Rightarrow \frac{\frac{d}{d\lambda}\bar{f}_c(\lambda)}{\bar{f}_c(\lambda)} = -\frac{\lambda}{2}$$

Integrating both sides with respect to λ

$\Rightarrow \log \bar{f}_c(\lambda) = -\frac{\lambda^2}{4} + \log k$, where $\log k$ is the constant of integration

$$\Rightarrow \bar{f}_c(\lambda) = K\mathrm{e}^{-\frac{\lambda^2}{4}} \quad .....③$$

$$\Rightarrow \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}\cos \lambda x\,dx = k\mathrm{e}^{-\frac{\lambda^2}{4}}$$

Putting $\lambda = 0$ on both sides

$$\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2}dx = k$$

$$\Rightarrow k = \sqrt{\frac{2}{\pi}}.\frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} \quad ..... ④$$

Using ④ in ③, we get

$$\bar{f}_c(\lambda) = \frac{1}{\sqrt{2}}\mathrm{e}^{-\frac{\lambda^2}{4}}$$

Example 13 Find Fourier transform of $xe^{-ax^2}, a > 0$

Solution: By definition, $F\{xe^{-ax^2}\} = \bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-ax^2}e^{i\lambda x}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-ax^2+i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-a\left(x^2-2\left(\frac{i\lambda}{2a}\right)x+\left(\frac{i\lambda}{2a}\right)^2-\left(\frac{i\lambda}{2a}\right)^2\right)}\,dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} xe^{-a\left(x-\frac{i\lambda}{2a}\right)^2+\frac{i^2\lambda^2}{4a}}\,dx$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}}\left[\int_{-\infty}^{\infty}\left(x-\frac{i\lambda}{2a}\right)e^{-a\left(x-\frac{i\lambda}{2a}\right)^2}\,dx + \frac{i\lambda}{2a}\int_{-\infty}^{\infty} e^{-a\left(x-\frac{i\lambda}{2a}\right)^2}\,dx\right]$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} \left[\int_{-\infty}^{\infty} te^{-at^2}\ dt + \frac{i\lambda}{2a}\int_{-\infty}^{\infty} e^{-at^2}\ dt\right], \text{Putting } \left(x - \frac{i\lambda}{2a}\right) = t$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} \left[0 + \frac{i\lambda}{a}\int_{0}^{\infty} e^{-at^2}\ dt\right]$$

$\because te^{-at^2}$ is odd function and e^{-at^2} is even function in t

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} . \frac{i\lambda}{a}\int_{0}^{\infty} e^{-(\sqrt{a}\,t)^2}\ dt$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} . \frac{i\lambda}{a\sqrt{a}}\int_{0}^{\infty} e^{-z^2}\ dz \text{ , Putting } \sqrt{a}\,t = z$$

$$= \frac{e^{\frac{-\lambda^2}{4a}}}{\sqrt{2\pi}} . \frac{i\lambda}{a\sqrt{a}} . \frac{\sqrt{\pi}}{2} \qquad \because \int_{0}^{\infty} e^{-z^2}\ dz = \frac{\sqrt{\pi}}{2}$$

$$\Rightarrow \overline{f}(\lambda) = \frac{i\lambda\, e^{\frac{-\lambda^2}{4a}}}{2a\sqrt{2a}}$$

Example 14 Find Fourier cosine integral representation of $f(x) = \begin{cases} x^2, 0 < x < a \\ 0 \quad , \quad x > a \end{cases}$

Solution: Taking Fourier Cosine transform of $f(x) = \begin{cases} x^2, 0 < x < a \\ 0 \quad , \quad x > a \end{cases}$

$$F_c\{f(x)\} \equiv \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos\lambda x\, dx$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^{a} x^2\cos\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\left[(x^2)\left(\frac{sin\lambda x}{\lambda}\right) - (2x)\left(\frac{-cos\lambda x}{\lambda^2}\right) + (2)\left(\frac{-sin\lambda x}{\lambda^3}\right)\right]_0^a$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\left[\left(\frac{a^2}{\lambda} - \frac{2}{\lambda^3}\right) sin\lambda a + \frac{2a}{\lambda^2} cos\lambda a\right]$$

Now taking Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi}\int_0^{\infty}\left[\left(\frac{a^2}{\lambda} - \frac{2}{\lambda^3}\right) sin\lambda a + \frac{2a}{\lambda^2} cos\lambda a\right]\cos\lambda x\, d\lambda$$

This is the required Fourier cosine integral representation of $f(x) = \begin{cases} x^2, 0 < x < a \\ 0 \quad , \quad x > a \end{cases}$

Example 15 If $f(x) = \begin{cases} \sin x\,, \ 0 < x < \pi \\ 0, \qquad otherwise \end{cases}$, prove that

$f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda$. Hence evaluate $\int_0^\infty \frac{\cos\frac{\pi t}{2}}{1-t^2}\,dt$

Solution: Given $f(x) = \begin{cases}\sin x\,, & 0 < x < \pi \\ 0, & otherwise\end{cases}$

To find Fourier cosine integral representation of $f(x)$, taking Fourier Cosine transform of $f(x)$

$$\bar{f_c}(\lambda) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x) cos\lambda x dx = \sqrt{\frac{2}{\pi}}\int_0^\pi \sin x\, cos\lambda x dx$$

$$= \sqrt{\frac{1}{2\pi}}\int_0^\pi (\sin(\lambda+1)x - \sin(\lambda-1)x)\,dx$$

$$= \sqrt{\frac{1}{2\pi}}\left[-\frac{\cos(\lambda+1)x}{(\lambda+1)} + \frac{\cos(\lambda-1)x}{(\lambda-1)}\right]_0^\pi$$

$$= \sqrt{\frac{1}{2\pi}}\left[-\frac{\cos(\lambda+1)\pi}{(\lambda+1)} + \frac{\cos(\lambda-1)\pi}{(\lambda-1)} + \frac{1}{(\lambda+1)} - \frac{1}{(\lambda-1)}\right]$$

$$= \sqrt{\frac{1}{2\pi}}\left[\frac{\cos\lambda\pi}{(\lambda+1)} - \frac{\cos\lambda\pi}{(\lambda-1)} + \frac{1}{(\lambda+1)} - \frac{1}{(\lambda-1)}\right]$$

$$= \sqrt{\frac{1}{2\pi}}\left[\frac{(\lambda-1)\cos\lambda\pi - (\lambda+1)\cos\lambda\pi}{(\lambda+1)(\lambda-1)} + \frac{\lambda-1-\lambda-1}{(\lambda+1)(\lambda-1)}\right]$$

$$\Rightarrow \bar{f_c}(\lambda) = \sqrt{\frac{1}{2\pi}}\left[\frac{-2\cos\lambda\pi - 2}{\lambda^2-1}\right] = \sqrt{\frac{2}{\pi}}\left[\frac{1+\cos\lambda\pi}{1-\lambda^2}\right]$$

Taking Inverse Fourier Cosine transform, $f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f_c}(\lambda)\cos\lambda x\,d\lambda$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \left[\frac{1+\cos\lambda\pi}{1-\lambda^2}\right]\cos\lambda x\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + 2\cos\lambda\pi\cos\lambda x}{1-\lambda^2}\,d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda$$

$$\Rightarrow f(x) = \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda$$

Putting $f(x) = \begin{cases}\sin x\,, & 0 < x < \pi \\ 0, & otherwise\end{cases}$

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \frac{2\cos\lambda x + \cos(\pi+x)\lambda + \cos(\pi-x)\lambda}{1-\lambda^2}\,d\lambda = \begin{cases}\sin x\,, & 0 < x < \pi \\ 0, & otherwise\end{cases}$$

Putting $x = \frac{\pi}{2}$ on both sides

$$\Rightarrow \frac{1}{\pi}\int_0^\infty \frac{2\cos\frac{\pi\lambda}{2} + \cos\left(\pi+\frac{\pi}{2}\right)\lambda + \cos\left(\pi-\frac{\pi}{2}\right)\lambda}{1-\lambda^2} d\lambda = 1$$

$$\Rightarrow \int_0^\infty \frac{\cos\frac{\pi\lambda}{2}}{1-\lambda^2} d\lambda = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\cos\frac{\pi t}{2}}{1-t^2} dt = \frac{\pi}{2}$$

Example 16 Solve the integral equation $\int_0^\infty f(x) cos\lambda x dx = \begin{cases} 1-\lambda, 0 \le \lambda \le 1 \\ 0, \quad \lambda > 1 \end{cases}$

Hence deduce that $\int_0^\infty \frac{sin^2 t}{t^2} dt = \frac{\pi}{2}$

Solution: Given that $\int_0^\infty f(x) cos\lambda x dx = \begin{cases} 1-\lambda, 0 \le \lambda \le 1 \\ 0, \quad \lambda > 1 \end{cases}$ ……①

$$\Rightarrow \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) cos\lambda x dx = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), 0 \le \lambda \le 1 \\ 0, \ \lambda > 1. \end{cases}$$

$$\Rightarrow \bar{f}_c(\lambda) = \begin{cases} \sqrt{\frac{2}{\pi}}(1-\lambda), 0 \le \lambda \le 1 \\ 0, \ \lambda > 1. \end{cases}$$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \bar{f}_c(\lambda) \cos \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi} \int_0^1 (1-\lambda) \cos \lambda x \ d\lambda$$

$$= \frac{2}{\pi}\left[(1-\lambda)\left(\frac{sin\lambda x}{x}\right) - (-1)\left(\frac{-cos\lambda x}{x^2}\right)\right]_0^1$$

$$= \frac{2}{\pi}\left[-\frac{cosx}{x^2} + \frac{1}{x^2}\right] = \frac{2}{\pi}\left[\frac{1-cosx}{x^2}\right] = \frac{2}{\pi}\frac{2sin^2\frac{x}{2}}{x^2}$$

$$\Rightarrow f(x) = \frac{4sin^2\frac{x}{2}}{\pi x^2} \ ....②$$

Using ②in ①, we get

$$\int_0^\infty \frac{4sin^2\frac{x}{2}}{\pi x^2} cos\lambda x dx = \begin{cases} 1-\lambda, 0 \le \lambda \le 1 \\ 0, \quad \lambda > 1 \end{cases}$$

Putting $\lambda = 0$ on both sides

$$\Rightarrow \frac{4}{\pi} \int_0^\infty \frac{sin^2\frac{x}{2}}{x^2} dx = 1$$

$$\Rightarrow \int_0^\infty \frac{sin^2\frac{x}{2}}{x^2} dx = \frac{\pi}{4}$$

Putting $\frac{x}{2} = t$, $dx = 2dt$

$$\therefore \int_0^\infty \frac{sin^2 t}{4t^2} 2dt = \frac{\pi}{4} \Rightarrow \int_0^\infty \frac{sin^2 t}{t^2} dt = \frac{\pi}{2}$$

Example 17 Find the function $f(x)$ if its Cosine transform is given by:

(i) $\frac{\sin a\lambda}{\lambda}$ (ii) $\begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\lambda}{2}\right), & \lambda < 2a \\ 0 \quad , & \lambda \geq 2a \end{cases}$

Solution: (i) Given that $\bar{f}_c(\lambda) = \frac{\sin a\lambda}{\lambda}$

Taking Inverse Fourier Cosine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f}_c(\lambda) \cos \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{\sin a\lambda}{\lambda} \cos \lambda x \, d\lambda$$

$$= \frac{1}{2}.\frac{2}{\pi}\int_0^\infty \frac{2\sin a\lambda \cos \lambda x}{\lambda} d\lambda$$

$$= \frac{1}{\pi}\int_0^\infty \frac{\sin(a+x)\lambda}{\lambda} d\lambda + \frac{1}{\pi}\int_0^\infty \frac{\sin\,(a-x)\,\lambda}{\lambda} d\lambda$$

Now $0 < x < \infty \quad \therefore \; a + x > 0$

$$\Rightarrow f(x) = \begin{cases} \frac{1}{\pi}\left[\frac{\pi}{2} + \frac{\pi}{2}\right], & a - x > 0 \;\; i.e. x < a \\ \frac{1}{\pi}\left[\frac{\pi}{2} - \frac{\pi}{2}\right], & a - x < 0 \;\; i.e. x > a \end{cases} \qquad \because \int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}, \lambda > 0$$

$$\Rightarrow f(x) = \begin{cases} 1 & , x < a \\ 0, & x > a \end{cases}$$

(ii) Given that $\bar{f}_c(\lambda) = \begin{cases} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\lambda}{2}\right), & \lambda < 2a \\ 0 \quad , & \lambda \geq 2a \end{cases}$

Taking Inverse Fourier Cosine transform

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{f}_c(\lambda) \cos \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \sqrt{\frac{2}{\pi}}\int_0^{2a} \frac{1}{\sqrt{2\pi}}\left(a - \frac{\lambda}{2}\right) \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi}\int_0^{2a} \left(a - \frac{\lambda}{2}\right) \cos \lambda x \, d\lambda$$

$$= \frac{1}{\pi}\left[\left(a - \frac{\lambda}{2}\right)\left(\frac{\sin \lambda x}{x}\right) - \left(-\frac{1}{2}\right)\left(-\frac{\cos \lambda x}{x^2}\right)\right]_0^{2a}$$

$$= \frac{1}{\pi}\left[-\frac{\cos 2ax}{2x^2} + \frac{1}{2x^2}\right] = \frac{1}{2\pi x^2}[1 - \cos 2ax] = \frac{\sin^2 ax}{\pi x^2}$$

Example 18 Find the function $f(x)$ if its Sine transform is given by:

(i) $e^{-a\lambda}$ (ii) $\frac{\lambda}{1+\lambda^2}$

Solution: (i) Given that $\bar{f_s}(\lambda) = e^{-a\lambda}$

Taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f_s}(\lambda) \sin \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty e^{-a\lambda} \sin \lambda x \, d\lambda = \frac{2}{\pi}.\frac{x}{a^2+x^2}$$

(ii) Given that $\bar{f_s}(\lambda) = \frac{\lambda}{1+\lambda^2}$

Taking Inverse Fourier Sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty \bar{f_s}(\lambda) \sin \lambda x \, d\lambda$$

$$\Rightarrow f(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda}{1+\lambda^2} \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\lambda^2}{\lambda(1+\lambda^2)} \sin \lambda x \, d\lambda = \frac{2}{\pi}\int_0^\infty \frac{(1+\lambda^2)-1}{\lambda(1+\lambda^2)} \sin \lambda x \, d\lambda$$

$$= \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda x}{\lambda} d\lambda - \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda$$

$$\Rightarrow f(x) = 1 - \frac{2}{\pi}\int_0^\infty \frac{\sin \lambda x}{\lambda(1+\lambda^2)} d\lambda \quad \ldots\ldots①$$

$$\because \int_0^\infty \frac{\sin \lambda x}{\lambda} d\lambda = \frac{\pi}{2}, x > 0$$

Differentiating with respect to x

$$\Rightarrow f'(x) = 0 - \frac{2}{\pi}\int_0^\infty \frac{\lambda \cos \lambda x}{\lambda(1+\lambda^2)} d\lambda$$

$$\Rightarrow f'(x) = -\frac{2}{\pi}\int_0^\infty \frac{\cos \lambda x}{(1+\lambda^2)} d\lambda \ \ldots\ldots②$$

Also $f''(x) = \frac{2}{\pi}\int_0^\infty \frac{\lambda \sin \lambda x}{(1+\lambda^2)} d\lambda = f(x)$

$$\Rightarrow f''(x) - f(x) = 0 \ldots.. ③$$

This is a linear differential equation with constant coefficients

③may be written as $(D^2 - 1)f(x) = 0$

Auxiliary equation is $m^2 - 1 = 0$

$\Rightarrow$ m $= \pm 1$

Solution of ③ is given by

$f(x) = c_1 e^{x} + c_2 e^{-x}$ ….. ④

$\Rightarrow f'(x) = c_1 e^{x} - c_2 e^{-x}$ …..⑤

Now from ①, $f(x) = 1$, at $x = 0$

Using in ④, we get $c_1 + c_2 = 1$ ….⑥

Again from ②, $f'(x) = -\frac{2}{\pi}\int_0^{\infty}\frac{1}{(1+\lambda^2)}d\lambda$, at $x = 0$

$\Rightarrow f'(x) = -\frac{2}{\pi}[\tan^{-1}\lambda]_0^{\infty} = -1$ at $x = 0$

Using in ⑤, we get $c_1 - c_2 = -1$ …. ⑦

Solving ⑥ and ⑦, we get $c_1 = 0$, $c_2 = 1$

Using in ④, we get $f(x) = e^{-x}$

Note: Solution of the differential equation $f''(x) - f(x) = 0$ may be written directly as $f(x) = e^{-x}$

Example 19 Find the Fourier transform of the function $f(x) = e^{-a|x|}$, $-\infty < x < \infty$

Solution: $f(x) = \begin{cases} e^{ax}, x < 0 \\ e^{-ax} x \geq 0 \end{cases}$

Fourier transform of $f(x)$ is given by $F\{f(x)\} \equiv \bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx$

$$\Rightarrow \bar{f}(\lambda) = \int_{-\infty}^{0} e^{ax}e^{i\lambda x}\,dx + \int_{0}^{\infty} e^{-ax}e^{i\lambda x}\,dx$$

$$= \int_{-\infty}^{0} e^{x(a+i\lambda)}dx + \int_{0}^{\infty} e^{-x(a-i\lambda)}dx$$

$$= \left[\frac{e^{x(a+i\lambda)}}{(a+i\lambda)}\right]_{-\infty}^{0} - \left[\frac{e^{-x(a-i\lambda)}}{(a-i\lambda)}\right]_{0}^{\infty}$$

$$\Rightarrow \bar{f}(\lambda) = \frac{1}{a+i\lambda} + \frac{1}{a-i\lambda} = \frac{2a}{a^2+\lambda^2}$$

$$\therefore F\{e^{-a|x|}\} = \frac{2a}{a^2+\lambda^2}$$

Result:

$$\boldsymbol{F\{e^{-a|x|}\} = \frac{2a}{a^2+\lambda^2} \Rightarrow F^{-1}\left[\frac{2a}{a^2+\lambda^2}\right] = e^{-a|x|}}$$

Example 20 Find $F^{-1}\left[\frac{1}{(9+\lambda^2)(4+\lambda^2)}\right]$

Solution: $F^{-1}\left[\frac{1}{(9+\lambda^2)(4+\lambda^2)}\right] = \frac{1}{5}F^{-1}\left[-\frac{1}{9+\lambda^2}+\frac{1}{4+\lambda^2}\right]$

$$= \frac{1}{5}F^{-1}\left[-\frac{1}{3^2+\lambda^2}+\frac{1}{2^2+\lambda^2}\right]$$

$$= \frac{-1}{30}F^{-1}\left[\frac{6}{9+\lambda^2}\right]+\frac{1}{20}F^{-1}\left[\frac{4}{4+\lambda^2}\right]$$

$$= \frac{-1}{30}e^{-3|x|}+\frac{1}{20}e^{-2|x|} \quad \because F^{-1}\left[\frac{2a}{a^2+\lambda^2}\right]=e^{-a|x|}$$

Example 21 Find the Fourier transform of the function $f(x)=e^{-ax}U(x),\ a>0$

where $U(x)$ represents unit step function

Solution: $f(x)=e^{-ax}\begin{cases}0\ , & x<0\\ 1, & x\geq 0\end{cases} = \begin{cases}0\ , & x<0\\ e^{-ax}, & x\geq 0\end{cases}$

Fourier transform of $f(x)$ is given by $F\{f(x)\}\equiv \bar{f}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x}f(x)dx$

$$\Rightarrow \bar{f}(\lambda) = \int_0^{\infty} e^{-ax}e^{i\lambda x}\,dx$$

$$= \int_0^{\infty} e^{-x(a-i\lambda)}dx$$

$$= -\left[\frac{e^{-x(a-i\lambda)}}{(a-i\lambda)}\right]_0^{\infty}$$

$$\Rightarrow \bar{f}(\lambda) = \frac{1}{a-i\lambda}$$

$$\therefore F\{f(x)\} = \frac{1}{a-i\lambda}$$

or $F\{e^{-ax}U(x)\} = \frac{1}{a-i\lambda}$

Result: $$\boldsymbol{F\{e^{-ax}U(x)\} = \frac{1}{a-i\lambda} \Rightarrow F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}U(x) = e^{-ax}H(x)}$$

Note: If Fourier transform of $\boldsymbol{f(x)=e^{-ax}U(x)}$ is taken as $\boldsymbol{\int_{-\infty}^{\infty} e^{-i\lambda x}e^{-ax}U(x)dx}$, then $\boldsymbol{F^{-1}\left[\frac{1}{a+i\lambda}\right] = e^{-ax}U(x) = e^{-ax}H(x)}$

Example 22 Find the inverse transform of the following functions:

i. $\frac{1}{2-3i\lambda-\lambda^2}$ ii. $\frac{1}{8+6i\lambda-\lambda^2}$ iii. $\frac{5}{6-5i\lambda-\lambda^2}$

Solution: i. $F^{-1}\left[\frac{1}{2-3i\lambda-\lambda^2}\right] = F^{-1}\left[\frac{1}{(1-i\lambda)(2-i\lambda)}\right] = F^{-1}\left[\frac{1}{(1-i\lambda)}-\frac{1}{(2-i\lambda)}\right]$

$$= F^{-1}\left[\frac{1}{(1-i\lambda)}\right] - F^{-1}\left[\frac{1}{(2-i\lambda)}\right]$$

$$= e^{-x}H(x) - e^{-2x}H(x) \qquad \because F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$$

$$\Rightarrow F^{-1}\left[\frac{1}{2-3i\lambda-\lambda^2}\right] = \begin{cases}(e^{-x} - e^{-2x}), x \geq 0 \\ 0\ , \quad x < 0\end{cases}$$

ii. $F^{-1}\left[\frac{1}{8+6i\lambda-\lambda^2}\right] = F^{-1}\left[\frac{1}{(4+i\lambda)(2+i\lambda)}\right] = F^{-1}\left[\frac{1}{(4+i\lambda)} - \frac{1}{(2+i\lambda)}\right]$

$$= F^{-1}\left[\frac{1}{(4+i\lambda)}\right] - F^{-1}\left[\frac{1}{(2+i\lambda)}\right]$$

$$= e^{-4x}H(x) - e^{-2x}H(x) \qquad \because F^{-1}\left[\frac{1}{a+i\lambda}\right] = e^{-ax}H(x)$$

$$\Rightarrow F^{-1}\left[\frac{1}{8+6i\lambda-\lambda^2}\right] = \begin{cases}(e^{-x} - e^{-2x}), x \geq 0 \\ 0\ , \quad x < 0\end{cases}$$

iii. $F^{-1}\left[\frac{5}{6-5i\lambda-\lambda^2}\right] = 5\,F^{-1}\left[\frac{1}{(2-i\lambda)(3-i\lambda)}\right] = 5F^{-1}\left[\frac{1}{(2-i\lambda)} - \frac{1}{(3-i\lambda)}\right]$

$$= 5F^{-1}\left[\frac{1}{(2-i\lambda)}\right] - 5F^{-1}\left[\frac{1}{(3-i\lambda)}\right]$$

$$= 5e^{-2x}H(x) - 5e^{-3x}H(x) \qquad \because F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$$

$$\Rightarrow F^{-1}\left[\frac{5}{6-5i\lambda-\lambda^2}\right] = \begin{cases}5(e^{-2x} - e^{-3x}), x \geq 0 \\ 0\ , \quad x < 0\end{cases}$$

Example 23 Find the Fourier transform of $f(x) = \frac{1}{2-ix}$

Solution: We know $F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$

$$\Rightarrow F^{-1}\left[\frac{1}{2-i\lambda}\right] = e^{-2x}\mathrm{H}(x)$$

$$\Rightarrow \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2-i\lambda}e^{-i\lambda x}\,d\lambda = e^{-2x}\mathrm{H}(x)$$

Interchanging x and λ, we get

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{1}{2-ix}e^{-i\lambda x}\,dx = e^{-2\lambda}\mathrm{H}(\lambda)$$

$$= \begin{cases}0\ , \quad \lambda < 0 \\ e^{-2\lambda}, \ \lambda \geq 0\end{cases}$$

$$\Rightarrow \int_{-\infty}^{\infty}\frac{1}{2-ix}e^{-i\lambda x}\,dx = \begin{cases}0\ , \qquad \lambda < 0 \\ 2\pi e^{-2\lambda}, \ \lambda \geq 0\end{cases}$$

$$\Rightarrow \mathrm{F}\left\{\frac{1}{2-ix}\right\} = \begin{cases} 0\,, & \lambda < 0 \\ 2\pi e^{-2\lambda}, & \lambda \geq 0 \end{cases}$$

2.3 Properties of Fourier Transforms

Linearity: If $\overline{f}(\lambda)$ and $\overline{g}(\lambda)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then

$$F\{af(x) + bg(x)\} = a\overline{f}(\lambda) + b\overline{g}(\lambda)$$

Proof: $F\{af(x) + bg(x)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [af(x) + bg(x)]\, e^{i\lambda x} dx$

$$= a\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{i\lambda x} dx + b\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g(x) e^{i\lambda x} dx$$

$$= a\overline{f}(\lambda) + b\overline{g}(\lambda)$$

Change of scale: If $\overline{f}(\lambda)$ is Fourier transforms of $f(x)$, then $F\{f(ax)\} = \frac{1}{a}\overline{f}\left(\frac{\lambda}{a}\right)$

Proof: $F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(ax).\, e^{i\lambda x}$

Putting $ax = t \Rightarrow adx = dt$

$$\therefore F\{f(ax)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\lambda \frac{t}{a}}.\frac{dt}{a} = \frac{1}{a}.\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\left(\frac{\lambda}{a}\right)t} dt$$

$$= \frac{1}{a}\overline{f}\left(\frac{\lambda}{a}\right)$$

Shifting Property: If $\overline{f}(\lambda)$ is Fourier transforms of $f(x)$, then $F\{f(x-a)\} = e^{i\lambda a}\, \overline{f}(\lambda)$

Proof: $F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-a).\, e^{i\lambda x}$

Putting $(x-a) = t \Rightarrow dx = dt$

$$\therefore F\{f(x-a)\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\lambda(t+a)}\, dt$$

$$= e^{i\lambda a} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t).\, e^{i\lambda t} dt = e^{i\lambda a}\, \overline{f}(\lambda)$$

Modulation Theorem: If $\overline{f}(\lambda)$ is Fourier transforms of $f(x)$, then

i. $F\{f(x)\cos ax\} = \frac{1}{2}\left\{\overline{f}(\lambda + a) + \overline{f}(\lambda - a)\right\}$

ii. $F_s[f(x)\cos ax] = \frac{1}{2}\left\{\overline{f}_s(\lambda + a) + \overline{f_s}(\lambda - a)\right\}$

iii. $F_c[f(x)\sin ax] = \frac{1}{2}\left\{\overline{f}_s(\lambda + a) - \overline{f_s}(\lambda - a)\right\}$

iv. $F_c[f(x)\cos ax] = \frac{1}{2}\left\{\overline{f}_c(\lambda + a) + \overline{f_c}(\lambda - a)\right\}$

v. $F_s[f(x)\sin ax] = \frac{1}{2}\{\overline{f}_c(\lambda-a) - \overline{f_c}(\lambda+a)\}$

Proof: i. $F\{f(x)\cos ax\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\cos ax \,. e^{i\lambda x}$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\frac{e^{iax}+e^{-iax}}{2} e^{i\lambda x}dx$$

$$= \frac{1}{2}\left[\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(\lambda+a)x}dx + \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{i(\lambda-a)x}dx\right]$$

$$= \frac{1}{2}\{\overline{f}(\lambda+a) + \overline{f}(\lambda-a)\}$$

ii. $F_s[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax \ \sin\lambda x \ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\sin(\lambda+a)x + \sin(\lambda-a)x \ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda-a)x\,dx\right]$$

$$= \frac{1}{2}\{\overline{f}_s(\lambda+a) + \overline{f_s}(\lambda-a)\}$$

iii. $F_c[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax \ \cos\lambda x \ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\sin(\lambda+a)x - \sin(\lambda-a)x \ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda+a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin(\lambda-a)x\,dx\right]$$

$$= \frac{1}{2}\{\overline{f}_s(\lambda+a) - \overline{f_s}(\lambda-a)\}$$

iv. $F_c[f(x)\cos ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos ax \ \cos\lambda x \ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\cos(\lambda+a)x + \cos(\lambda-a)x \ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\lambda+a)x\,dx + \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\lambda-a)x\,dx\right]$$

$$= \frac{1}{2}\{\overline{f}_c(\lambda+a) + \overline{f_c}(\lambda-a)\}$$

v. $F_s[f(x)\sin ax] = \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\sin ax \ \sin\lambda x \ dx$

$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\,[\cos(\lambda - a)x - \cos(\lambda + a)x\ \ ]dx$$

$$= \frac{1}{2}\left[\sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\lambda - a)x\,dx - \sqrt{\frac{2}{\pi}}\int_0^{\infty} f(x)\cos(\lambda + a)x\,dx\right]$$

$$= \frac{1}{2}\left\{\overline{f}_c(\lambda - a) - \overline{f_c}(\lambda + a)\right\}$$

Convolution theorem: Convolution of two functions $f(x)$ and $g(x)$ is defined as

$$f(x) *\ g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

If $\overline{f}(\lambda)$ and $\overline{g}(\lambda)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively, then Convolution theorem for Fourier transforms states that

$$F\{f(x) * g(x)\} = F\{f(x)\}\,.\,F\{fg(x)\} \equiv \overline{f}(\lambda).\,\overline{g}(\lambda)$$

Proof: By definition $\bar{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i\lambda x} f(x)dx$ and $\bar{g}(\lambda) = \int_{-\infty}^{\infty} e^{i\lambda x} g(x)dx$

Now $\quad f(x) *\ g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$

$$\therefore F\{f(x) * g(x)\} = \int_{-\infty}^{\infty} e^{i\lambda x}\left[\int_{-\infty}^{\infty} f(u)g(x-u)du\right]dx$$

Changing the order of integration, we get

$$\therefore F\{f * g\} = \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{i\lambda x} g(x-u)dx\right]du$$

Putting $x - u = t \ \Rightarrow dx = dt$ in the inner integral, we get

$$F\{f * g\} = \int_{-\infty}^{\infty} f(u)\left[\int_{-\infty}^{\infty} e^{i\lambda(u+t)} g(t)dt\right]du$$

$$= \int_{-\infty}^{\infty} e^{i\lambda u} f(u)\left[\int_{-\infty}^{\infty} e^{i\lambda t} g(t)dt\right]du$$

$$= \int_{-\infty}^{\infty} e^{i\lambda u} f(u)\overline{g}(\lambda)\,du$$

$$= \overline{g}(\lambda)\int_{-\infty}^{\infty} e^{i\lambda u} f(u)du$$

$$= \overline{f}(\lambda).\,\overline{g}(\lambda)$$

Example 24 Find the Fourier transform of e^{-x^2}. Hence find Fourier transforms of

i. $e^{-ax^2}, a > 0$ ii. $e^{\frac{-x^2}{2}}$ iii. $e^{\,2(x-3)^2}$ iv. $e^{-x^2}\cos 2x$

Solution: F $\left\{e^{-x^2}\right\} = \overline{f}(\lambda) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2} e^{i\lambda x}\,dx$

$$= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-x^2 + i\lambda x}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x^2 - 2\left(\frac{i\lambda}{2}\right)x + \left(\frac{i\lambda}{2}\right)^2 - \left(\frac{i\lambda}{2}\right)^2\right)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{i\lambda}{2}\right)^2 + \frac{i^2\lambda^2}{4}} dx$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\left(x - \frac{i\lambda}{2}\right)^2} dx$$

$$= \frac{e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-z^2} dz \qquad \text{By putting } z = \left(x - \frac{i\lambda}{2}\right)$$

$$= \frac{2e^{-\frac{\lambda^2}{4}}}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-z^2} dz \qquad e^{-z^2} \text{ being even function of } z$$

$$\therefore \overline{f}(\lambda) = \frac{2e^{\frac{-\lambda^2}{4}}}{\sqrt{2\pi}} \cdot \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}} e^{\frac{-\lambda^2}{4}} \text{ .....①}$$

$\therefore$ We have $F\{f(x)\} = \overline{f}(\lambda) = \frac{1}{\sqrt{2}} e^{\frac{-\lambda^2}{4}}$ if $f(x) = e^{-x^2}$

i. Now $F\{e^{-ax^2}\} = F\left\{e^{(\sqrt{a}\sqrt{x})^2}\right\}$

$$= \frac{1}{\sqrt{a}} \overline{f}\left(\frac{\lambda}{\sqrt{a}}\right) \qquad \text{By change of scale property.... ②}$$

$$\therefore F\{e^{-ax^2}\} = \frac{1}{\sqrt{a}} \cdot \frac{1}{\sqrt{2}} e^{-\frac{1}{4}\left(\frac{\lambda}{\sqrt{a}}\right)^2} = \frac{1}{\sqrt{2a}} e^{-\frac{\lambda^2}{4a}} \qquad \text{Using ①in ②}$$

ii. Putting $a = \frac{1}{2}$ in i.

$$F\left\{e^{-\frac{x^2}{2}}\right\} = \frac{1}{\sqrt{2 \cdot \frac{1}{2}}} \cdot e^{-\frac{\lambda^2 \cdot}{\frac{4}{\cdot 2}}} = e^{-\frac{\lambda^2}{2}}$$

iii. To find $F\{e^{-2(x-3)^3}\}$, Put $a = 2$ in i.

$$F\{e^{-2x^2}\} = \frac{1}{2} e^{-\frac{\lambda^2}{8}}$$

$$\therefore F\{e^{-2(x-3)^2}\} = e^{3i\lambda} \cdot \frac{1}{2} e^{-\frac{\lambda^2}{8}} \quad \because \text{By shifting property } F\{f(x-k)\} = e^{i\lambda k} \overline{f}(\lambda)$$

iv. To find Fourier transform of $F\{ e^{-x^2} cos 2x\}$

$$F\{f(x)\cos ax\} = \frac{1}{2} \overline{f}(\lambda + a) + \overline{f}(\lambda - a) \quad \text{By modulation theorem}$$

Now $F\{e^{-x^2}\} \equiv \overline{f}(\lambda) = \frac{1}{\sqrt{2}} e^{-\frac{\lambda^2}{4}}$.

$$\therefore\ F\{e^{-x^2}cos2x\} = \frac{1}{2}\left[\frac{1}{\sqrt{2}}e^{-\frac{(\lambda+2)^2}{4}} + \frac{1}{\sqrt{2}}e^{-\frac{(\lambda-2)^2}{4}}\right]$$

Example 25 Using Convolution theorem, find $F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right]$

Solution: $F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right] = F^{-1}\left[\frac{1}{(4-i\lambda)(3-i\lambda)}\right] = F^{-1}\left[\frac{1}{(4-i\lambda)}.\frac{1}{(3-i\lambda)}\right]$

Now by Convolution theorem

$$F\{f(x)*g(x)\} = \overline{f}(\lambda).\,\overline{g}(\lambda) \Rightarrow F^{-1}\left[\overline{f}(\lambda).\overline{g}(\lambda)\right] = f(x)*g(x)$$

$$\therefore F^{-1}\left[\frac{1}{(4-i\lambda)}.\frac{1}{(3-i\lambda)}\right] = F^{-1}\left[\frac{1}{(4-i\lambda)}\right] * F^{-1}\left[\frac{1}{(3-i\lambda)}\right]$$

$$= e^{-4x}H(x) * e^{-3x}H(x) \qquad \because F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$$

$$= \int_{-\infty}^{\infty} e^{-4u}H(u)e^{-3(x-u)}H(x-u)du$$

$$\because f(x) * g(x) = \int_{-\infty}^{\infty} f(u)g(x-u)du$$

$$= e^{-3x}\int_{-\infty}^{\infty} e^{-u}H(u)H(x-u)du$$

Now $H(u)H(x-u) = \begin{cases}1, & u \geq 0, x-u \geq 0, \quad i.e. 0 \leq u \leq x\\ 0, u<0, x-u<0, & i.e. u<0 \text{ and } u>x\end{cases}$

$$\therefore F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right] = e^{-3x}\int_0^x e^{-u}du = -e^{-3x}[e^{-u}]_0^x = -e^{-3x}[e^{-x}-1], x \geq 0$$

$$= e^{-3x} - e^{-4x}, x \geq 0$$

$$\Rightarrow F^{-1}\left[\frac{1}{12-7i\lambda-\lambda^2}\right] = \begin{cases}e^{-3x}-e^{-4x}, x \geq 0\\ \quad 0\ , \quad x<0\end{cases}$$

Example 26 Find the inverse Fourier transforms of $\frac{e^{3i\lambda}}{2-i\lambda}$

Solution: i. We know that $F^{-1}\left[\frac{1}{a-i\lambda}\right] = e^{-ax}H(x)$

$$\therefore F^{-1}\left[\frac{1}{2-i\lambda}\right] = e^{-2x}H(x)$$

Now By shifting property $F\{f(x-k)\} = e^{i\lambda k}\overline{f}(\lambda)$

$$\Rightarrow F^{-1}\left[e^{i\lambda k}\overline{f}(\lambda)\right] = f(x-k)$$

$$\therefore F^{-1}\left[\frac{e^{3i\lambda}}{2-i\lambda}\right] = e^{-2(x-3)}H(x-3)$$

2.4 Fourier Transforms of Derivatives

Let $u(x,t)$ be a function of two independent variables x and t, such that Fourier transform of $u(x,t)$ is denoted by $\overline{u}(\lambda,t)$ i.e $\overline{u}(\lambda,t) = \int_{-\infty}^{\infty} e^{i\lambda x} u(x,t)dx$

Again let $u, \frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots \to 0$ as $x \to \pm\infty$,

Then Fourier transforms of $\frac{\partial u}{\partial x}, \frac{\partial^2 u}{\partial x^2}, \ldots$ with respect to x are given by:

1. $$F\left\{\frac{\partial u}{\partial x}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial u}{\partial x}dx = \left[e^{i\lambda x}u\right]_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}udx = -i\lambda\,\overline{u}(\lambda,t)$$

   $$F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial^2 u}{\partial x^2}dx = \left[e^{i\lambda x}\frac{\partial u}{\partial x}\right]_{-\infty}^{\infty} - i\lambda\int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial u}{\partial x}dx = (-i\lambda)^2\,\overline{u}(\lambda,t)$$

   $\vdots$

   $$\boldsymbol{F\left\{\frac{\partial^n u}{\partial x^n}\right\} = (-i\lambda)^n\,\overline{u}(\lambda,t)}$$

2. Fourier sine transform of $\frac{\partial^2 u}{\partial x^2}$ is given by:

   $$F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty}\frac{\partial^2 u}{\partial x^2}\sin\lambda x\,dx = \left[\sin\lambda x\frac{\partial u}{\partial x}\right]_0^{\infty} - \lambda\int_0^{\infty}\cos\lambda x\frac{\partial u}{\partial x}dx$$

   $$= 0 - \lambda[\cos\lambda x\,.\,u(x,t)]_0^{\infty} - \lambda^2\int_0^{\infty}\sin\lambda x\frac{\partial^2 u}{\partial x^2}dx$$

   $$\therefore \boldsymbol{F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \lambda u(0,t) - \lambda^2\overline{u}_s(\lambda,t)}$$

3. Fourier cosine transform of $\frac{\partial^2 u}{\partial x^2}$ is given by:

   $$F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \int_0^{\infty}\frac{\partial^2 u}{\partial x^2}\cos\lambda x\,dx = \left[\cos\lambda x\frac{\partial u}{\partial x}\right]_0^{\infty} + \lambda\int_0^{\infty}\sin\lambda x\frac{\partial u}{\partial x}dx$$

   $$= -\left[\frac{\partial u}{\partial x}\right]_{x=0} + \lambda[\sin\lambda x\,.\,u(x,t)]_0^{\infty} - \lambda^2\int_0^{\infty}\cos\lambda x\frac{\partial^2 u}{\partial x^2}dx$$

   $$\therefore \boldsymbol{F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - \lambda^2\overline{u}_c(\lambda,t)}$$

4. Fourier transforms of $\frac{\partial u}{\partial t}$ with respect to x are given by:

   $$F\left\{\frac{\partial u}{\partial t}\right\} = \int_{-\infty}^{\infty} e^{i\lambda x}\frac{\partial u}{\partial t}dx = \frac{d}{dt}\int_{-\infty}^{\infty} e^{i\lambda x}u(x,t)dx$$

   $$\therefore \boldsymbol{F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}(\lambda,t)}$$

   Similarly $$\boldsymbol{F_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_s(\lambda,t)}$$

   $$\boldsymbol{F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_c(\lambda,t)}$$

2.5 Applications of Fourier Transforms to boundary value problems

Partial differential equation together with boundary and initial conditions can be easily solved using Fourier transforms. In one dimensional boundary value problems, the partial differential equations can easily be transformed into an ordinary differential equation by applying a suitable transform and solution to boundary value problem is obtained by applying inverse transform. In two dimensional problems, it is sometimes required to apply the transforms twice and the desired solution is obtained by double inversion.

Algorithm to solve partial differential equations with boundary values:

1. Apply the suitable transform to given partial differential equation. For this check the range of x
   i. If $-\infty < x < \infty$, then apply Fourier transform.
   ii. If $0 < x < \infty$, then check initial value conditions
      a) If value of $u(0,t)$ is given, then apply Fourier sine transform
      b) If value of $\left[\frac{\partial u}{\partial x}\right]_{x=0}$ is given, then apply Fourier cosine transform

   An ordinary differential equation will be formed after applying the transform.
2. Solve the differential equation using usual methods.
3. Apply Boundary value conditions to evaluate arbitrary constants.
4. Apply inverse transform to get the required expression for $u(x,t)$.

Example 27 The temperature $u(x,t)$ at any point of an infinite bar satisfies the equation $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, -\infty < x < \infty,\ t > 0$ and the initial temperature along the length of the bar is given by $u(x,0) = \begin{cases} 1 \ for\ |x| < 1 \\ 0 \ for\ |x| > 1 \end{cases}$

Determine the expression for $u(x,t)$.

Solution: As range of x is $(-\infty, \infty)$, applying Fourier transform to both sides of the given equation :

$$F\left\{\frac{\partial u}{\partial t}\right\} = F\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt}\overline{u}(\lambda,t) = -\lambda^2\overline{u}(\lambda,t) \ \because F\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}(\lambda,t) \text{ and } F\left\{\frac{\partial^2 u}{\partial x^2}\right\} = (-i\lambda)^2\,\overline{u}(\lambda,t)$$

Rearranging the ordinary differential equation in variable separable form:

$$\Rightarrow \frac{d\overline{u}}{\overline{u}} = -\lambda^2 dt \ldots ① \qquad \text{where } \overline{u} \approx \overline{u}(\lambda,t)$$

Solving ① using usual methods of variable separable differential equations

$$\log \overline{u} = -\lambda^2 t + \log A$$

$$\Rightarrow \log\frac{\overline{u}}{A} = -\lambda^2 t$$

$$\Rightarrow \overline{u}(\lambda, t) = A\, e^{-\lambda^2 t} \ \ldots \ ②$$

Putting $t = 0$ on both sides

$$\Rightarrow \overline{u}(\lambda, 0) = A \ \ldots \ ③$$

Now given that $u(x,0) = \begin{cases} 1 \ for\ |x| < 1 \\ 0 \ for\ |x| > 1 \end{cases}$

Taking Fourier transform on both sides, we get

$$\Rightarrow \overline{u}\,(\lambda, 0) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} u(x,0)\, e^{\,i\lambda x} dx$$

$$= \frac{1}{\sqrt{2\pi}}\int_{-1}^{1} e^{\,i\lambda x}\, dx$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\lambda}\left[e^{\,i\lambda x}\right]_{-1}^{1}$$

$$= \frac{1}{\sqrt{2\pi}}\,\frac{1}{i\lambda}\left[e^{\,i\lambda} - e^{-i\lambda}\right] = \frac{1}{\sqrt{2\pi}}\,\frac{2i}{i\lambda}\left[\frac{e^{\,i\lambda} - e^{-i\lambda}}{2i}\right]$$

$$\Rightarrow \ \overline{u}\,(\lambda, 0) = \frac{2}{\sqrt{2\pi}}\,\frac{sin\lambda}{\lambda} \ \ldots \ ④$$

From ③and ④, we get

$$A = \frac{2}{\sqrt{2\pi}}\,\frac{sin\lambda}{\lambda} \ldots ⑤$$

Using ⑤ in ②, we get

$$\overline{u}(\lambda, t) = \frac{2}{\sqrt{2\pi}}\,\frac{sin\lambda}{\lambda}\ e^{-\lambda^2 t}$$

Taking Inverse Fourier transform

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-i\lambda x}\overline{u}(\lambda, t) d\lambda$$

$$\Rightarrow u(x,t) = \frac{2}{2\pi}\int_{-\infty}^{\infty} \frac{sin\lambda}{\lambda}\ e^{-\lambda^2 t} e^{-i\lambda x} d\lambda$$

$$\Rightarrow u(x,t) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{sin\lambda}{\lambda} e^{-\lambda^2 t}(\cos\lambda x - i\sin\lambda x) d\lambda$$

$$\Rightarrow u(x,t) = \frac{2}{\pi}\int_{0}^{\infty} e^{-\lambda^2 t}\left(\frac{\sin\lambda\cos\lambda x}{\lambda}\right) d\lambda \qquad \because \left(\frac{\sin\lambda\sin\lambda x}{\lambda}\right) \text{ is odd function of } \lambda$$

Example 28 Using Fourier transform, solve the equation $\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, 0 < x < \infty,\ t > 0$ subject to conditions:

i. $u(0,t) = 0, \quad t > 0$

ii. $u(x,0) = e^{-x}, \ \ x > 0$

iii. u and $\frac{\partial u}{\partial x}$ both tend to zero as $x \to \pm\infty$

Solution: As range of x is $(0,\infty)$, and also value of $u(0,t)$ is given in initial value conditions, applying Fourier sine transform to both sides of the given equation:

$$F_s\left\{\frac{\partial u}{\partial t}\right\} = kF_s\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt}\overline{u}_s(\lambda,t) = k\lambda u(0,t) - k\lambda^2\overline{u}_s(\lambda,t)$$

$$\because F_s\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_s(\lambda,t) \text{ and } F_s\left\{\frac{\partial^2 u}{\partial x^2}\right\} = \lambda u(0,t) - \lambda^2\overline{u}_s(\lambda,t)$$

$$\Rightarrow \frac{d}{dt}\overline{u}_s(\lambda,t) = -k\lambda^2\overline{u}_s(\lambda,t) \qquad \because u(0,t) = 0$$

Rearranging the ordinary differential equation in variable separable form:

$$\Rightarrow \frac{d\overline{u}}{\overline{u}} = -k\lambda^2 dt \ldots ① \qquad \text{where } \overline{u} \approx \overline{u}_s(\lambda,t)$$

Solving ① using usual methods of variable separable differential equations

$$\log\overline{u} = -k\lambda^2 t + \log A$$

$$\Rightarrow \log\frac{\overline{u}}{A} = -k\lambda^2 t$$

$$\Rightarrow \overline{u}_s(\lambda,t) = A\, e^{-k\lambda^2 t} \ldots ②$$

Putting $t = 0$ on both sides

$$\Rightarrow \overline{u}_s(\lambda,0) = A \ \ldots ③$$

Now given that $u(x,0) = e^{-x}$

Taking Fourier sine transform on both sides, we get

$$\Rightarrow \overline{u}_s(\lambda,0) = \sqrt{\frac{2}{\pi}}\int_0^\infty u(x,0)\sin\lambda x\, dx$$

$$= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin\lambda x\, dx$$

$$\Rightarrow \overline{u}_s(\lambda,0) = \sqrt{\frac{2}{\pi}}\,\frac{\lambda}{1+\lambda^2} \ldots ④$$

From ③and ④, we get

$$A = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} \ldots ⑤$$

Using ⑤ in ②, we get

$$\overline{u}_s(\lambda, t) = \sqrt{\frac{2}{\pi}} \frac{\lambda}{1+\lambda^2} e^{-k\lambda^2 t}$$

Taking Inverse Fourier sine transform

$$u(x,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \overline{u}_s(\lambda, t) \sin \lambda x \, d\lambda$$

$$\Rightarrow u(x,t) = \frac{2}{\pi} \int_0^\infty \frac{\lambda}{1+\lambda^2} e^{-k\lambda^2 t} \sin \lambda x \, d\lambda$$

Example 29 The temperature $u(x,t)$ in a semi-infinite rod $0 < x < \infty$ is determined by the differential equation $\frac{\partial u}{\partial t} = 2\frac{\partial^2 u}{\partial x^2}$ subject to conditions:

i. $u = 0$, when $t = 0$, $x \geq 0$
ii. $\frac{\partial u}{\partial x} = -k \ (a\ costant)$, when $x = 0, t > 0$

Solution: As range of x is $(0, \infty)$, and also value of $\left[\frac{\partial u}{\partial x}\right]_{x=0}$ is given in initial value conditions, applying Fourier cosine transform to both sides of the equation:

$$F_c\left\{\frac{\partial u}{\partial t}\right\} = 2F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt}\overline{u}_c(\lambda, t) = -2\left[\frac{\partial u}{\partial x}\right]_{x=0} - 2\lambda^2 \overline{u}_c(\lambda, t)$$

$$\because F_c\left\{\frac{\partial u}{\partial t}\right\} = \frac{d}{dt}\overline{u}_c(\lambda, t) \text{ and } F_c\left\{\frac{\partial^2 u}{\partial x^2}\right\} = -\left[\frac{\partial u}{\partial x}\right]_{x=0} - \lambda^2 \overline{u}_c(\lambda, t)$$

$$\Rightarrow \frac{d}{dt}\overline{u}_c(\lambda, t) = 2k - 2\lambda^2 \overline{u}_c(\lambda, t)$$

$$\Rightarrow \frac{d\overline{u}}{dt} + 2\lambda^2 \overline{u} = 2k \ldots ① \qquad \text{where } \overline{u} \approx \overline{u}_c(\lambda, t)$$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

where $P = 2\lambda^2$, $Q = 2k$

Integrating Factor (IF) = $e^{\int P dt} = e^{\int 2\lambda^2 dt} = e^{2\lambda^2 t}$

Solution of ① is given by

$$\overline{u}.e^{2\lambda^2 t} = \int 2k.e^{2\lambda^2 t} dt + A$$

$$\Rightarrow \overline{u}.e^{2\lambda^2 t} = \frac{2ke^{2\lambda^2 t}}{2\lambda^2} + A$$

$$\Rightarrow \overline{u}_c(\lambda, t) = \frac{k}{\lambda^2} + Ae^{-2\lambda^2 t} \text{ ... ②}$$

Putting $t = 0$ on both sides

$$\Rightarrow \overline{u}_c(\lambda, 0) = \frac{k}{\lambda^2} + A \text{ ... ③}$$

Now given that $u(x, 0) = 0$

Taking Fourier cosine transform on both sides, we get

$$\Rightarrow \overline{u}_c(\lambda, 0) = \int_0^\infty u(x, 0) \cos \lambda x \, dx = 0$$

$$\Rightarrow \overline{u}_c(\lambda, 0) = 0 \text{ ... ④}$$

From ③and ④, we get

$$A = -\frac{k}{\lambda^2} \text{ ... ⑤}$$

Using ⑤ in ②, we get

$$\overline{u}_c(\lambda, t) = \frac{k}{\lambda^2}\left(1 - e^{-2\lambda^2 t}\right)$$

Taking Inverse Fourier cosine transform

$$u(x, t) = \frac{2}{\pi}\int_0^\infty \overline{u}_c(\lambda, t) \cos \lambda x \, d\lambda$$

$$\Rightarrow u(x, t) = \frac{2k}{\pi}\int_0^\infty \left(\frac{1 - e^{-2\lambda^2 t}}{\lambda^2}\right) \cos \lambda x \, d\lambda$$

Example 30 Using Fourier transforms, solve the equation $\frac{\partial y}{\partial t} = k\frac{\partial^2 y}{\partial x^2}, x > 0, \ t > 0$ subject to conditions:

i. $y = \alpha, \ when \ x = 0, \ t > 0$
ii. $y = 0, when \ t = 0, \ x > 0$

Solution: As range of x is $(0, \infty)$, and also value of $y(0, t)$ is given in initial value conditions, applying Fourier sine transform to both sides of the given equation:

$$F_s\left\{\frac{\partial y}{\partial t}\right\} = kF_s\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d}{dt}\overline{y}_s(\lambda, t) = k\lambda y(0, t) - k\lambda^2 \overline{y}_s(\lambda, t)$$

$$\because F_s\left\{\frac{\partial y}{\partial t}\right\} = \frac{d}{dt}\overline{y}_s(\lambda, t) \text{ and } F_s\left\{\frac{\partial^2 y}{\partial x^2}\right\} = \lambda y(0, t) - \lambda^2 \overline{y}_s(\lambda, t)$$

$\Rightarrow \frac{d}{dt}\overline{y}_s(\lambda, t) = k\alpha\lambda - k\lambda^2\overline{y}_s(\lambda, t)$ $\quad \because y(0, t) = \alpha$

$\Rightarrow \frac{d\overline{y}}{dt} + k\lambda^2\overline{y} = k\alpha\lambda$ ... ① $\quad$ where $\overline{y} \approx \overline{y}_s(\lambda, t)$

This is a linear differential equation of the form $\frac{dy}{dx} + Py = Q$

where $P = k\lambda^2$, $Q = k\alpha\lambda$

Integrating Factor (IF) = $e^{\int P dt} = e^{\int k\lambda^2 dt} = e^{k\lambda^2 t}$

Solution of ① is given by

$\overline{y}.e^{k\lambda^2 t} = \int k\alpha\lambda.e^{k\lambda^2 t}dt + A$

$\Rightarrow \overline{y}.e^{k\lambda^2 t} = \frac{k\alpha\lambda e^{k\lambda^2 t}}{k\lambda^2} + A$

$\Rightarrow \overline{y}_s(\lambda, t) = \frac{\alpha}{\lambda} + Ae^{-k\lambda^2 t}$ ... ②

Putting $t = 0$ on both sides

$\Rightarrow \overline{y}_c(\lambda, 0) = \frac{\alpha}{\lambda} + A$ ... ③

Now given that $y(x, 0) = 0$

Taking Fourier sine transform on both sides, we get

$\Rightarrow \overline{y}_s(\lambda, 0) = \int_0^\infty y(x, 0) \sin \lambda x \, dx = 0$

$\Rightarrow \overline{y}_s(\lambda, 0) = 0$ ... ④

From ③and ④, we get

$A = -\frac{\alpha}{\lambda}$ ... ⑤

Using ⑤ in ②, we get

$\overline{y}_s(\lambda, t) = \frac{\alpha}{\lambda}\left(1 - e^{-k\lambda^2 t}\right)$

Taking Inverse Fourier sine transform

$y(x, t) = \frac{2}{\pi}\int_0^\infty \overline{y}_s(\lambda, t) \sin \lambda x \, d\lambda$

$\Rightarrow y(x, t) = \frac{2\alpha}{\pi}\int_0^\infty \left(\frac{1 - e^{-k\lambda^2 t}}{\lambda}\right) \sin \lambda x \, d\lambda$

Example 31 An infinite string is initially at rest and its initial displacement is given by $f(x), -\infty < x < \infty$. Determine the displacement $y(x, t)$ of the string.

Solution: The equation of the vibrating string is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

Initial conditions are

i. $\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0$

ii. $y(x,0) = f(x)$

Taking Fourier transform on both sides

$$F\left\{\frac{\partial^2 y}{\partial t^2}\right\} = c^2 F\left\{\frac{\partial^2 y}{\partial x^2}\right\}$$

$$\Rightarrow \frac{d^2}{dt^2}\overline{y}(\lambda,t) = -c^2\lambda^2\,\overline{y}(\lambda,t) \quad \text{where } F\{y(x,t)\} \equiv \overline{y}(\lambda,t)$$

$$\Rightarrow \frac{d^2\overline{y}}{dt^2} + c^2\lambda^2\,\overline{y} = 0 \ldots ① \qquad \text{where } \overline{y} \approx \overline{y}(\lambda,t)$$

Solution of ① is given by

$$\overline{y}(\lambda,t) = A\cos cpt + B\sin cpt \ldots ②$$

Putting $t = 0$ on both sides

$$\overline{y}(\lambda,0) = A \ldots ③$$

Given that $y(x,0) = f(x)$

$$\Rightarrow \overline{y}(\lambda,0) = \overline{f}(\lambda) \ldots ④$$

From ③ and ④

$$A = \overline{f}(\lambda) \ldots ⑤$$

Using ⑤ in ②

$$\overline{y}(\lambda,t) = \overline{f}(\lambda)\cos cpt + B\sin cpt \ldots ⑥$$

$$\Rightarrow \frac{\partial y}{\partial t} = -cp\overline{f}(\lambda)\sin cpt + cpB\cos cpt$$

$$\Rightarrow \left.\frac{\partial y}{\partial t}\right]_{t=0} = cpB \ldots ⑦$$

Also given that $\left.\frac{\partial y}{\partial t}\right]_{t=0} = 0 \ldots$ ⑧

From ⑦ and ⑧, we get $B = 0 \ldots$ ⑨

Using ⑨ in ⑥, we get

$$\overline{y}(\lambda,t) = \overline{f}(\lambda)\cos cpt$$

Taking inverse Fourier transform

$$y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{y}(\lambda,t)e^{-i\lambda x}d\lambda$$

$$\Rightarrow y(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\overline{f}(\lambda)\cos cpt\, e^{-i\lambda x}d\lambda$$

Exercise 2A

1. Find the Fourier transform of $f(x) = \begin{cases} a-|x|, & |x| \le a \\ 0, & |x| > a \end{cases}$

   Hence prove that $\int_0^{\infty}\frac{\sin^2 x}{x^2}dx = \frac{\pi}{2}$

2. Solve the integral equation $\int_0^{\infty} f(x)\cos\lambda x\, dx = e^{-\lambda}, \lambda > 0$

3. Obtain Fourier sine integral of the function $f(x) = \begin{cases} x, & 0 < x < 1 \\ 2-x, & 1 < x < 2 \\ 0, & x > 2 \end{cases}$

4. Prove that Fourier integral of the function $f(x) = \begin{cases} 1, & |x| \le 1 \\ 0, & otherwise \end{cases}$ is given by

   $f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{\sin\lambda\cos\lambda x}{\lambda}d\lambda.$

5. Find the Fourier sine and cosine transforms of xe^{-ax}

   Hence show that $\int_0^{\infty}\frac{\sin x}{x}dx = \frac{\pi}{2}$

6. The temperature $u(x,t)$ at any point of a semi infinite bar satisfies the equation

   $\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2},\ 0 < x < \infty,\ t > 0$, subject to conditions

   i. $u(0,t) = 0, \mathrm{t} > 0$

   ii. . $u(x,0) = \begin{cases} 1, & 0 < x < 1 \\ 0, & x > 1 \end{cases}$ Determine the expression for $u(x,t)$

7. Determine the distribution of temperature in the semi infinite medium, $x \ge 0$, when the end at $x = 0$ is maintained at zero temperature and initial distribution of temperature is $f(x)$.

Answers

1. $\frac{2(1-\cos a\lambda)}{\lambda^2}$ 2. $f(x) = \frac{2}{\pi(1+x^2)}$ 3. $f(x) = \frac{2}{\pi}\int_0^{\infty}\frac{(2\sin\lambda - \sin 2\lambda)}{\lambda^2}\sin\lambda x\, d\lambda$

5. $\frac{2a\lambda}{(a^2+\lambda^2)^2}$, $\frac{a^2-\lambda^2}{(a^2+\lambda^2)^2}$ 6. $u(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{1-\cos\lambda}{\lambda}e^{-\lambda^2 t}\sin\lambda x\, d\lambda$

7. $u(x,t) = \frac{2}{\pi}\int_0^\infty \bar{f}_s(\lambda) e^{-c^2\lambda^2 t} \sin\lambda x \, d\lambda$